Workshop on Affine Convex Geometric Analysis

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1. Overview of the Field

While affine differential geometry originated precisely a century ago in two of Tzitzeica's papers (*"Sur une nouvelle classe des surfaces"*, 1908, 1909), its full development is due to Blaschke and his School. Initially, affine invariants were systematically studied within affine differential geometry and then also by methods of partial differential equations of Monge-Ampère type. This area of research was initiated by Calabi, Pogorelov, Chen and Yau, among others. More recently, affine invariants started to play a central role in convex geometry and geometric analysis, and certain partial differential equations, both parabolic and elliptic, became inherently related to them. Subsequently, there have been numerous applications of affine invariants in the asymptotic theory of convex bodies, differential geometry, ordinary and partial differential equations, and even in seemingly unrelated fields like complex analysis, number theory, geometric tomography and image processing (see [5], [15], [20], [21]). Convex geometric analysis is undoubtedly now a flourishing area of research generating a lot of interest from adjacent fields as evidenced by the recent ICM talks of Alesker, Bárány, Barthe, Klartag, Szarek, Tomczak-Jaegermann and others.

Within the last few years, a substantial amount of research was devoted to investigate in depth the invariant valuations (or additive functions) occurring in the theory of convex bodies through the work of Alesker, Klain, Ludwig, Reitzner and Schneider. These results show that there is a small class of $\mathrm{SL}(n)$ invariant functionals and $\mathrm{SL}(n)$ covariant operators and underline their importance. Besides their intrinsic interest, these functionals and operators are naturally and essentially related to affine isoperimetric inequalities. Examples here include the Petty projection inequality (less known, but more powerful, than the Euclidean isoperimetric inequality), the Zhang-Sobolev inequality [24] and the Cramer-Rao inequality for star bodies, one of the basic inequalities in information theory [12]. Though there has been substantial progress concerning these questions, fundamental problems remain open: Petty's conjectured projection inequality, the reverse Blaschke-Santaló inequality, the reverse centroid body inequality, the slicing problem, to mention just a few.

Affine surface area, originally a basic affine invariant from the field of affine differential geometry, has recently attracted increased attention too.

It is the subject of the affine Plateau problem solved in $\mathbb{R}^3$ by Trudinger and Wang [22]. The classical affine isoperimetric inequality, which gives an upper bound for the affine surface area in terms of volume, proved to be the key ingredient in many problems. In particular, it was used to show the uniqueness of self-similar solutions of the affine curvature flow and to study the asymptotic behavior of this flow by Andrews [1], Sapiro and Tannenbaum [14]. On its turn, as motion by curvature provides an efficient way to smooth curves representing boundaries of objects, the affine curvature flow has successfully been employed in computer vision and image processing by Olver, Sapiro and Tannenbaum. In addition, since many basic problems in discrete and stochastic geometry are equi-affine invariant, affine surface area has found numerous applications in these fields.

Affine surface area in its classical definition involves the curvature function of a smooth convex body. Extensions of affine surface area to *arbitrary* convex bodies were obtained in the work of Lutwak, Leichtweiss and Schütt and Werner. Extensions of affine surface area to L_p-affine surface area were made available in the last decade through the work of Lutwak within the frame of the rapidly developing *L_p-Brunn Minkowski theory*. A focus of research in recent years was centered around the following problem: Are there "natural" geometric interpretations of L_p-affine surface area for *arbitrary* convex bodies? It is desirable to have such interpretations as affine surface area and L_p-affine surface area appear naturally in e.g. approximation of convex bodies by polytopes which in turn has important consequences in areas like computer vision. Recent contributions here include new interpretations of L_p-affine surface areas by Schütt and Werner [16] and characterization theorems by Ludwig and Reitzner [9].

In fact, as a central problem of the L_p-Brunn Minkowski theory, known as L_p-Minkowski problem, concerns the existence, uniqueness and stability of convex bodies with prescribed L_p-surface area, it is naturally related to the study of solutions of nontrivial ordinary and, respectively, partial differential equations (see for example Chou and Wang [3], Stancu [17], [18]). Additionally, one should note the growing body of work in the area of L_p-affine inequalities from which we quote only Lutwak, Yang and Zhang [11], [13] and Werner and Ye [23]. These inequalities are more powerful than their better known Euclidean relatives and the interest which they are generating is to be expected, as sharp affine inequalities have a wide applicability to partial differential equations, functional analysis, Minkowski geometry and other areas.

2. Presentation Highlights

The main lectures of the meeting provide an overview of the field and describe recent advancement.

Franck Barthe: Remarks on conservative spin systems and related questions in convexity

Abstract: The study of the dynamical aspects of non interacting conservative spin systems motivates the question of estimating the spectral gap and logarithmic constants of product measures conditioned to the sum of the coordinates being given:

$$\mu^{n|\rho} := \mu^n\Big(\cdot \,\Big|\, \sum_{i=1}^{n} x_i = n\rho \Big).$$

We give a complete treatment of the particular case of gamma distributed spins. The method relies on a representation of the conditioned measure as a push forward of a product measure, and uses recent convexity results of E. Milman. Our result actually confirms the Kannan-Lovász-Simonovits conjecture for the regular simplices (this conjecture predicts the value of the spectral gap of uniform measures on convex sets), and a slight modification of it recovers in a unified way the fact that the unit balls of ℓ_p^n also verify the conjecture. We then discuss related isoperimetric inequalities for unit balls and rotation invariant measures. On a technical level, a recurrent problem in the proofs is that the natural approaches lead to spectral gap or log-Sobolev inequalities with a weighted energy term. Luckily enough the convexity helps in removing the weight although it may be unbounded.

Peter Olver: Moving Frames, Differential Invariants and Surface Geometry

Abstract: I will begin with a brief discussion of the classical theory of differential invariants, followed by a survey, for the non-expert, of the new, equivariant approach to the method of moving frames that has been extensively developed with various collaborators over the past decade. The moving frame approach can be applied to arbitrary group actions, and algorithmically reveals the full structure of the associated algebra of differential invariants, pinpointing generating differential invariants as well as their differential syzygies. I will then present some surprising new results on differential invariant algebras in classical surface geometries, including equi-affine surfaces. I will close by discussing applications to equivalence problems, symmetry detection, image processing, and, time permitting, invariant surface flows.

Matthias Reitzner: Classification of SL(n) invariant Valuations

Abstract: Let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$. A functional $\Phi : \mathcal{K}^n \to \mathbb{R}$ that satisfies

$$\Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L)$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{K}^n$, is called a (real-valued) *valuation.* A valuation is $\mathrm{SL}(n)$ invariant, if $\Phi(AK) = \Phi(K)$ for every $A \in \mathrm{SL}(n)$ holds.

In 1937 Blaschke proved that every continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathcal{K}^3$ is a linear combination of volume and the Euler characteristic. In a joint paper with Ludwig we weakened the assumption of continuity. Every semi-continuous, $\mathrm{SL}(n)$ and translation invariant valuation is a linear combination of volume and the Euler characteristic and the affine surface area $\Omega(K)$. This raises the following general question:

Is it possible to classify all $SL(n)$ invariant valuations on $\mathcal{K}_0^n$?

Here $\mathcal{K}_0^n$ denotes the space of convex bodies that contain the origin in their interiors. A complete answer is contained in the following theorem (jointly with M. Ludwig):

A functional $\Phi : \mathcal{K}_0^n \to R$ is an upper semicontinuous and $SL(n)$ invariant valuation that is homogeneous of degree q if and only if there are constants c_0 and $c_1 \geq 0$ such that

$$\Phi(K) = \begin{cases} c_0\, V_0(K) + c_1\, \Omega_n(K) & \text{for } q = 0 \\ c_1\, \Omega_p(K) & \text{for } -n < q < n \text{ and } q \neq 0 \\ c_0\, V_n(K) & \text{for } q = n \\ c_0\, V_n(K^*) & \text{for } q = -n \\ 0 & \text{for } q < -n \text{ or } q > n \end{cases}$$

for every $K \in \mathcal{K}_0^n$ where $p = n(n-q)/(n+q)$.

Here Φ is called homogeneous of degree q, if $\Phi(t\,K) = t^q\,\Phi(K)$. And K^* denotes the polar body of K. The valuations $\Omega_p(K)$ in this theorem are the L_p *affine surface area* of a convex body K. This theorem shows that L_p affine surface areas are the natural definition for an $\mathrm{SL}(n)$ invariant surface area on $\mathcal{K}_0^n$.

Rolf Schneider: The role of the volume product in stochastic geometry

Abstract: The volume product $\mathrm{vp}(K) = \mathrm{vol}(K)\mathrm{vol}(K^o)$ of a (say, 0-symmetric) convex body in $\mathbb{R}^d$ is a fundamental invariant of affine convex geometry, with many applications. Its interpretation as a limit of the expected vertex number of a certain type of random polytope led S. Reisner to his first proof of the sharp lower bound of the volume product for zonoids. Together with the Blaschke–Santaló inequality, Reisner's inequality characterizes the range of the volume product for zonoids, with the extremal cases given by parallelepipeds and ellipsoids. This fact has applications in stochastic geometry, since a useful auxiliary convex body, Matheron's 'Steiner compact', is

a zonoid, and its volume product appears in expressions for some geometric parameters in different contexts.

After some historical remarks on inequalities for the volume product and on earlier applications in stochastic geometry, we present new results about typical faces in stationary Poisson hyperplane tessellations of $\mathbb{R}^d$. Typical k-dimensional faces of such a random mosaic can be defined in different ways. Heuristically, a typical k-face is obtained if one picks out a k-face at random, with equal chances. The weighted typical k-face is obtained if the random choice is done with weights proportional to the k-volume. For the typical k-face, it is known that the expected number of vertices is always 2^k, irrespective of the directional distribution of the random hyperplanes. For the weighted typical k-faces ($k \in \{2, \dots, d\}$), we establish sharp bounds for the expected vertex number, using the volume product. (An unpublished result of Matthias Reitzner, of which I learned at the workshop, allows one to do the same for expected facet numbers.) The minimum is attained precisely by parallel mosaics, generated by hyperplanes of d fixed directions; the maximum is attained by hyperplane processes with rotation invariant distributions. The discussion of the equality cases requires new geometric results on characterizations of parallelepipeds and ellipsoids within the class of zonoids, in terms of their projections.

Nicole Tomczak-Jaegermann: Random points uniformly distributed on an isotropic convex body

Abstract: The talk described a series of recent theorems concerning random matrices with i.i.d. columns uniformly distributed on an isotropic convex body, or more generally, distributed according to a log-concave measure on $\mathbb{R}^n$ with the identity as covariance matrix.

The results are contained in two papers: [ALPT], by Radoslaw Adamczak, Alexander Litvak, Alain Pajor and the speaker, and [AGLPT], by Adamczak, Olivier Guédon, Litvak, Pajor and the speaker.

The following question was investigated by R. Kannan, L. Lovász and M. Simonovits in 1995, motivated by a problem of complexity in computing volume in high dimension: *Let K be an isotropic convex body in $\mathbb{R}^n$. Given $\varepsilon > 0$, how many independent points X_i uniformly distributed on K are needed for the empirical covariance matrix to approximate the identity up to ε with overwhelming probability?* Kannan, Lovász and Simonovits proved that for an i.i.d sequence X_i of random vectors in $\mathbb{R}^n$ with uniformly bounded 4-th moments, $N \geq Cn^2/\varepsilon\delta$ vectors give the required approximation with probability $\geq 1 - \delta$.

A major step was then achieved by Bourgain who proved in 1996 that for any $\varepsilon, \delta \in (0, 1)$, there exists $C(\varepsilon, \delta) > 0$ such that $N = C(\varepsilon, \delta)n \log^3 n$ i.i.d. uniformly distributed points on an isotropic convex body satisfy the required approximation up to ε with probability $1 - \delta$. The powers of logarithm were decreased by Rudelson (still in the late 1990's), and already in this century by Giannopoulos-Hartzoulaki-Tsolomitis, and by Paouris. Several further papers by these and other authors developed related aspects of the problem.

Paper [ALPT] completely answers the Kannan-Lovász-Simonovits question. More precisely, let $X \in \mathbb{R}^n$ be a centered random vector with a log-concave distribution and with the identity as covariance matrix. An example of such a vector X is a random point in an isotropic convex body. We show that *for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$, such that if $N \sim C(\varepsilon)\, n$ and $(X_i)_{i \le N}$ are i.i.d. copies of X, then $\left\| \frac{1}{N} \sum_{i=1}^N X_i \otimes X_i - \mathrm{Id} \right\| \le \varepsilon$, with probability larger than $1 - \exp(-c\sqrt{n})$.* In the talk we provide some comments on the proof of this theorem.

An important related direction concerns norms of random matrices with independent log-concave columns (or rows). Again let $X \in \mathbb{R}^n$ be a centered random vector as above, consider N independent random vectors $(X_i)_{i \le N}$ distributed as X and define $A = A^{(N)}$ to be the $n \times N$ matrix with $(X_i)_{i \le N}$ as columns. For n, N arbitrary (and N not too large, actually, $n = N$ being the central case) the question is to prove an estimate for the norm $\|A\|$ as an operator $A : \ell_2^N \to \ell_2^n$, valid with overwhelming probability. This problem can be viewed as an "isomorphic form" of an upper estimate in the approximation of the identity discussed above, (for $n = N$, say), and the papers already mentioned provided some answers – with "parasitic" logarithmic factors – to this question as well. [ALPT] gives optimal estimates for $\|A\|$; for example, for the square matrix, if $n = N$, we have $\|A\| \le C\sqrt{n}$, with overwhelming probability. More details on this direction, including a sketch of the proof, was given in the talk by Sasha Litvak.

The paper [AGLPT] develops a related problem of estimating the smallest singular value of the random matrix $A = A^{(N)}$ introduced above. This is recently an active direction with major results for random matrices with i.i.d entries (by Litvak-Pajor-Rudelson-Tomczak, Rudelson, Tao-Vu and Rudelson-Vershynin). The theorem in [AGLPT], even in its preliminary form, pretty close recovers estimates recently established in the case of i.i.d entries. We refer to the talk of Olivier Guédon and to the forthcoming paper for more details and most recent developments.

3. Participants

Frank Barthe	Alexander Litvak	Dmitry Ryabogin
Gautier Berck	Monika Ludwig	Eugenia Saorín Gómez
Gabriele Bianchi	Mark Meckes	Rolf Schneider
Ted Bisztriczky	Mathieu Meyer	Franz Schuster
Andrea Colesanti	Efren Morales Amaya	Carsten Schütt
Matthieu Fradelizi	Carlo Nitsch	Alina Stancu
Yehoram Gordon	Peter J. Olver	Christian Steineder
Olivier Guédon	Carla Peri	Nicole Tomczak-Jaegerman
Christoph Haberl	Fedor Petrov	Elisabeth Werner
Daniel Hug	Peter Pivovarov	Marina Yaskina
Justin Jenkinson	Matthias Reitzner	Vladyslav Yaskin
Alexander Koldobsky	Omar Rivasplata	Artem Zvavitch
Joseph Lehec	Mark Rudelson	Marisa Zymonopoulou

4. Scientific Progress Made

One of the directions of discussion of our workshop centered around the affine invariant functional known as the volume product. It started with the talk of Rolf Schneider who showed that this functional appears naturally in stochastic geometry (see Schneider's abstract). The celebrated Blaschke-Santaló inequality establishes an upper bound for the volume product. Joseph Lehec presented new proofs for various functional Blaschke-Santaló inequalities, originally due to Keith Ball, to Artstein, Klartag and Milman and to Fradelizi and Meyer [4]. Unlike the original ones, Lehec's proofs are direct, in the sense that they do not rely on the classical Blaschke-Santaló inequality.

Similarly, it was interesting to see that the Blaschke-Santaló inequality was employed in a reverse Faber-Kahn inequality for the Monge-Ampère equation. Carlo Nitsch (see [2]) used a symmetrization procedure of Pólya-Szegö type to compare the eigenvalue of the Dirichlet problem of the Monge-Ampère operator on an arbitrary domain U with one where U is an ellipsoid. Implicitly, the fact that the first eigenvalue of the Monge-Ampère operator is maximal on ellipsoids is a consequence of the Blaschke-Santaló inequality. It was concluded that there are probably other spectral problems for elliptic operators in divergence form where one can employ similar techniques to get bounds on their eigenvalues.

The reverse Blaschke-Santaló inequality, known also as the Mahler's conjecture, remains an open problem in dimension greater than two. However, a number of talks offered new perspectives on its approach. Mathieu Meyer (talking about work done with K. Böröczky Jr, E. Makai and S. Reisner) gave a new proof of the reverse Blaschke-Santaló inequality in dimension two. Its novelty lies in the fact that this is the first proof which is tied up with a stability result. It not only shows that the minimum of the volume product functional on convex bodies containing the origin, and whose dual polar is taken with respect to the centroid, is reached on the simplex, but Meyer succeeded in getting an estimate of the gap for arbitrary convex bodies. Matthieu Fradelizi presented joint work with Franck Barthe on a lower bound for the volume product of convex bodies with many symmetries.

A second important direction of discussion of our workshop centered around valuations in affine convex geometric analysis. It started with the talk of Matthias Reitzner on the classification of $\mathrm{SL}(n)$ invariant valuations and the characterization of affine surface areas (see [9], [23]). The main theme of Reitzner's talk was taken up by Fedor Petrov and Daniel Hug. For a convex body $K \subset \mathbb{R}^n$ and an integer m, Petrov presented asymptotic estimates (as $m \to \infty$) for the number of integer points lying on the boundary of $m\,K$ and for the number of integer points in a small neighborhood of $m\,K$. These asymptotic estimates depend on the affine surface area of K. Hug presented results on random polytopes where the affine surface area is again determining the asymptotic behavior.

The classification of convex body valued valuations within affine convex geometric analysis motivated new affine isoperimetric inequalities by Christoph Haberl and Franz Schuster [7], [8]. Schuster presented these results and showed that they are stronger than Zhang's affine Sobolev inequality [24] (which in turn implies the sharp Sobolev inequality by Federer-Fleming and Maz′ya) and Lutwak-Yang-Zhang's affine L_p Sobolev inequality [13] (which in turn implies the sharp L_p Sobolev inequality by Aubin and Talenti).

Haberl presented a new classification of convex body valued valuations with respect to Blaschke addition compatible with the general linear group. Haberl proved that the only non-trivial such valuation is the curvature image operator on symmetric bodies. The classification of convex body valued valuations also underlines the importance of projection bodies. Christian Steineder [19] presented a surprising new application of this classical notion in symbolic dynamics. Steineder proves that the asymptotic complexity of a coding sequence of a convex polytope P is equal to the volume of the projection body P.

The classification of star body valued valuations provides characterizations of intersection bodies and L_p intersection bodies [6]. Whereas intersection bodies are by now classical (see [10]), new results on k-intersection bodies were presented by Vlad Yaskin and on non-intersection bodies by Marisa Zymonopoulou. A fundamental theorem of Busemann states that the intersection body of a centrally symmetric convex body is convex. Gautier Berck showed that Busemann's result is a special case of the more general result that for all p, the L_p intersection body of a centrally symmetric body is convex. Berck's result also provides an elegant approach to Busemann's original theorem. Moreover, the talks of Marina Yaskin, Vlad Yaskin and Marisa Zymonopoulou rely heavily on Fourier analytic methods and thus establish that methods from this area can be beneficial to affine convex geometry and vice versa.

In a similar spirit, a number of talks focused on interactions between probabilistic and affine geometric aspects. Alexander Litvak gave an outline of the proof of one of the theorems announced in Nicole Tomczak-Jaegermann's lecture. In his talk - based on a joint work with R. Adamczak, A. Litvak, A. Pajor and N. Tomczak-Jaegermann - Olivier Guédon investigated the behaviour of the smallest singular value of random matrices with i.i.d. random entries. The main result states that with large probability, this smallest singular value can not be significantly smaller than $n^{-1/2}$ (as it is the case for any matrices with independent subgaussian entries). Mark Rudelson addressed the question of how many points of the discrete cube are close to a subspace. This problem, which originally arose in the theory of ± 1 random matrices, has been extensively studied in the last 20 years. The answer depends on the arithmetic structure of the space E and is measured by the smallest Euclidean norm of a non-zero integer point, which belongs to a certain neighborhood of $E^{\perp}$. His results lead to the optimal upper bound of the number of the vertices of the discrete cube.

Mark Meckes studied the measure-theoretic analogue of Dvoretzky's theorem. He presents results that identify which marginals of K are guaranteed to be nearly Gaussian in the case that K is 1-unconditional or possesses the symmetries of a simplex. Yehoram Gordon, reporting on a joint work with M. Meyer, gave a geometric and probabilistic analysis of convex bodies with unconditional structures, and associated spaces of operators. In particular, his results lead to estimates on best random embeddings of k-dimensional Hilbert spaces in the spaces of nuclear operators.

More classical aspects of affine convex geometry were explored in the talks of Efren Morales Amaya and Eugenia Saorín Gómez. Morales Amaya, together with D. Larman and L. Montejano, gave a characterization of ellipsoids by means of parallel translated sections. Saorín Gómez, together with M. A. Hernandez Cifre, gave necessary and sufficient conditions when a convex body K in $\mathbb{R}^n$ can be written as a sum of an inner parallel body and its form body.

The close relation between affine convex geometry and differential equations was apparent throughout the workshop. In particular, it was the case in the talks of Andrea Colesanti and Marina Yaskina. A common theme of their talks was the Christoffel problem which asks to find a convex body in $\mathbb{R}^n$ given the mean radius of curvature as a function of the unit normal vectors to the boundary of the convex body.

Colesanti presented recent contributions to this problem coming from partial differential equations, due to Guan and Ma, while Yaskina presented a new approach to the Christoffel problem using Fourier transforms.

5. Outcome of the Meeting

Our meeting brought together researchers from various areas where affine convex geometric analysis plays an important role. Participants from the different fields were able to get insight in the current research trends of the connected areas. A total of 9 graduate students and new PhDs participated in the workshop. For those researchers in the early stages of their career, the workshop provided an excellent opportunity to learn about the most recent results in the field, to present their own research and to make new connections. Also, the timing of the workshop was symbolic: It was the 100 years anniversary of the first publications of the germs of affine differential geometry.

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