# NUMBER THEORY AND PHYSICS AT THE CROSSROADS

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This is the fifth gathering of this series of workshops in the interface of number theory and string theory. The first one was held at the Fields Institute in 2001. The subsequent workshops were all held at BIRS, and this one was the fourth BIRS workshop. The workshop met from May 8th to May 13th for five days. Altogether 40 mathematicians (number theorists and geometers) and physicists (string theorists) converged at BIRS for the five-days' scientific endeavors. About half of the participants were familiar faces, while the other half were new participants. There were 24 one-hour talks, and our typical working day started at 9:30am and ended at 9:30pm. Indeed we worked very hard, and the workshop was a huge success!

This fourth workshop was partially dedicated to Don Zagier on the occasion of his completing the first life cycle and reaching the 60 years of age.

This series of workshops has been getting very prominent in the community and attracted more applications than the workshop could accommodate.

There was overwhelming urge from the participants to organize a five-day workshop in two or three years time at BIRS. We are indeed planning to submit a proposal for the next BIRS workshop in due course.

## 1 Overview of the Field

One of the most significant discoveries in the last two decades in the theoretical physics is, arguably, string theory and mirror symmetry. Spectacular discoveries in string theory have inspired many new developments in mathematics. However, mathematicians and physicists have been living in the parallel universes, and evidently there have been very little interactions between the two sets of researchers, with some exceptions.

The original purpose of organizing this series of workshops was to promote and enhance communications between the two camps. This series of workshops brought together researchers to BIRS in number theory, algebraic/arithmetic/differential/symplectic/toric geometry, and physics (string theory) whose common interests were/are centered around modular forms, but not restricted to it. At the fourth workshop, we witnessed very active and intensive interactions of both camps from early mornings to late nights. These interactions proved to be very futile and fruitful, and each workshop at BIRS brought the two camps closer and closer. At the end of workshops, we all felt that all things modular have came together at BIRS from both sides: number theory and physics (in particular, string theory). At the end of the workshop, all participants felt that both camps have finally crossed boundaries and established relatively comfortable rapport.

The Proceedings of the 2003 BIRS workshop in this series entitled "Calabi–Yau Varieties and Mirror Symmetry" (eds. N. Yui et al.) has been published as "Mirror Symmetry V" in Studies in Advanced Mathematics Series from American Mathematical Society and International Press (2006). Since the 2003 workshop, many articles dealing with subjects in the interface of number theory, geometry and physics (string theory) have appeared, and time has come to have a mathematical journal devoted to articles in this area. Thus, in 2007, the new journal "Communications in Number Theory and Physics" (CNTP) was launched from International Press with editors-in-chief: R. Dijkgraaf, D. Kazdhan, M. Kontsevich and S.-T. Yau. The journal has been publishing excellent articles, providing a venue for dissemination of results in this interface well into the future. The journal has proved to be an enormous success. Many participants of this series of workshops have published articles, and many have served as referees for the journal.

For quite while now, we have witnessed that modular forms, quasi-modular forms and automorphic forms play central roles in many areas of physics, e.g., quantum field theory, conformal field theory, black holes, mirror symmetry, $F$-theory, and 4D gauge theory. Most prominently, generating functions counting the number of curves on Calabi–Yau manifolds (e.g., Gromov–Witten invariants), elliptic genera/partition functions of conformal field theory, and generating functions in 4D gauge theory are all characterized by some kinds of modular forms (classical modular forms, quasi-modular forms, mock modular forms, Jacobi forms, Siegel modular forms. etc.)

Physical duality symmetries relate special limits of the various consistent string theories (Types I, II, Heterotic string and their cousins, including F-theory) one to another. By comparing the mathematical descriptions of these theories, one reveals often quite deep and unexpected mathematical conjectures. The best known string duality to mathematicians, Type IIA/IIB duality also called *mirror symmetry*, has inspired many new developments in algebraic and arithmetic geometry, number theory, toric geometry, Riemann surface theory, and infinite dimensional Lie algebras. Other string dualities such as Heterotic/Type II duality and F-Theory/Heterotic string duality have also, more recently, led to series of mathematical conjectures (and results), many involving elliptic curves, K3 surfaces, modular forms of one variable, and Siegel modular forms.

The principal theme of this workshop was centered around modular forms in the broadest sense, though not restricted to it. Many types of modular forms enter the physics scenes, and one of the main goals of the field is to understand conceptually "why" modular forms appear often in physics landscapes.

# 2 Recent Developments and Open Problems

Since the last BIRS workshop in 2008, we have seen some new developments: a number of problems have been solved. Many, though not all, articles have been published in the Communication in Number Theory and Physics Volume 3 (2009) and Volume 4 (2010). Some sample results are listed below.

**(A)** Topological quantum field theory.

• The article of Dimofte, Gukov, Lenells and Zagier [3] developed several new methods for computing all-loop partition functions in perturbative Chern–Simons theory with complex gauge group $G_{\mathbf{C}}$. A notion of "arithmetic topological quantum field theory" was introduced and a conjecture (with supporting evidence) was formulated that $SL(2, \mathbf{C})$ Chern–Simons theory is an example of such a theory.

**(B)** Hodge structures and renormalization in physics.

• The article of Bloch and Kreimer [1] observed that the renormalization of Feynman amplitudes in physics is closely related to the theory of limiting mixed Hodge structures in Mathematics. Using Hodge theory, the Feynmann amplitudes are studied and classified, and the renormalization problem is reduced to the study of logarithmically divergent projective integrals. This paper has opened the floodgates and subsequently many articles on graph hypersurfaces and Feynman integrals have been published, for instance, [2].

**(C)** Elliptic genera, mock modular forms and Jacobi forms, and moonshine.

• The theory of mock modular forms are used to study elliptic genera of hyperKähler manifolds in terms of the representations of $\mathcal{N} = 4$ supercomformal algebra. For instance, the article of T. Eguchi and K. Hikami [6] obtained the exact formula for the Fourier coefficients of the elliptic genus for K3 surfaces, which counts the number of non-BPS representations. For three-dimensional cases, Jacobi forms are considered instead of mock modular forms.

• Subsequently, Eguchi, Ooguri and Tachikawa [7], they pointed out that the elliptic genus of the K3 surface has a natural decomposition in terms of dimensions of irreducible representations of the Mathieu group $M_{24}$. The reason is still a mystery. The recent article of Miranda Cheng [3] gave further evidence for this connection studying the elliptic genus of K3 surfaces twisted by some symplectic automorphisms. This gave a new “moonshine” for $M_{24}$.

Understanding the monstrous moonshine in connection with string theory is and will be a major open problem.

**(D)** The modularity of the Galois representations arising from Calabi–Yau varieties over $\mathbf{Q}$.

• The modularity of rigid Calai–Yau threefolds over $\mathbf{Q}$ has been established in Gouvêa and Yui [8]. These are $2$-dimensional Galois representations, and proof is built on the resolution of the Serre conjectures by Khare–Wintenberger [10] and Kisin [11] on the modularity of $2$-dimensional residual Galois representations. The result is that the Galois representation associated to a rigid Calabi–Yau threefold over $\mathbf{Q}$ is equivalent to a Galois representation of a modular form of weight $4$.

• For the modularity of higher ($> 2$) dimensional Galois representations arising from Calabi–Yau varieties over $\mathbf{Q}$, the article of Livné–Schütt–Yui [12] has established the (motivic) modularity of $16$ K3 surfaces defined over $\mathbf{Q}$ with non-symplectic group actions acting trivially on algebraic cycles (these K3 surfaces are studied by Vorontsov, Nikulin and Kondo). The Galois representations have dimension between $2$ and $20$ excluding $8$ and $14$. The main results are that these K3 surfaces are dominated by Fermat surfaces and thus of CM type. The modularity of the Galois representations is established by automorphic induction process.

The modularity of higher dimensional Galois representations arising from Calabi–Yau varieties remains an open problem. This is regarded as a concrete realization of the Langlands Program.

**(E)** The modularity of families of Calabi–Yau varieties.

Contrary to the modularity questions discussed in (**D**), the objects here are families of Calabi–Yau varieties, e.g., one-parameter families of $K3$ surfaces, or families of Calabi–Yau threefolds parametrized by Shimura varieties.

• Consider families of K3 surfaces which are lattice polarized by lattices of large ranks. Clingher and Doran [4] classified lattice polarized K3 surfaces by the rank 17 lattice $H \oplus E_8 \oplus E_7$, and described the coarse moduli space and the inverse of the period map in terms of Siegel modular forms, i.e., Eisenstein series $E_4, E_6$ of weight $4$ and $6$ and Igusa cusp forms of weight $10$ and $12$.

• The paper of Hashimoto and Terasoma [9] studied the period map of the family $\{\mathcal{X}_t\}$ with $t = (t_0, t_1) \in \mathbf{P}^1$ of quartic family of K3 surfaces defined by $\mathcal{X}_t : x_1 + \cdots + x_5 = t_0(x_1^4 + \cdots + x_5^4) + t_1(x_1^2 + \cdots + x_5^2)^2 = 0$ in $\mathbf{P}^4$ with homogeneous coordinates $(x_1 : \cdots : x_5)$. This family admits a symplectic group action by the symmetric group $S_5$. The Picard number of a generic fiber is equal to $19$. The inverse of the period map was constructed using automorphic forms of one variable. They use different method from that of Clingher and Doran. In fact, automorphic forms are constructed as the pull-backs of the fourth power of theta constants of genus $2$.

**(F)** Non-existence of mirror Calabi–Yau threefolds.

• The mirror principle states that given a Calabi–Yau threefold $X$, there exists a Calabi–Yau threefold $Y$, the mirror of $X$, with $h^{1,1}(X) = h^{2,1}(Y)$ and $h^{2,1}(X) = h^{1,1}(Y)$. A further refinement is that the variation of Hodge structures on the third cohomology group in the deformations of $X$ is related, in specific way, to the Kähler cone in the second cohomology group of the deformation of $Y$. The expected relation between deformations of $X$ and the Kähler cone of $Y$ requires that there exist boundary points in the (complex structure) moduli space of $X$ where the variation of Hodge structures on the $H^3$ has maximal unipotent monodromy. Recently, Rohde [13] found examples of families which do not admit such boundary points. Examples of families of these Calabi–Yau threefolds are product type $S \times E$ where $S$ is a K3 surface and $E$ an elliptic curve, equipped with automorphisms of order $3$, and are of CM type. The moduli spaces of these families of Calabi–Yau threefolds are Shimura varieties. For these families, the mirror symmetry principle fails.

Is there any modification of mirror symmetry principle to accommodate these examples?

# 3 Presentation Highlights

There are new recent developments in the field, many of which are reported at the workshop, but have not yet been published. We had a very diverse spectrum of talks and topics at this workshop more so than the previous workshops. Topics of lectures ranged from various aspects of modular forms, Calabi–Yau differential equations, wall-crossings formulas, Donaldson–Thomas invariants, topological strings and Gromov–Witten invariants, Eynard–Orantin recursion formulas, holomorphic anomaly equations, mirror symmetry, among others. Subject area of interest might be classified into not clearly disjoint sets of the following subjects:

(a) (Classical) Modular forms, quasimodular forms, vector-valued modular forms, mock modular forms, Igusa, Siegel, Hilbert, and Jacobi modular forms.

(b) Moduli spaces of Calabi–Yau varieties, Hodge theory.

(c) Mirror symmetry and modular forms.

(d) Topological string theory, Gromov–Witten invariants, Eynard–Orantin recursion formulas.

(e) Feynman integrals, quantum field theory and modular forms.

(f) Conformal field theory and modular forms.

(g) Holomorphic anomaly equations.

(h) Calabi–Yau Differential equations.

(i) Wall-crossing formula, Donaldson–Thomas invariants.

(j) New kinds of moonshine.

(k) Other topics in the interface of number theory and physics not covered above.

There were essentially no talks devoted to mock modular forms, nor to multiple zeta-values. Also the modularity of Galois representations arising from Calabi–Yau varieties defined over $\mathbf{Q}$ was not covered in this workshop.

Here are some highlights of the talks. For detailed descriptions of these talks, the reader is referred to the section 6 "Abstracts of Talks Presented at the Workshop" below.

• **V. Bouchard** talked about the "remodelling conjecture" which asserts that the generating functions of Gromov–Witten invariants of a toric Calabi–Yau threefold $X$ are completely determined in terms of topological recursion. The topological recursion considered in this talk was the Eynard–Orantin recursion, and it was applied to the complex curve $\Sigma$ mirror to $X$. The full free energies $F_g$ was computed including the constant term, and the result was that *the constant terms computed through the Eynard–Orantin recursion are precisely those of Gromov–Witten theory*. This established the remodelling conjecture for the full free energies including the constant term.

• **F. Brown** talked about modular forms in quantum field theory. He defined Feynman integrals associated to graph hypersurfaces, and then related Feynman integrals to the theory of motives. Kontsevich conjectured in 1997 that the point counting functions of graph hypersurfaces over a finite field $\mathbf{F}_q$ ($q = p^n$) are (quasi) polynomials in $q$. A counter-example was given to the Kontsevich conjecture that the counting functions are given modulo $pq^2$ by modular form arising from a singular K3 surface associated to the modular form of weight 3, i.e., the eta-product $(\eta(q)\eta(q^7))^3$. Also, counterexamples to several conjectures in mathematics and physics (e.g., Kontsevich's conjecture that the Euler characteristic of graph hypersurfaces are mixed Hodge–Tate type) were also presented.

• **R. Song** introduced a new holomorphic system of differential equations, called the tautological system, which govern the period integrals of Calabi–Yau hypersurfaces in a partial flag variety. The construction is an imitation of the GKZ system for toric Calabi–Yau hypersurfaces, and indeed the new system coincides with the GKZ in toric Calabi–Yau case.

• **D. Morrison** gave interpretations from physics points of view to lattice polarized K3 surfaces of large Picard rank $18 - \ell$ with $\ell \in \{0, 1, 2\}$ (e.g., the rank 17 case was discussed earlier by Clingher, and also by Kerr). Dreams (Plans) were (1) to determine normal forms for equations of elliptically fibered K3 surfaces, (2) to understand family over $\Gamma \setminus O(2, 18)/O(2) \times O(18)$ by $\Gamma$-modular forms, and (3) to establish dictionary between physics of monodromy and mathematics of monodromy. Two examples ($F$-theory compactification in IIB theory in 10 dimensions, and heterotic-string on $T^k \times \mathbf{R}^{1,9-k}$ at large radius) were discussed.

• **M. Mulase** discussed mathematics and geometry behind the Eynard–Orantin topological recursion formula in random matrix theory. An attempt was made to answer the question "what does the E–O formula calculate?" The Eynard–Orantin type topological recursion formula was calculated for the canonical Euclidean volume of the combinatorial moduli space of pointed smooth algebraic curves. The recursion comes from the edge removal operation on the space of ribbon graphs. As an application, a new proof was given for the Kontsevich constants for the ratio of the Euclidean and the symplectic volumes of the moduli space of curves.

• **B. Szendroi** presented refined Donaldon–Thomas (DT) theory. DT theory is the enumerative theory of sheaves on Calabi–Yau threefolds. It is used, for instance, to calculate Gromov–Witten invariants. It also counts D-branes. A $q$-refinement of DT theory was suggested by works of physicists. This talk reported on a recent computation on refined DT theory using motivic invariants of Kontsevich–Soibelman, and of Behrend–Bryan–Szendroi.

• **J. Walcher** presented his work in progress on *New normal functions for Calabi-Yau threefolds*. The basic principle of classical mirror symmetry is the correspondence between enumerative geometry (Gromov–Witten theory, A-model) and complex algebraic geometry (variation of Hodge structure, B-model). A few years ago, it was understood that this correspondence can also be applied in the context of D-branes, linking enumerative geometry of Lagrangian submanifolds with algebraic cycles. In this context, the A-model was understood first, and the B-model slightly later. There are then a few cases in which the correspondence can be formulated as a theorem. In most cases studied now, however, the B-model is much more powerful, and the A-model is lacking, due mostly to the lack of methods of constructing suitable Lagrangian submanifolds.

Thus, the theme of the talk was the computation of analytic (Hodge theoretic) invariants of algebraic cycles, specifically normal functions for codimension $2$ cycles on families of Calabi–Yau threefolds, and their interpretation via mirror symmetry as generating functions of enumerative invariants. He started with a review of the case in which a theorem has been proven (together with D. Morrison as far as B-model is concerned, and with Pandharipande and Solomon for the A-model)– the enumerative geometry of real rational curves on the quintic as mirror to a special family of conics on the mirror quintic. He then described the generalizations that have been obtained, by various collaborations. Firstly to the class of $14$ one-parameter hypergeometric Calabi–Yau differential equations (in the terminology of Almqvist, Enckevort, van Straten, Zudilin), and their inhomogeneous extensions. $10$ of the $14$ work as for the quintic, the remaining $4$ do not seem to admit the similar hypergeometric extension. He then described further hypergeometric extensions (using points of order $3$ and $4$ instead of $2$), as well as a number of non-hypergeometric extensions (such as Pfaffian Calabi–Yaus studied by Shimizu–Suzuki). He summarized his work on multi-parameter models. He ended with a return to the mirror quintic. The main novelty of the most recent calculations are two results of arithmetic flavor: Firstly, the Ooguri–Vafa multi-cover formula in general involves a di-logarithm twisted by a Dirichlet character (tentatively styled a D-logarithm). Secondly, in a way anticipated by Kerr and collaborators, the expansion of the normal function in the large complex structure degeneration limit in general involves an algebraic number field, even if the cycle itself is defined over the integers. This has potentially profound implications for the enumerative geometry.

Talks by Ruifang Song on *Picard–Fuchs system of Calabi–Yau complete intersections in partial flag varieties*, and by Balazs Szendroi on *Motivic Donaldson–Thomas theory of some ocal Calabi–Yau threefolds* were video-taped for dissemination to wider audience.

# 4 Scientific Progress Made

Many of the participants have been collaborating in joint projects, and BIRS offered an ideal environment to pursue their joint projects. Also some new collaborations were initiated at this meeting. It is expected that results of these scientific endeavors will be reported in the next meeting at BIRS.

The most noteworthy progress of the workshop might be that mathematicians and physicists finally felt at ease with each other trying to understand problems in the interface of number theory (modular forms), geometry (Hodge theory), and physics (string theory), combining expertise from both camps.

# 5 Outcome of the Meeting

The meeting was a huge success. It has encouraged further communications among mathematicians and physicists who share common interests on subjects in the interface of mathematics and string theory. Many participants mentioned that they liked the informal and friendly atmosphere of the workshop, which contributed to the success of the workshop. Indeed mathematicians (resp. physicists) benefited tremendously from face to face discussions with physicists (resp. mathematicians). Audience were encouraged to ask questions to speakers during their talks, and indeed there were lots of questions! With many interesting talks, workshop was very intense and exhaustive. Everybody has learned something new from attending the workshop. There are number of joint continuing projects and/or some new projects were originated at this workshop. Participants are encouraged to submit their articles for publication in “Communications in Number Theory and Physics (CNTP)”.

There was a strong desire among participants to have the next workshop in this series at BIRS in two (or three) years time.

# 6 Abstracts of Talks Presented at the Meeting

Speaker: **M. Ballard** (UPenn, Math.)
Title: **Matrix factorization categories for complete intersections with applications to Orlov spectra of triangulated categories**
Abstract: We provide a description of the category of singularities of a (graded) complete intersection in terms of a category that is a natural extension of category of matrix factorizations. This description utilizes work of L. Positselski on derived categories of curved differential graded algebras. In the style of T. Dyckerhoff, we give an analogous description of the dg-category of functors between the category of singularities of two (graded) complete intersection singularities. This allows us to compute the Hochschild cohomology of such categories. We also use these descriptions to prove that the Rouquier dimensions of the derived categories of a coherent sheaves on the self-product of the Fermat elliptic curve and on a closely associated K3 surface are two. This verifies a conjecture of Orlov in these cases. This is joint work with D. Favero and L. Katzarkov.

Speaker:**F. Brown** (Jussieu, Math.)
Title: **Modular forms in quantum field theory**
Abstract: In perturbative Quantum Field theory, physical predictions are obtained by computing the Feynman integrals associated to the graphs in the theory. These integrals are periods of the complement of a certain hypersurface associated to each graph, and, at least in low orders, are expressible in terms of the Riemann zeta function.

In the first part of the talk I will give an overview of recent work relating Feynman amplitudes to the theory of motives, and in the second part I will report on joint work with Oliver Schnetz in which we find a range of examples of graph hypersurfaces which are modular. This unexpected discovery disproves a certain number of conjectures about the arithmetic nature of graph hypersurfaces.

Speaker: **M. Bogner** (Mainz, Math.)
Title : **Symplectically rigid monodromy tuples induced by fourth order differential Calabi–Yau operators**
Abstract: We classify all $Sp_4(\mathbf{C})$-rigid, quasi-unipotent monodromy tuples having a maximally unipotent element and show that all of them can be constructed via tensor- and Hadamard-products of rank one tuples. Furthermore, we translate those constructions to the level of differential operators and investigate whether such a monodromy tuple is induced by a fourth order differential Calabi–Yau operator. We also obtain closed formulae for special solutions of those operators. This is joint work with Stefan Reiter.

Speaker: **V. Bouchard** (Alberta, Math.)
Title: **The geometry of mirror curves**
Abstract: According to the ”remodeling conjecture”, the generating functions of Gromov–Witten invariants of toric Calabi–Yau threefolds are fully determined in terms of a topological recursion. At the root of the recursion is the geometry of the corresponding mirror curves. In this talk I will describe the geometry of mirror curves and the remodeling conjecture, focusing on the fate of ”constant terms”. In particular, I will explain how the ”pair of pants” decomposition of mirror curves plays a role in the topological recursion, in

mirror analogy to the topological vertex formalism on the Gromov–Witten side. This is joint work with Piotr Sulkowski.

Speaker: **A. Clingher** (Missouri–St. Louis, Math.)
Title: **On a Family of K3 Surfaces of Picard Rank 16**
Abstract: I will report on a classification for the K3 surfaces polarized by the lattice $H + E7 + E7$. In terms of periods, the moduli space of these objects is a quotient of a four-dimensional bounded symmetric domain of type IV. Explicit normal forms will be presented, as well as a discussion of modular forms associated to this family.

Speaker: **S. Galkin** (IMPU, Math.)
Title: **Fano and Mathieu**
Abstract: There is a correspondence between $G$-Fano threefolds and conjugacy classes in Mathieu group $M_{24}$. Construction of cusp-forms from conjugacy classes in Mathieu group is well-known. It is less known that A-model on $G$-Fano threefolds also naturally produce modular forms. Why these two lists of modular forms are so similar is yet another moonshine.

Speaker: **T. Gannon** (Alberta, Math.)
Title : **Vector-valued automorphic forms and the Riemann-Hilbert problem**
Abstract: In this talk I'll sketch the basic theory of vector-valued automorphic forms for arbitrary finite-index subgroups of any genus-zero Fuchsian group of the first kind, and arbitrary representation and arbitrary weight. I'll describe the analogues here of Grothendieck's Theorem, Riemann–Roch, Serre duality, etc and show they can be sharpened into effective tools (e.g. for finding dimensions and basis vectors). A crucial role is played by Fuchsian differential equations. I'll focus on the most familiar case of $SL(2, \mathbf{Z})$, where there are plenty of direct applications to physics, geometry and algebra. This is joint work with Peter Bantay.

Speaker: **P. Gunnells** (UMass, Math.)
Title: **Metaplectic Whittaker functions and lattice models**
Abstract: Whittaker functions are special functions on algebraic groups that play an important role in number theory and representation theory. Just as the usual exponential function is the basic ingredient for Fourier expansions, Whittaker functions provide the special functions needed to do nonabelian harmonic analysis in the theory of automorphic forms.

In this talk we will discuss the structure of spherical Whittaker functions on finite covers of $GL(n)$ over $p$-adic fields (i.e. metaplectic groups). We will show how such functions are related to certain two-dimensional lattice models from statistical physics. In particular, we will show that metaplectic Whittaker functions can be described using partition functions attached to six-vertex lattice models. We will define and give examples of all the relevant objects.

This is joint work with Ben Brubaker, Dan Bump, Gautam Chinta, and Sol Friedberg.

Speaker: **S. Hosono** (Tokyo, Math.)
Title: **Mirror symmetry and projective geometry of Reye congruences**
Abstract: A line congruence is a congruence of lines given by a variety in Grassmannian $G(2, n + 1)$. Reye congruence is a line congruence defined by a linear system of quadrics on $\mathbf{P}^n$, and for $n = 3$ it's relation to Enriques surfaces is a well-studied subject in projective geometry.

In this talk, we will consider the Reye congruence for $n = 4$, where we naturally come to a Calabi–Yau three fold $X$, (called generalized Reye congruence in [Oliva,1994]). We make a sutable mirror family to $X$ by Batyrev–Borisov construction supplemented by a $\mathbf{Z}_2$ quotient. We will then observe that there appear two different large complex structure limits in the complex structure moduli space. We identify one of them with the mirror to the Reye congruence $X$ and for the other we find a new Calabi–Yau threefold $Y$, which we construct as the double cover of a determinantal quintic in $\mathbf{P}^4$ branched over a curve of genus 26 and degree 20. By using mirror symmetry, we will calculate the BPS numbers for both $X$ and $Y$. It is observed that some of them have nice explanations as the numbers of curves on $X$ and $Y$. It is conjectured that $X$ and $Y$ are derived equivalent although they are not birational. We announce a proof of this fact briefly, referring to a paper which will appear soon.

References:

[1] S. Hosono and H. Takagi, "Mirror symmetry and projective geometry of Reye congruences I", arXiv:1101.2746v1[mathAG].

[2] S.Hosono and H. Takagi, ”Mirror symmetry and projective geometry of Reye congruences II – Derived equivalence”.

Speaker: **M. Kerr** (Washington University St. Louis, Math.)
Title: **On isotrivial families of $K3$ surfaces**
Abstract: We describe an explicit construction of K3-fibered Calabi–Yau threefolds, together with their period mappings into appropriate Mumford-Tate domains. This is joint work with A. Clingher and C. Doran, and based on their modular families of M-polarized K3 surfaces. Part of the talk will review their construction as well as the analogous story for elliptically-fibered K3's.

Speaker: **A. Klemm** (Bonn, Physics)
Title : **Omega backgrounds and generalized holomorphic anomaly equation**
Abstract: We derive an anomaly equation which incorporates the general Omega background in the B-model. We discuss applications to topological string theory on Calabi–Yau backgrounds and $N = 2$ gauge theory with (massive) flavors. Using geometric engineering on the Enriques Calabi–Yau we derive Seiberg–Witten curves for the conformal cases, which are compatible with Nekrasovs partition function.

Speaker: **J. Manschot** (CEA, Physics)
Title: **The Betti numbers of moduli spaces of sheaves on the projective plane**
Abstract: Electric-magnetic duality of gauge theory implies modular properties for generating functions of invariants of moduli spaces of sheaves. I'll explain the computation of the generating functions of Betti numbers of moduli spaces of sheaves with rank $1, 2$ and $3$ on the projective plane in terms of indefinite theta functions. The main ingredients are wall-crossing and the blow-up formula.

Speaker: **D. Morrison** (UC Santa Barbara, Math. & Physics)
Title: **K3 surfaces, modular forms, and non-geometric heterotic compactifications**
Abstract: Type IIB string theory has an $SL(2, \mathbf{Z})$ symmetry and a complex scalar field $\tau$ valued in the upper half plane, on which $SL(2, \mathbf{Z})$ acts by fractional linear transformations; this naturally suggests building models in which $\tau$ is allowed to vary. Although the $SL(2, \mathbf{Z})$-invariant function $j(\tau)$ can reveal some of the structures of these models, for their full construction and study we need $SL(2, \mathbf{Z})$ modular forms, particularly the Eisenstein series $E_4(\tau)$ and $E_6(\tau)$ and the corresponding Weierstrass equations. The Weierstrass equations can also be analyzed in algebraic geometry via the theory of elliptic curves. This approach leads to the ”F-theory” compactifications of type IIB theory.

Similarly, the heterotic string compactified on $T^2$ has a large discrete symmetry group $SO(2, 18; \mathbf{Z})$, which acts on the scalars in the theory in a natural way; there have been a number of attempts to construct models in which these scalars are allowed to vary by using $SO(2, 18; \mathbf{Z})$-invariant functions. In our new work, we give (in principle) a more complete construction of these models, using $SO(2, 18; \mathbf{Z})$-modular forms analogous to the Eisenstein series. In practice, we restrict to special cases in which either there are no Wilson lines – and $SO(2, 2; \mathbf{Z})$ symmetry – or there is a single Wilson line – and $SO(2, 3; \mathbf{Z})$ symmetry. In those cases, the modular forms can be analyzed in detail and there turns out to be a precise theory of K3 surfaces with prescribed singularities which corresponds to the structure of the modular forms. Using these two approaches – modular forms on the one hand, and the algebraic geometry of the K3 surfaces on the other hand – we can construct non-geometric compactifications of the heterotic theory.

This is a report on two joint projects: one with McOrist and Sethi and the other with Malmendier.

Speaker: **H. Movasati** (IMPA, Math.)
Title: **Mirror quintic Calabi–Yau modular forms**
Abstract: In this talk we first reintroduce the classical (quasi) modular forms using algebraic de Rham cohomology of elliptic curves and the corresponding Gauss–Manin connections. We then apply the same ideas to the one parameter family of mirror quintic Calabi–Yau threefolds and we get a new (quasi) modular form theory generated by seven series algebraically independent over the field of complex numbers. The modular group is the monodromy group of such a family and it is generated by two explicit matrices in the four dimensional symplectic group with integer coefficients. The automorphy factor in this case has image inside an algebraic group of dimension six which is generated by two multiplicative and four additive subgroups. We present the functional equation of such (quasi) modular forms, however, we emphasis that the characterization of such functions in the algebraic geometric context and through polynomial ordinary differential equations is much more convenient for calculations. At the end we present some conjectures following some similar

statements in the case of elliptic curves and classical modular forms. The talk is based on the following articles which can be found in my homepage:

(1) Quasi-modular forms attached to elliptic curves, I, Lecture notes at Besse, France 2010

(2) Eisenstein type series for Calabi–Yau varieties , Nuclear Physics B, 847, 2011, 460-484.

(3) Quasi-modular forms attached to Hodge structures, Preprint.

Speaker: **M. Mulase** (UC Davis, Math.)
Title: **A topological recursion in B-model as the Laplace transform of a combinatorial equation**
Abstract: The topological recursion formula discovered by Eynard and Orantin in random matrix theory has been applied to Gromov–Witten theory by string theorists (Bouchard, Klemm, Marino, and Pasquetti), and has produced an effective conjectural formula that calculates both open and closed Gromov–Witten invariants of toric Calabi–Yau threefolds. Recently some special cases of this conjecture have been solved by mathematicians. In this talk, the key idea of these mathematical work, the Laplace transform playing the role of the mirror symmetry transformation, will be explained. This talk is based on my joint papers with Chapman, Eynard, Penkava, Safnuk, and Zhang.

Speaker: **B. Pioline** (LPTHE, Jussieu, Physics)
Title: **Automorphy in hypermultiplet moduli spaces**
Abstract: The hypermultiplet moduli space in type II string theories compactified on a Calabi–Yau threefold provides a unifying framework for Gromov–Witten invariants (worldsheet instantons), Donaldson–Thomas invariants (D-instantons) with a new type of invariants (NS5-instantons). String dualities require that this moduli space should be invariant under $SL(2, \mathbf{Z})$, or larger arithmetic groups obtained by combining $SL(2, \mathbf{Z})$ with monodromies and large gauge transformations. I will review recent progress in understanding quantum corrections to the perturbative moduli space metric consistently with these automorphic symmetries.

Speaker: **J. C. Rohde** (Hamburg, Math.)
Title: **Maximal automorphisms of Calabi–Yau manifolds versus maximally unipotent monodromy**
Abstract: Let $X$ denote a Calabi–Yau 3-manifold. Moreover let $p$ denote the period map of the $F^2$ bundle in the variation of Hodge structures of weight $3$ of the local universal deformation of $X$. There are examples of Calabi–Yau 3-manifolds $X$ satisfying that $p$ is constant. In the case of these examples $X$ cannot be a fiber of a maximal family of Calabi–Yau 3-manifolds with maximally unipotent monodromy. This contradicts the assumptions of a classical formulation of the mirror symmetry conjecture. Almost all known examples of this kind arise from the observation that the $F^2$ bundle is an eigenspace of the non-trivial action of an automorphism of order $3$ or $4$ of the local universal deformation over its base space. Moreover the associated period domain is a complex ball containing a dense set of complex multiplication points in all known examples of this kind.

Speaker: **A. Sebbar** (Ottawa, Math.)
Title: **On the critical points of modular forms**
Abstract: In this talk, we study the critical points of modular forms. In particular, we prove that for each modular form $f$ for a discrete group $G$, its derivative $f'$ has infinitely many non-equivalent zeros, and all, but a finite number, are simple. Applications will also be provided.

Speaker: **R. Song** (Harvard, Math.)
Title: **The Picard–Fuchs systems of Calabi–Yau complete intersections in partial flag varieties**
Abstract: We introduce a system of differential equations associated to a smooth algebraic variety $X$ with the action of a complex Lie group $G$ and an ample $G$-linearized line bundle $L$ on $X$. Assuming $G$ acts on $X$ with finitely many orbits, we show that this system is holonomic (in particular, its solutions form a locally constant sheaf of finite rank over a Zariski open dense subset). This construction recovers the GKZ systems when $X$ is a toric variety. When $G = SL_n$, $X = G/P$ where $P$ is a parabolic subgroup of $G$ and $L = K_X^{-1}$, we get a holonomic system of differential equations to which period integrals on Calabi–Yau hypersurfaces in $X$ are solutions. This can also be generalized to the case of Calabi–Yau complete intersections in $X$. This is based on a joint work with Bong H. Lian and S.-T. Yau.

Speaker: **J. Stienstra** (Utrecht, Math.)
Title: **Dimer models and hypergeometric systems**
Abstract: This talk will be an updated review of the relation between dimer models and hypergeometric systems. First the definition of a dimer model will be given and illustrated with nice pictures. Then it will be

shown how these pictures contain the equations for toric compactifications. Finally there will be comments on what this might tell about hypergeometric systems.

Speaker: **B. Szendroi** (Oxford, Math.)
Title: **Motivic DT theory of some local Calabi–Yau threefolds**
Abstract: Donaldson-Thomas theory is the enumerative theory of sheaves on Calabi–Yau threefolds, or more generally objects in CY3 categories. Work of Nekrasov and Hollowood–Iqbal–Kozcaz–Vafa suggested a $q$-refinement of this theory. This was realized mathematically using motivic invariants by Kontsevich–Soibelman and Behrend–Bryan–Szendroi, with closely related work done also by Dimofte–Gukov.

We aim to explain this theory, and a recent computation of the motivic invariants on the resolved conifold geometry, in all chambers of the space of stability conditions. This is joint work with Andrew Morrison, Sergey Mozgovoy and Kentaro Nagao.

Speaker: **R. Rodriguez-Villegas** (Texas–Austin, Math.)
Title: **The A-polynomial at $q = 1$, the dilogarithm and the asymptotics of q-series**
Abstract: I will discuss an approach to the value at $q = 1$ of the A-polynomial of a general quiver. (This polynomial counts the number of absolutely irreducible representations of the quiver over $F_q$.) The truncations of a formula of Hua for these polynomials yield $q$-series of the form

$$\sum_m q^{Q(m)} x^m/(q)_m$$

where $Q$ is a quadratic form and $m$ runs over a lattice. The asymptotics of this series as $q$ approaches 1 is then related to a truncated form (conjecturally polynomial) of the A-polynomial at $q = 1$.

Speaker: **J. Walcher** (McGill, Math. & Physics)
Title: **New Normal Functions for Calabi–Yau Threefolds**
Abstract: The expansion of normal functions associated with families of algebraic cycles on Calabi–Yau threefolds around the large complex structure point has been given enumerative interpretation via mirror symmetry. This is an update on the most recent calculations, which exhibit several new interesting features.

Speaker: **U. Whitcher** (Harvey Mudd, Math.)
Title: **K3 Surfaces with $S_4$ Symmetry**
Abstract: Hypersurfaces in toric varieties offer a rich source of examples of K3 surfaces and Calabi–Yau varieties. We use a toric residue map to study variation of complex structure for families of K3 hypersurfaces with a high degree of symmetry. This talk describes joint work with Dagan Karp, Jacob Lewis, and two Harvey Mudd College undergraduates, Daniel Moore and Dmitri Skjorshammer.

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