Transversal and Helly-type Theorems in Geometry, Combinatorics and Topology

Imre Barany (Renyi Institute)
Ted Bisztriczky (University of Calgary)
Jacob E. Goodman (City College, City University of New York)
Luis Montejano (Universidad Nacional Autónoma de México)
Deborah Oliveros (Universidad Nacional Autónoma de México)

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A prominent role in combinatorial geometry is played by Helly's theorem, which states the following:

Helly's Theorem (1913 [13]). *Let $\mathcal{A}$ be a finite family of at least $d+1$ convex sets in R^d. If every $d+1$ members of $\mathcal{A}$ have a point in common, then there is a point common to all members of $\mathcal{A}$.*

Helly's theorem also holds for infinite families of compact convex sets, and has stimulated numerous generalization and variants. Results of the type "if every m members of a family of objects have property $\mathcal{P}$ then the entire family has the property $\mathcal{P}$" are called Helly-type theorems. The minimum positive integer m that makes this theorem possible is called the Helly number. Helly-type theorems have been the object of active research, inspired by many of the problems posed in "Helly's Theorem and Its Relatives" [7].

In the past ten years, there has been a significant increase in research activity and productivity in the area. (For an excellent survey in the area, see [24].) Notable advances have been made in several subareas including the development of a theory of hyperplane transversals (see [15]); the proofs of interesting colorful theorems generalizing classical results (see [23]); and many others such as the problem of finding a line transversal to a family of mutually disjoint congruent disks in the plane.

There are many interesting connections between Helly's theorem and its relatives, the theorems of Radon, of Caratheodory and of Tverberg. In fact, one of the most beautiful theorems in combinatorial convexity is Tverberg's theorem, which is the r-partite version of Radon's theorem, and it is very closely connected with the multiplied, or colorful versions of the theorems of Helly, Hadwiger and Caratheodory. The first of these colorful versions was discovered by Barany and Lovasz and has many applications (see [3]).

This workshop brought together senior and junior researchers in the area with the objective of interchanging ideas and assessing recent advances, of fostering awareness of the inter-disciplinary aspects of the field such as geometry, topology, combinatorics, and computer science, and of mapping future directions of research.

On Helly-type theorems, we had some talks dealing with colorful type theorems. In particular, Tamon Stephen spoke about colorful simplicial depth, joint work with A. Deza, S. Huang, T. Terlaky and H. Thomas. The simplicial depth of a point p in R^d with respect to a finite set S of points is the number of $(d+1)$-sets from S whose convex hull contains p. In statistics, this is a measure of how well p represents S.

A natural generalization is to consider colors on the points of S and to pay attention only to the colorful simplices containing p. In this presentation, the speaker exhibited a configuration where p is in the convex hull of each of $d+1$ colors, but is only in d^2+1 colorful simplices, conjecturing that this is minimal and proving a quadratic lower bound. This result sharpens Bárány's Colorful Carathéodory Theorem and gives an improved lower bound for monochrome simplicial depth.

Ricardo Strausz, reporting on joint work with J. Bracho and J. Bokowski, also spoke about some generalization of the famous Barany-Carathedory theorem to oriented matroids of Euclidian dimension 3. David Csaba Toth presented a study of the worst-case complexity of the union of rainbow simplices determined by a set of colorful points in Euclidean space. He also considered possible data structures to support efficient containment queries.

A very nice natural generalization of Helly's theorem is the piercing problem, also known as the (p,q) problem, and was first investigated by Hadwiger and Debrunner [12]. On this topic, Pablo Soberon gave interesting and deep bounds to the piercing number of a general family of convex sets in terms of the $(p,q)_r$ property, that is, a finite family of convex sets with the property that for every p convex sets there are at least r q-tuples that intersect.

On the same lines, Deborah Oliveros introduced the notion of k-tolerance and presented several results related to the tolerance versions of the Helly, Caratheodory and Tverberg classical theorems.

One of the problems that has received much attention in this field has been the problem of finding a line transversal to a family of mutually disjoint compact convex sets. (For an excellent survey on transversal theorey, see [14].) In 1945, Vincessini asked whether there is a Helly-type theorem for line transversals to a family of convex sets in R^2. In other words, is there a number m such that if every m members of the family are simultaneously intersected by a line, then there exists a single line that intersects all members of the family? The answer is no, even for line transversals to families of pairwise disjoint line segments.

In 1955 Hadwiger [11] posed the problem of determining the smallest number k with the property that if every collection of k members of the family of $n \leq k$ pairwise disjoint unit disks in the plane are met by a line, then all the disks are met by a line; that is, he proposed to find a Helly number for the problem of finding a line transversal to a family of disjoint unit disks in the plane. There is an example proposed in the same paper by Hadwiger, consisting of 5 almost touching disks centered at the vertices of a regular pentagon in such a way that every four of them have a line transversal but the set of all of them does not. Hadwiger's problem was solved by Danzer [8], showing that a Helly number does exist for $k=5$. In 1989 Tverberg [22] gave a generalization of Danzer's theorem on unit disks for disjoint translations of a compact convex set in the plane.

Let us denote by $\mathcal{F}$ a family of *ovals* (compact convex sets with non-empty interior) in the euclidean plane and let us say that $\mathcal{F}$ has property T if there is a line that intersects all members of $\mathcal{F}$. If there is a line that meets not all but at most k members of $\mathcal{F}$, then $\mathcal{F}$ has the property $T-k$. Finally, if each k-element subfamily of $\mathcal{F}$ has a transversal line, then $\mathcal{F}$ has property $T(k)$. In fact it is known that there is a family $\mathcal{F}$ of congruent ovals such that $T(5)$ does not imply T in general.

Later, in 1980, M. Katchalski and T. Lewis [17] proved that if $\mathcal{F}$ is a family of translations of an oval, then $T(3)$ implies $T-k$ for some universal constant k. This theorem generates several open branches for interesting research options. In this workshop, problems of this type were presented, such as the one presented by Ferenc Fodor who gave an an outline of a proof that $T(4)$ implies $T-1$ for a family $\mathcal{F}$ of n mutually disjoint unit disks in the plane (joint work with T. Bisztriczky and D. Oliveros).

Aladar Heppes talked about line transversals in super-disjoint $T(3)$-families of translations of an oval. Two translates, K_i and K_j, are said to be φ-disjoint, $\varphi > 0$, if the concentric φ-enlarged copies of K_i and K_j are disjoint. It is well known that in a φ-disjoint family of congruent discs, $T(3) \Rightarrow T$ if $\varphi > \sqrt{2}$, and $T(3) \not\Rightarrow T$ if $\varphi < \sqrt{2}$. In his talk, Professor Aladar Heppes discussed finite φ-disjoint $T(3)$-families of translates of an oval different from the disk.

T. Bistriczky, A. Heppes and K. Boroczky presented a preliminary report on the $T(5)$ property for families of overlapping unit disks, where they consider a finite family $\mathcal{F}$ of unit disks in the plane with the properties $T(k)$: Any k-element subfamily of $\mathcal{F}$ has a (line) transversal, and $O(d)$: The distance between the centres of any two elements of $\mathcal{F}$ is greater than d. It is well known that $\mathcal{F}$ has a transversal in each of the following cases:

$$\begin{array}{rll} & k=3 & \text{and } d > 2(\sqrt{2}) \text{ (sharp)}, \\ & k=4 & \text{and } d > 4/\sqrt{3} \text{ (sharp)}, \\ \text{and} & k=5 & \text{and } d \geq 2. \end{array}$$

In particular, they presented results for the case when $k=5$ and $d=\sqrt{3}$.

Andreas Holmsen presented new and interesting results concerning $T(k)$-families related to classic results and conjectures of Katchalski and Eckhoff. For a family of convex sets in the plane satisfying $T(k)$, it is known that every $T(k)$ family has a partial line transversal of size at least $a(k)|F|$, where $a(k)$ is a function that tends to 1 as k tends to infinity. Previous bounds on this function are due to Katchalski-Liu (1980) and Eckhoff (2008). In this workshop, he presented some new (and sharper) bounds on $a(k)$.

Line transversals to convex polytopes were also treated during the workshop by Otfried Cheong and Natan Rubin. O. Cheong gave a very interesting talk about isolated line transversals to convex polytopes in R^3, where instead of treating general convex sets, he focuses on a family $\mathcal{F}$ of convex polytopes in R^3 satisfying property T. If in addition, such line transversals satisfy the property of being an isolated point in the space of line transversals to $\mathcal{F}$, we say $\mathcal{F}$ is a *pinning* of the line transversal.

In his talk, O. Cheong showed that any minimal pinning of a line by polytopes in R^3 such that no face of a polytope is coplanar with the line has size at most eight. Moreover, if in addition the polytopes are disjoint, then it has size at most six. He completely characterized configurations of disjoint polytopes that form minimal pinnings of a line.

Natan Rubin presented a joint work with Haim Kaplan and Micha Sharir. They showed some combinatorial and algorithmical properties of line transversals of convex polytopes. In his presentation he discussed an upper bound of $O(n^2k^{1+e})$, for any $e > 0$, on the combinatorial complexity of the set $T(P)$ of line transversals of a collection P of k convex polyhedra in R^3 with a total of n facets. Thus when $k << n$, this is an improvement on the previously best known bounds, which are nearly cubic in n. Their analysis was curiously related to the three-dimensional variant of the following fundamental problem in geometric transversal theory:

Given a collection $\mathcal{C}$ of k pairwise-disjoint convex sets (of arbitrary description of complexity) in R^d, find a bound for the maximum number $g_d(k)$ of geometric permutations, i.e., the maximum number of distinct orders in which the transversal lines visit individual elements of $\mathcal{C}$. He also presented a related algorithmic ray-shooting problem for the above collection P of convex polyhedra, showing how to pre-process P into a data structure which uses $O^*(n\ poly(k))$ storage and answers ray-shooting queries in polylogarithmic time, provided that the ray origins are restricted to lie on a fixed line. This is a substantial improvement over previously known techniques which require super-quadratic storage (as a function of n, the number of facets). This result can be generalized to a number of other cases when the lines containing query rays have three degrees of freedom. Once again, handling distinct transversal orders is a key ingredient of the eventual solution.

Instead of relating transversal hyperplanes to families of convex bodies we can also consider having supporting hyperplanes of convex bodies and ask about Helly-type theorems for this case. On this topic, we had an excellent exposition given by Valeriu Soltan, in which he presented "Helly-type Results on Common Supports of Convex Bodies." Following Dawson and Edelstein, he introduces the following definition: A family $\mathcal{F}$ of convex bodies in R^n has the property S provided there is a hyperplane H that supports every member of $\mathcal{F}$. Similarly, $\mathcal{F}$ has the property $S(k)$ if every k-membered subfamily of $\mathcal{F}$ has the property S. He discussed some results and problems related to the Helly-type condition $S(k) \Rightarrow S$.

Although many of the talks at this workshop on transversals and Helly-type theorems dealt essentially with classical subjects related to this area, many of the talks also had deep relationships with other areas of discrete and non-discrete mathematics, such as algebraic topology, polytopes, convex bodies, packing, geometric permutations, and algebraic geometry.

In the area of well known problems, K. Bezdek presented a survey of the Tarski plank problem that included new results and a list of interesting open research problems on the discrete geometric side of the plank problem. In the 1930s, Tarski introduced his plank problem at a time when the field of Discrete Geometry was about to be born. It is quite remarkable that Tarski's question and its variants still continue to generate interest in the geometric and analytic aspects of coverings by planks today.

Gergely Ambrus gave a talk on recent developments in the polarization problem. The polarization problem states that for any system $u_1, \ldots, u_n$ of unit vectors in an n-dimensional real Hilbert space, there exists a unit vector v such that $(u_1, v) \ldots (u_n, v) \geq n^{-n/2}$. In his talk, he presented an overview of recent results. In particular, he posed a natural, stronger conjecture, and transformed both problems into a geometric setting. By this means, it turns out that the strong polarization problem serves as the "proper" real analogue of the complex plank problem, which was proved by K. Ball in 2001.

Arseniy Akopyan spoke on some generalizations of a diameter of sets and Jung's problem. Let M be a metric space. For each p-element set $W \subset M$, there exists a q-element subset $U \subset W$ of diameter 1. Then M can be divided into parts, $M_1, M_2, \ldots, M_{p-1}$, in a such way that $\mathrm{diam} M_1 + \mathrm{diam} M_2 + \cdots + \mathrm{diam} M_{p-1} \leq [\frac{p-1}{q-1}]$.

Antoine Deza presented some work on the Hirsch conjecture and more bounds on the diameter of convex polytopes. He introduced the notion of $\Delta(d, n)$ as the maximum possible edge diameter over all polytopes defined by n inequalities in dimension d. Hirsch's conjecture, formulated in 1957, states that $\Delta(d, n)$ is not greater than $n - d$. No polynomial bound is known for $\Delta(d, n)$, the best one being quasipolynomial and due to Kalai and Kleitman in 1992. Goodey showed in 1972 that $\Delta(4, 10) = 5$ and $\Delta(5, 11) = 6$.

Recently, Bremner and Schewe proved that $\Delta(4,11) = \Delta(6,12) = 6$. In this follow-up work, he showed that $\Delta(4,12) = 7$ and presented evidence that $\Delta(5,12) = \Delta(6,13) = 7$.

At this workshop we had two talks about packing problems, too. The first was given by Andreas Bezdek on finite packings and the second one by W. Kuperberg on packing densities of convex cones. In this talk he considered the family C of convex bodies K in R^3, each of which is a cone over a convex plane disk, and packing densities of all members of this family. If the admissible packings allow translations of K or translations of K and $-K$ (the symmetric image of K) only, then he showed that there is a supremum smaller than 1 and an infimum greater than 0 for the packing densities of all K in C, and these extreme values are attained at certain members of C. Since these densities are affine invariants, the packing density of each K in C depends only on its base. All four problems of finding the convex plane regions that produce cones whose packing densities of this sort are extreme, and remain open. The analogous four problems on covering are open as well. In his talk, he gave a motivation for considering arrangements of convex bodies consisting of translates of K and $-K$ only, and he discussed some partial results and related ideas.

David Larman gave a nice presentation about skeleta and shadow boundaries of convex bodies. In his talk, he defined the s-skeleton of a convex body K in euclidean space as the set of points of K which are not at the centre of an $s+1$ dimensional disc contained in K. So for a polytope, the 1-skeleton is the usual set of edges and vertices. In his talk, D. Larman described many of the unsolved problems relating to the s-skeleton as well.

Geometric permutations are an important subject in Helly-type theorems and geometric transversal which could not, of course, be neglected during this workshop. Several interesting talks were given in this area: A geometric permutation of a family $\mathcal{F}$ of convex sets is a pair of k-orderings induced by a k-transversal of the family $\mathcal{F}$. The technique of double permutation sequences applied to the subject of arrangements of pseudolines was presented by R. Pollack. In his presentation, he gave varius ideas and talked about his joint work with several of his colleagues such as Jacob E. Goodman, Raghavan Dhandapani, Andreas Holmsen, Shakhar Smorodinsky, Rephael Wenger, and Tudor Zamfirescu. He (re)introduced double permutation sequences, which provide a combinatorial encoding of arrangements of convex sets in the plane, and recalled the notion of a topological affine plane and several of its properties (some new). In particular, he mentioned that there is a universal topological affine plane P (i.e. any finite arrangement of pseudolines is isomorphic to some arrangement of finitely many lines of P).

Helge Tverberg also gave an interesting talk about two problems from the Asinowski-H-K-T paper published in 2003. The first problem deals with a pair of geometric permutations of disjoint translates $A, B, \ldots$ of an oval K in the plane. For which K can one have both permutations $ABCX$ and $BXAC$ simultaneously, i.e. four given sets that admit both of these permutations? The second problem deals with a sharpening of Theorem 7b. Here one wishes to find a more complete characterization of those K for which there are arbitrarily large families of translates of K, admitting three geometric permutations of the forms $WBACXW', WABCXW', WBXACW'$.

Xavier Goaoc spoke about the growth rate of families of (geometric) permutations in a joint work with Otfried Cheong (KAIST, Korea) and Cyril Nicaud (Univ. Marne-La-Vallee, France). In this interesting talk, he showed how the asymptotic behavior of $P(m,k,n)$ as n goes to infinity depends on m and k, where $P(m,k,n)$ denotes the maximum size of a family of permutations on $[n] = \{1,\ldots,n\}$ that has at most k distinct restrictions to any m elements of $[n]$. He described some results in this direction.

Applications of algebraic topology to discrete geometry was an especially interesting topic. The common theme of several of the talks given in this workshop relating algebraic topology and discrete geometry, transversals and Helly-type theorems is the topology of the space of Grassmannians and its canonical vector bundle together with the structure of the cohomology ring of these Grassmanian spaces. (See [6] and [21].)

In the 1990s, techniques and ideas of algebraic topology began to be used in a relevant and deep manner to solve purely combinatorial problems. In fact, Laszo Lovasz used algebraic topology [19], particularly the theory of Z_2-equivariant maps and the Borsuk-Ulam theorem to solve a conjecture of Kneser [18] concerning the chromatic number of the graph of k-subsets of an n-set. See [20].

After the publication of Lovasz paper, several different proofs were published [3]. One of these proofs related computation of the chromatic number of the Kneser graphs, which is a purely combinatorial problem, with the following geometric problem: What is the maximum number n such that any finite set $N \subset R^d$ of size n has a hyperplane transversal to the family of all convex hulls of k-set of N? It turns out that this number is related to the chromatic number of the Kneser graph $G^2(n,k)$.

In his talk, Ramirez-Alfonsin defined $M(k,d,\lambda) =$ the maximum positive integer n such that every set of n points in R^d has the property that the convex hull of all k-sets have a transversal $(d-\lambda)$-plane, and he introduced a special *Kneser hypergraph* establishing a close connection between its *chromatic number* and $M(k,d,\lambda)$. In fact, he defined the *Kneser hypergraph* $KG^{\lambda+1}(n,k)$ as the hypergraph whose vertices are $\binom{[n]}{k}$ and a collection of vertices $\{S_1,\dots,S_\rho\}$ is a hyperedge of $KG^{\lambda+1}(n,k)$ if and only if $2 \leq \rho \leq \lambda+1$ and $S_1 \cap \cdots \cap S_\rho = \phi$. He remarked that $KG^{\lambda+1}(n,k)$ is the Kneser graph when $\lambda = 1$. Furthermore he noted that the Kneser hypergraph defined by him is different from that defined in [2] and using the cohomology structure of the space of Grasmannians and following the spirit of Dolnikov [9]. It is possible to prove that

$$\chi(KG^{\lambda+1}(n,k)) \leq d-\lambda+1, \text{ then } n \leq M(k,d,\lambda).$$

Finally, he conjectured that $M(k,d,\lambda) = (d-\lambda)+k+\lceil \frac{k}{\lambda} \rceil - 1$.

In the first talk by R. N. Karasev, he discussed some results on the topology of the real Grassmannian and its canonical vector bundle. These topological claims are mostly formulated in terms of the cohomology index of the antipodal Z_2-action on the sphere space of the canonical bundle.

One corollary of these topological results is a theorem that establishes the existence of a k-flat transversal for a family $\mathcal{F}$ of $d+1$ convex compact sets in R^d, provided that for any $K \in \mathcal{F}$ the intersection $K \cap \partial\,(\operatorname{conv} \bigcup \mathcal{F})$ has no antipodal points, and any $(d-k)$-dimensional linear image of $\bigcup \mathcal{F}$ is convex. Omitting the requirement of convexity of any $(d-k)$-dimensional image, we obtain the existence of an equidistant k-flat instead of the transversal.

A corollary on partitioning d measures in R^d into parts of prescribed measure (compare the ham sandwich theorem) by a single hyperplane was also discussed, generalizing some results of [5]. Another talk in this workshop about the topology of Grassmannians and its canonical vector bundle was the talk given by Luis Montejano. He defined when a family $\mathcal{F}$ of convex sets in R^d has a *topological ρ-transversal of index* (m,k). He established that $\mathcal{F}$ has a *topological ρ-transversal of index* (m,k), $\rho < m$, $0 < k \leq d-m$, if there are, homologically, as many transversal m-planes to $\mathcal{F}$ as m-planes through a fixed ρ-plane in R^{m+k}. More precisely, the family $\mathcal{F}$ has a topological ρ-transversal if

$$i^*([0,\dots,0,k,\dots,k]) \in H^{(m-\rho)k}(\mathcal{T}_m(\mathcal{F}), Z_2)$$

is not zero, where $i^*\colon H^{(m-\rho)k}(G(m+1,d+1),Z_2) \to H^{(m-\rho)k}(\mathcal{T}_m(\mathcal{F}),Z_2)$ is the cohomology homomorphism induced by the inclusion $\mathcal{T}_m(\mathcal{F}) \subset M(d,m) \subset G(d+1,m+1)$ and

$$[0,\dots,0,k,\dots,k] \in H^{(m-\rho)k}(G(d+1,m+1),Z_2).$$

Clearly, if $\mathcal{F}$ has a ρ-transversal plane, then $\mathcal{F}$ has a topological ρ-transversal of index (m, k) for $\rho < m$ and $k \leq d - m$. The converse is not true. It is easy to give examples of families with a topological ρ-transversal but without a ρ-transversal plane. He conjectured that for a family $\mathcal{F}$ of $k + \rho + 1$ compact, convex sets in euclidean d-space R^d, there is a ρ-transversal plane if and only if there is a topological ρ-transversal of index (m, k). The purpose of this was to state some cases of this conjecture and to use them, together with the Lusternik-Schnirelmann category and several versions of the colorful Helly Theorem of Lovasz, to obtain geometric results.

Some of these geometric results are the following:

For integers $n, m > 1$, let us now consider the configuration of points and lines in the plane that consists of nm points and $n + m$ lines, in which the first n lines, $\ell_1, \ldots, \ell_n$ are parallel and vertical and the next m lines $L_1, \ldots, L_m$ are parallel and horizontal. So every vertical line has exactly m points and every horizontal line has n points. Let us denote by $L_{n,m}$ the simplicial complex describing this configuration, in which we have n $(m-1)$-simplices, corresponding to vertical lines and m $(n-1)-$simplices corresponding to vertical lines. If $[n] = \{1, \ldots, n\}$, then the vertices of $L_{n,m}$ are $[n] \times [m]$ and for every $i = 1, \ldots, n$, the subset $\{(i, 1), \ldots, (i, m)\}$ is an $(m-1)$-simplex of $L_{n,m}$ and for every $j = 1, \ldots, m$, the subset $\{(1, j), \ldots, (n, j)\}$ is an $(n - 1)-$simplex of $L_{n,m}$.

In his talk, Montejano stated that for every linear embedding of $L_{n,m} \subset R^{n+m-3}$, either there is an $(n - 2)$-plane transversal to the $(m - 1)$-simplices of $L_{n,m}$ or there is an $(m - 2)$-plane transversal to the $(n - 1)$-simplices of $L_{n,m}$.

He also stated the following result in the spirit of the colorful Helly results [4]: Let $\mathcal{F}$ be a family of $(n + 1)(\rho + 2)$ compact, convex sets in $R^{n+\rho+1}$ painted with $n + 1$ colors, in which we have $\rho + 2$ convex sets of each color, $n \geq 1$, $\rho \geq 1$. Suppose that every heterochromatic $(n + 1)$-subset of F is intersecting. Then there is a color and a ρ-transversal plane to all convex sets of $\mathcal{F}$ painted with this color.

Homotopy Theory was also presented at the workshop. In his talk, Michel Pocchiola showed that any two arrangements of double pseudolines of the same size are homotopic via a finite sequence of mutations during which the only moving curves are the curves that belong to the set difference of the two arrangements. He showed us how the proof is based on a enhanced version of the Pumping Lemma of [10] of independent interest. He also discussed a second application of this enhanced version of the Pumping Lemma.

As mentioned above, there are many interesting connections between Helly's theorem and its relatives. At the workshop, Roman Karasev discussed some theorems focusing mostly on the Tverberg theorem. He proved a dual Tverberg theorem for hyperplanes based on the following notion of separation.

Let F_i be a family of $d+1$ hyperplanes in R^d in general position, $i = 1, \ldots, n$. We say that the n families $\{F_i\}_1^n$ are separated if the intersection of the n simplices generated by each one of the families is empty.

The dual Tverberg Theorem stated by Karasev is that a family F of $(d + 1)n$ hyperplanes in general position in R^d can be partitioned into n subfamilies which are non-separated, where n is a prime power. Karasev claimed that the main tool of his proof is to find the obstruction to the existence of a counterexample. This obstruction is defined as the Euler class of a equivariant vector bundle. He claimed that new results can be obtained by the multiplying the obstructions as in the following technical result:

Let $0 \leq m \leq d - 1$ and let r_i $(i = 0, \ldots, m)$ be powers of the same prime $r_i = p^{k_i}$. If p is odd, let $d - m$ be even. For each $i = 0, \ldots, m$ let f_i map continuously an $(r_i - 1)(d - m + 1)$-dimensional simplex

$\Delta_i = \Delta^{(r_i-1)(d-m+1)}$ to R^d. Then every simplex Δ_i has r_i points $x_{i1}, x_{i2}, \ldots, x_{ir_i} \in \Delta_i$ with pairwise disjoint supports so that all the points $f_i(x_{ij})$ are contained in a single m-flat.

Javier Bracho used the ruling structure of symmetric hyperboloids to study flat transversals to flats and convex sets of a fixed dimension. In his talk he recalled that Hadwiger considered in [11] the possibility of a Helly-type theorem for transversal lines to a family of convex sets. He also observed that an extra hypothesis about the hitting order of transversal lines to subfamilies of a given size must be assumed. His result was generalized by Goodman and Pollack (see [16]) to one of transversal hyperplanes with the notion of order type, which generalizes the notion of order for lines given by J. Arocha, L. Montejano and the speaker ([1]) and proves a Hadwiger-type theorem for transversal lines to convex sets of dimension 1. This is closely related to a Helly-type theorem for transversal lines to a family of lines in projective space (of any dimension). That is, if every six of them have a transversal line, they all do. In this talk, these ideas were extended to results about transversal flats to finite families of convex sets or projective flats of any pair of dimensions where an extra hypothesis is made concerning "general position" of the families. In this talk, a family of flats is defined to be k-generic if every $k+1$ of them are in general position, or equivalently, if no $k+1$ of them have a transversal $(k-1)$-flat.

The main results of these talks are that a family of m-generic n-flats has a transversal m-flat if every subfamily with $\frac{1}{2}(3n+2m+7)$ or fewer elements has a transversal m-flat and if a family of m-generic convex sets of dimension n has a transversal m-flat if they correspond to an order type of dimension m such that every subfamily with $2n+m+3$ elements has a transversal m-flat compatible with that order type.

The workshop was successful in many ways, bringing together old and new colleagues from all over the world. We had participants from many countries including Russia, England, France, USA, Mexico, Italy, Canada, Hungary, and Denmark, among others. The talks were far from being the only academic activity of the workshop. We had many formal and informal mathematical discussions and all these activities have given rise to many new research projects and new collaboration.

We had the chance to share nice anecdotes in the lounge of Corvett Hall, such as the one by Professor Tverberg about how he created his famous theorem, which has been mentioned several times above.

Walks along the river and the wonderful view of the mountains, especially the hike to the top of Sulfur Mountain on the free Wednesday afternoon, no doubt generated friendships and good mathematical discussions.

The workshop was so successful that many of the participants agreed to submit their work so that a special volume dedicated to this workshop could be published in the Journal of Discrete and Computational Geometry.

Unfortunately due to health issues, Professor Ted Bistriczky and Professor Eli Goodman were unable to come, but they were both missed and behind the organization at all times. We also had other absences, particularly some Mexican students who were unable to demonstrate their financial solvency to the Canadian Embassy. We strongly believe that something has to be done in terms of gender equity since only one female organizer was able to attend.

We appreciate and would like to say thank you for the support we have received from BIRS. The excellent facilities and environment that it provides are perfect for creative interaction and the exchange of ideas, knowledge, and methods within the Mathematical Sciences. We would like to thank programme coordinator Wynne Fong and Station Manager Brenda Williams for all their support in the organization of the conference. We would like to thank as well all the participants of the Transversal and Helly-type theorems in Geometry, Combinatorics and Topology Workshop 09w5047 for all their enthusiasm and the productive, enjoyable environment that was created.

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