Measure algebras and their second duals

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1 Overview of the Field

Let A be a Banach algebra. Then there are two natural products on the second dual A'' of A; they are called the *Arens products*; we here denote the products by $\Box$ and $\Diamond$, respectively. For definitions and discussions of these products, see [2, 4, 5], for example. We briefly recall the definitions. As usual, A' and A'' are Banach A-bimodules. For $\lambda \in A'$ and $\Phi \in A''$, define $\lambda \cdot \Phi \in A$ and $\Phi \cdot \lambda \in A'$ by

$$\langle a, \lambda \cdot \Phi \rangle = \langle \Phi, a \cdot \lambda \rangle, \quad \langle a, \Phi \cdot \lambda \rangle = \langle \Phi, \lambda \cdot a \rangle \quad (a \in A).$$

For $\Phi, \Psi \in A''$, define

$$\langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle \ (\lambda \in A'),$$

and similarly for $\Diamond$. The *left topological centre* of A'' is defined by

$$Z^{(\ell)}(A'') = \{\Phi \in A'' : \Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in A'')\},$$

and similarly for the *right topological centre* $Z^{(r)}(A'')$. The algebra A is said to be *Arens regular* if $Z^{(\ell)}(A'') = Z^{(r)}(A'') = A''$ and *strongly Arens irregular* if $Z^{(\ell)}(A'') = Z^{(r)}(A'') = A$. For example, every C^*-algebra is Arens regular [2].

There has been a great deal of study of these two algebras, especially in the case where A is the group algebra $L^1(G)$ for a locally compact group G. Results on the second dual algebras of $L^1(G)$ are given in [2, 16, 17], for example.

More recently, the three participants have studied [5] the second dual of a semigroup algebra; here S is a semigroup, and our Banach algebra is $A = (\ell^1(S), \star)$. We see that the second dual A'' can be identified with the space $M(\beta S)$ of complex-valued, regular Borel measures on βS, the Stone–Cech compactification of S. In fact, $(\beta S, \Box)$ is itself a subsemigroup of $(M(\beta S), \Box)$. (See [15] for background on $(\beta S, \Box)$.)

Let A be a Banach algebra which is strongly Arens irregular, and let V be a subset of A''. Then V is *determining for the topological centre* if $\Phi \in A$ for each $\Phi \in A''$ such that $\Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in V)$. Recently it has become clear that various 'small' subsets of A'' are determining for the topological centre in the case of some of the above algebras. For example, in [5], we showed that, for a wide class of semigroups including all cancellative semigroups, there are just two points in the space βS that are determining for the topological centre of $\ell^1(S)''$. For an extension of these results to the case of various weighted convolution algebras, see [4] and [3].

Let G be a locally compact group. The measure algebra $M(G)$ of G has also been much studied. This algebra is the multiplier algebra of the group algebra $L^1(G)$. Even in the case where G is the circle group $\mathbb{T}$, the Banach algebra $M(G)$ is very complicated; its character space is 'much larger' than the dual group $\mathbb{Z}$ of $\mathbb{T}$ [14].

Starting at a BIRS 'Research in Teams' in September, 2006, the three participants have been studying the algebras $(M(G)'', \Box)$ and $(L^1(G)'', \Box)$. The report on that week discussed our progress in 2006. Work by the participants continued, and in 2007 and 2008 we established a number of other results that are now all contained in a memoir [6].

The first part of our memoir studies the second dual space of $C_0(\Omega)$, where Ω is a locally compact space. This second dual is identified with $C(\widetilde{\Omega})$ for a certain compact hyper-Stonean space $\widetilde{\Omega}$. The seminal paper on this space is the classic [10] of Dixmier, but we were able to establish some results in this setting that go beyond [10]. We then turn to the algebras $(M(G)'', \Box)$ and $(L^1(G)'', \Box)$ when G is a locally compact group. For example, [6] contains many results on the semigroup structure of $\widetilde{G}$, which is the natural analogue of βS in the non-discrete case. Indeed it is shown in [6, Chapter 8] that $(\widetilde{G}, \Box)$ is semigroup if and only if G is discrete, and in [6, Chapter 7] that the space $\widetilde{G}$ determines the locally compact group G, a result that was already known in the case where G is compact [13].

The plan for the present workshop had two aspects: (1) to complete the memoir [6]; (2) to study the spaces $AP(M(G))$ and $WAP(M(G))$ of almost periodic and weakly almost periodic functionals, respectively, on $M(G)$, where G is a locally compact group.

2 Recent Developments and Open Problems

There have recently been three very striking advances in our area. These all occurred after our proposal to BIRS was written, and so that that document does not take account of them.

1) Let G be a locally compact group. M. Daws of Leeds has made a dramatic advance [8] on the study of $AP(M(G))$ and $WAP(M(G))$: by using results on Hopf–von Neumann algebras and his earlier work [7], he showed that both of these spaces are C^*-subalgebras of the space $M(G)'$, so resolving a central problem that had been raised in our proposal.

Nevertheless, many problems about $AP(M(G))$ and $WAP(M(G))$ remain open. For example, we know that

$$X_G \subset AP(G) \subset AP(M(G)) \subset WAP(M(G)) \subset M(G)' = C(\widetilde{G}),$$

where X_G is the closed linear span of the character space of $M(G)$. Is it true that $AP(G) = AP(M(G))$ only if G is discrete? If H is a subgroup of G, what is the relation of $AP(M(H))$ to $AP(M(G))$? Several related questions are stated in [6], and may form the basis of a future proposal to BIRS.

2) As stated above, it is known that the group algebra $L^1(G)$ is strongly Arens irregular for each locally compact group G. It is a next obvious question to determine some 'small sets' that determine the topological centre of $L^1(G)''$. Let Φ be the character space, or spectrum, of the commutative C^*-algebra $L^\infty(G)$.

First suppose that G is compact. Then a proof in [16] shows that the family of right identities in $(M(\Phi), \Box)$, a subset of Φ, is determining for the topological centre of $L^1(G)''$.

Second, suppose that G is a locally compact, non-compact group. Then a set which is determining for the topological centre of $L^1(G)''$ is specified by Neufang in [18, Theorem 1.1] (with a certain set-theoretic condition). A further paper of Filali and Salmi [11] establishes in an attractive way that $L^1(G)$ is strongly Arens irregular, and unifies this result with several related results.

Third, our memoir [6] proves that the space Φ (together with two further points in the non-compact case) is determining for the topological centre.

Shortly before the meeting in Banff, we received the very impressive paper of Budak, Işik, and Pym [1] that proves a much stronger result in the case where G is not compact, namely that there are just two points $\varphi_a, \varphi_b \in \Phi$ such that $\{\varphi_a, \varphi_b\}$ is determining for the topological centre of $L^1(G)''$.

The above all leave open the question in the case where the group is compact. Let G be a compact group (such as $\mathbb{T}$). Could it be that a smaller set than Φ is sufficient to determine the topological centre? In fact, at least in the case where G has a at most $\mathfrak{c}$ Borel subsets, it is shown in [6] that we only need $\mathfrak{c}$ points, whereas the fibre has cardinality at least $2^{\mathfrak{c}}$. (Here $\mathfrak{c}$ is the cardinal of the continuum.) However the main open **question** is: *Is there always a finite or countable set S of points in Φ such that S is determining for the topological centre of $L^1(G)$?*

3) The question whether or not the Banach algebra $M(G)$ is strongly Arens irregular for each locally compact group was raised in [12]. This question was resolved positively for non-compact groups G by Neufang in [19], leaving open the question for compact groups. Our proposal stated that we planned to study the Banach algebra $M(G)$, and in particular to seek to show that $M(G)$ is strongly Arens regular for each compact group G. We were not able to resolve this question, although some partial results are given in [6, Chapter 10].

Very shortly before the meeting in Banff, we received from Matthias Neufang an announcement of the following result [20]. Let G be a compact, infinite group of non-measurable cardinality at most $\mathfrak{c}$. Then $M(G)$ is strongly Arens irregular. As yet, we have not had an opportunity to study the proof of this exciting result. Again it suggests the quest of finding small subsets of $\widetilde{G}$ that are determining for the topological centre of $M(G)$.

There is a variety of open questions at the end of [6]. One which we find attractive is the following. Let X be a compact space such that $C(X)$ is isometrically isomorphic to the second dual space of a Banach space. Is it necessarily true that there is a locally compact space Ω such that $X = \widetilde{\Omega}$? Some partial results are given in [6].

3 Presentation Highlights

Since this was a workshop for three people assembled for 'Research in teams', there were no formal presentations.

4 Scientific Progress Made

We made progress in two related areas.

First, we studied the draft of the memoir [6] carefully, and made a number of minor corrections, clarifications, and extensions; we included also references to recent results. This memoir has now been submitted to the *Memoirs of the American Mathematical Society*, and is available at the website:

http://www.amsta.leeds.ac.uk/ pmt6hgd/dales.html.

Second, we made further study of the space $\widetilde{\Omega}$ mentioned above. There is an equivalence relation $\sim$ on the set $\widetilde{\Omega}$, defined by saying that $\varphi \sim \psi$ if $\varphi, \psi \in \widetilde{\Omega}$ are not separated by the images of the bounded Borel functions on Ω. The subset U_Ω of $\widetilde{\Omega}$ is the union of the sets Φ_μ, where Φ_μ is the character space of the C^*-algebra $L^1(\mu)'$ for μ a positive measure, and $[U_\Omega]$ is the collection of points of $\widetilde{\Omega}$ that are equivalent to a point in U_Ω. We established the following theorem, and several similar results; it seems that results of this type were not considered before, perhaps rather surprisingly.

Theorem Let Ω be an uncountable, compact, metrizable space. Then

$$|\beta\Omega_d \setminus [U_\Omega]| = |[U_\Omega]| = \left|[U_\Omega] \cap \widetilde{\Omega}_c\right| = \left|\widetilde{\Omega}_c \setminus [U_\Omega]\right| = 2^{2^{\mathfrak{c}}}\,.$$

Further, suppose that $\varphi \in \widetilde{\Omega}_c$. Then

$$\left|[\varphi] \cap \widetilde{\Omega}_c\right| = 2^{2^{\mathfrak{c}}}\,.$$

Here $\widetilde{\Omega}_c$ and $\beta\Omega_d$ are the subset of $\widetilde{\Omega}$ corresponding to the space of continuous and discrete measures on Ω, resepctively.

5 Outcome of the Meeting

The three participants have submitted for publication the memoir [6] that they produced.

All three participants will attend a meeting in Cambridge in July, 2009, in honour of the 75th anniversary of Dr. Dona Strauss; we shall meet several experts in area, including Pym and Neufang, and we expect to have useful discussions. Dales will speak on some recent work in [6].

Dales and Daws will attend the 19th International Conference on Banach algebras in Bedlewo, Poland, in July, 2009, and will have discussions on their work there.

Lau will visit the other two authors in England in November 2009; we expect to discuss further related topics during his visit. We expect that at that time we shall prepare a proposal for a future visit to BIRS for 'Research in Teams'. Further, Lau will speak on work in [6] at a conference in Taiwan in December, 2009.

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