ALGEBRAIC STACKS: PROGRESS AND PROSPECTS
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Overview of the field

The theory of stacks originated with the work of Grothendieck and his students in the 60s (see [Gir65]) as an extension of the notion of an algebraic variety and is closely related to the notion of Grothendieck topologies, introduced by Grothendieck. An affine algebraic variety may be viewed as the zero locus of a collection of polynomial functions. For this to make sense over any commutative ring rather than a field, one needs to allow nilpotents and zero divisors in the ring. These are called affine schemes, and schemes are obtained by gluing together affine schemes making use of the Zariski topology. It is possible to consider objects that look locally like schemes, but globally are not schemes, by performing the above gluing in a different Grothendieck topology. One may view such objects as contravariant functors from the category of schemes to sets satisfying certain gluing conditions. The algebraic spaces introduced by Michael Artin are examples of such functors. Good examples of such functors are the moduli-functors appearing in moduli problems. Such a functor will associate to each scheme U, a family of certain algebraic objects over U satisfying certain conditions. Unfortunately, the fact that these are functors to sets means one has to ignore all automorphisms of these algebraic objects, or in other words one is considering only isomorphism classes of such objects rather than the objects themselves. Since the category of schemes is not closed under colimits, it means that many such functors are usually not representable by schemes or even realized as algebraic spaces.

The idea of algebraic stacks is to consider similar functors, but without modding out by the automorphisms. This makes it necessary to consider what are functors *upto natural isomorphism to the category of groupoids* or *lax-functors to groupoids*. Such lax functors that satisfy certain gluing conditions are stacks and those that look *locally like affine schemes, in a suitable Grothendieck topology* are algebraic stacks. The gluing conditions correspond to *descent data* as in Grothendieck's theory of faithfully flat descent. When the topology used is the étale topology, one obtains what are called Deligne-Mumford stacks: the objects called orbifolds are special cases of Deligne-Mumford stacks that are *smooth* (in a certain sense) and which are generically schemes. For example, if G is a finite group acting on a smooth scheme X, with stabilizers that are trivial generically, the resulting quotient stack $[X/G]$ is an orbifold. Observe that the quotient stack $[X/G]$ is not simply the quotient space X/G, which ignores the stabilizers: instead the quotient stack $[X/G]$ should be viewed as an object sitting over X/G which also keeps track of all the stabilizers. The theory of orbifolds and their cohomology has been of significant interest in recent years, especially to the algebraic topologists.

The class of Artin stacks is much bigger than Deligne-Mumford stacks. As a simple example, if G is a linear algebraic group acting on a smooth scheme X, one obtains the quotient stack $[X/G]$ which classifies principal G-bundles together with a G-equivariant map to X. If the stabilizers for the G-action are not finite, the resulting stack is an Artin stack that is not Deligne-Mumford. As the above example of quotient stacks shows, Artin stacks occur far more commonly than Deligne-Mumford stacks: yet it is only in the last 10 years or so that such stacks have begun to be studied in detail.

Recent developments

After a rather slow beginning, with a few notable papers in the 60s by Mumford and Deligne and Mumford (see [Mum65], [DM69]), and by Michael Artin (see [Ar74]) in the 70s, the theory of algebraic stacks gained acceptance as a main-stream mathematical tool in the last 15 years or so, because of the wide-spread applications that were discovered in diverse fields as mathematical physics, geometric representation theory (especially centered around the geometric Langlands' correspondence), higher topoi and categories, differential graded algebraic geometry and also algebraic topology. In retrospect this is hardly surprising, since algebraic varieties and schemes are often too restrictive to contain solutions of many important problems. The notion of algebraic stacks being a vast generalization of schemes, it is in fact possible to solve many problems in this more general framework which cannot be solved in a more classical setting. Several objects of fundamental importance in mathematics have been constructed only in the framework of stacks.

- For example, many moduli-spaces that cannot be constructed in the framework of schemes have been constructed in the setting of algebraic stacks as in the work of Deligne and Mumford, Mumford and Knudsen (see [MFK94], [MK76]) as well as Kontsevich [Kont95].
- Connections with mathematical physics and Gromov-Witten theory. This is an area where the use of stacks has been highly successful and useful. For example, construction of a virtual fundamental class associated to certain moduli-spaces of stable maps, has been carried out elegantly in stack-theoretic contexts as in the work of Behrend and Fantechi (and also Li and Tian): see [BF97] and [LT98]. Mathematical physics continues to be a source of interesting and challenging problems and recent work of Witten and others have established new connections of this area with mathematical physics: for example between the geometric Langlands' program which requires algebraic stacks just to get off the ground with mathematical physics.

- Some of the applications, notably to Gromov-Witten theory and mathematical physics, necessitated the development of cohomology-homology theories for algebraic stacks, extending them from schemes. The last 15 years or so, saw the development of various cohomology theories for Deligne-Mumford stacks as in the work of Toen, Chen and Ruan: see [T99] and [CR04]. In addition various technical tools to study Deligne-Mumford stacks have been sharpened: for example, the Quot functor, which is extremely useful in algebraic geometry, has only been recently constructed for Deligne-Mumford stacks by Olsson and Starr: see [OS03]. Considerable work on intersection theory on algebraic stacks was done also in view of these applications: currently there is a workable theory of Chow-groups and higher-Chow groups for all Artin stacks. (See [EG], [J02], [J07] and [Kr99].

The study of vector bundles on any geometric object, whether it is a topological space, an algebraic variety or an algebraic stack is of fundamental importance. K-theory is the cohomology theory that studies vector bundles. Several results on the algebraic K-theory of schemes have been extended in recent years to algebraic stacks. For example, in a series of papers, Angelo Vistoli and his collaborators studied this for quotient stacks: see [VV1], [VV2]. Joshua then was able to extend Thomason's basic theorem relating étale cohomology and algebraic K-theory with finite coefficients to fairly general Artin tacks: see [J03]. Moreover [J10] extends several of the basic results in K-theory and G-theory to stacks and differential graded stacks.

- l-adic Derived categories, t-structures and perverse sheaves for algebraic stacks. Though there exist several works from the 90s on this, for example see [J93] and [Be03], the whole machinery of Grothendieck-Verdier duality, t-structures perverse sheaves in the setting of l-adic derived categories were extended in full generality to Artin stacks satisfying certain (mild) finiteness conditions in recent work of Laszlo and Olsson: see [LO08]. Applications to the Langlands' program made this development essential and quite useful.

- Derived and higher stacks and higher categories This is an area of active research by several groups. The manuscript, *Pursuing stacks* by Grothendieck has been a guiding force in the development of this area. Another source of motivation comes from study of *virtual phenomena* in Gromov-Witten theory while another distinct source of motivation was the work of Hopkins and Miller on topological modular forms followed upon by Lurie's work The work of Hirschowitz and Simpson on higher categories (especially higher Segal categories) is a somewhat different approach.
- Connections and techniques of an arithmetical nature. Here the main technique seems to be using the Brauer groups and important recent work in this area is due to Johann Dejong and Max Lieblich. Several impressive results have been obtained using this methods in recent years.
- Toric stacks Just like toric varieties form a nice class of algebraic varieties that can be studied rather easily using combinatorial data, toric stacks have emerged in recent years as a nice class of Artin stacks which can be studied combinatorially. Work in this area seems closely connected with logarithmic geometry: in Martin Olsson's thesis (see [Ol]) provides an explicit connection between log-schemes and algebraic stack, where he constructed a moduli stack of log-structures.

Participation

The workshop planned to and succeeded in bringing together several of the leading experts in related fields along with several post-docs and advanced graduate students working in these areas. In fact most of the recent Ph. Ds working related fields were invited and participated in the workshop. In addition, several graduate students working on related areas also participated.

One of our goals was to combine various camps of mathematicians working in aspects of geometric representation theory, differential graded algebraic geometry, algebraic topology and mathematical physics and who make use of algebraic stacks and stack theoretic techniques from possibly diverse points of view so as to promote exchange of ideas between these various camps. We believe we succeeded in this based on the comments we have received from the participants.

Presentation Highlights

Toric and quotient stacks: The talks by Satriano, Geraschenko and Krishna fell into this area. Geraschenko considered the (old) question of deciding whether a variety with quotient singularities is the quotient of a smooth variety by a finite group. He spoke on results from his joint work with Satriano which answers the question affirmatively for torus quotients.

Satriano considered the notion of "stacky resolution" of a scheme X. Roughly speaking this is a morphism $\mathcal{X} \to X$ from an Artin stack $\mathcal{X}$ to X which is an isomorphism over the smooth locus of X and exhibits X as a good moduli space of $\mathcal{X}$ (in the sense of J. Alper). He gave results proving that stacky resolutions exists for certain X. He also gave some applications of stacky resolutions, the general nature of which is proving a statement for X by proving it for the stacky resolution $\mathcal{X}$ and "push the result to X".

A. Krishna's talk was on equivariant algebraic cobordism which could be viewed as a variant of algebraic cobordism for quotient stacks.

Higher categories and stacks. Both Simpson and Kapranov (who replaced Rydh who could not attend) spoke on closely related results on higher Segal categories.

K-theory and G-theory of stacks. Joshua's talk was on the K-theory and G-theory of algebraic and dg-stacks. After giving an overview, he discussed several more recent results. This involved recent and ongoing computations on the K-theory of toric stacks joint with A. Krishna and also several results of a basic nature on K-theory and G-theory of dg-stacks as well. He concluded with results on a site called the iso-variant étale site that is a replacement for the étale site of a coarse moduli space which may not always exist. This talk had close connections with the talk of Jarod Alper on his construction of a good moduli space for Artin stacks.

Edidin's talk was on producing a new λ-ring structure on the rational K-theory of the inertia stack of toric Deligne-Mumford stacks. Kai Behrend discussed an operator on the Grothendieck groups of algebraic stacks defined by taking the inertia stacks of a given stack. He presented results towards understanding the eigenvalue spectrum of this operator.

Moduli problems. Alper talked about a weak generalization of the Keel-Mori theorem giving conditions on a non-separated algebraic stack which guarantee the existence of a good moduli space. Given a finite group G, if $M_g(G)$, denotes the locus in M_g, consisting of curves which admit an effective action by G, Perroni discussed numerical invariants of the G-action to distinguish irreducible components of $M_g(G)$.

Gerbes, Deformation theory. Vistoli discussed an extension of Nori's fundamental group schemes to schemes/stacks over fields which need not have a base rational point. Tseng discussed a duality between étale gerbes and a pair of disconnected space and a $U(1)$-gerbe on it.

Wise considered the question of classifying deformations and obstructions by cohomology groups. He showed that the ideal result valid for deforming smooth schemes still holds true in the singular case if one uses a suitable Grothendieck topology.

Results of an arithmetic nature. Lieblich discussed a work in progress which shows that the moduli space of supersingular $K3$ surfaces is uniruled. His method involves producing rational curves via moduli spaces of twisted sheaves, which is quite surprising.

Gromov-Witten theory, connections with mathematical physics and quantum geometry. Bryan discussed some results of his Ph.D. student Simon Rose which gives a formula for the number of hyperelliptic curves in an Abelian surface A in terms of quasi-modular form. The formula was derived by first relating the counts of hyperelliptic curves in A with genus 0 Gromov-Witten invariants of the *orbifold* $[A/\pm 1]$, which is then related to Gromov-Witten invariants of the Kummer $K3$ surface via the so-called crepant resolution conjecture. Since the Gromov-Witten theory of $K3$ is explicitly solved by the Yau-Zaslow formula, the aforementioned relations combined to give the result.

Mann talked about a problem involving quantum D-modules for toric nef hypersurfaces. Jarvis discussed recent work with Drew Johnson, Amanda Francis, and Rachel Suggs on the Landau-Ginzburg Mirror Symmetry Conjecture for orbifolded Frobenius algebras for a large class of invertible singularities, including arbitrary sums of loops and Fermats with arbitrary symmetry groups.

Other talks. Noohi discussed classification of forms of an algebraic stack in terms of certain cohomology of a 2-group. He also explained the classification of forms of weighted projective stacks using his result. Vakil talked on stabilization of discriminants in the Grothendieck ring.

Participant Feedback

(Dhillon) There were many valuable interactions and great talks but probably the most valuable for me was some conversations with Zinovy and Angelo regarding essential dimension. In particular their latest paper (arXiv:1103.1611v2). I expect to be able adapt the arguments in the paper to prove the genericity theorem for the moduli stack of vector bundles. The cited arxiv paper provides a powerful tool for controlling the essential dimension of polystable loci.

(Edidin) This is the second Banff conference I've attended. At the first conference in 2008 I met Jarvis and Kimura and we began a collaboration that continues to date. Once again, it looks like I'll be able to gain new collaborators by attending a Banff conference. I think that having participants share a dormitory and common meals definitely helps foster mathematical discussion.

(Kimura) The workshop gave me an opportunity to get together with my collaborators, Dan Edidin and Tyler Jarvis, to continue our work on orbifold and virtual K-theory, power operations, and hyperKahler resolutions. This is generally quite difficult for us to do since we are geographically separated.

(Reichstein) I am a recent convert to algebraic stacks. I attended the workshop to learn more about them from the experts. My own background is in algebraic groups and invariant theory. I connected best to the talks related to these areas, but enjoyed a number of the others as well.

There were two surprising things I learned during the workshop. One was from Dan Edidin, who has reworked my 1988 Ph.D. thesis in a very satisfying way (some of this work Dan did jointly with his postdoc Yogesh More). My thesis was inspired by earlier work of Francis Kirwan. Given the action of a reductive linear group G on an algebraic variety X, the idea is to construct a sequence of blow ups $X_n - \to \dots \to X_2 \to X_1 = X$ with smooth G-equivariant centers, which "improves" the properties of the GIT quotient $X^{ss}//G$. (Here X^{ss} denotes the open subset of semistable points in X). For example, the initial quotient $X_1^{ss}//G$ may only be categorical, while we may want the final quotient X_n^{ss}/G to be geometric. The final theorem I got was quite nice, but it required some (common in GIT but nevertheless) awkward choices/assumptions along the way, having to do with a linearization of the original action, and carrying this linearization up the chain of blow ups. Dan adopted my resolution procedure to a more general (non-GIT) framework, where the conclusion has to do with the existence of a "good quotient" for a certain G-invariant open subset of X, and no linearizations are involved at any stage.

Another surprise for me was an application of the work I have been involved in on classifying (stably) Cayley simple and semisimple linear algebraic groups. A group G is defined to be Cayley if admits a (partial) algebraic analogue of the exponential map; G is stably Cayley, if $G \times T$ is Cayley for some split algebraic torus T. Stable Cayliness is equivalent to the character lattice of G being quasi-permutation. Surprisingly, this same condition came up in Kai Behrend's work on the inertia operator on the K-group of stacks. Kai is my colleague at UBC but I was not previously aware of this connection; I intend to think about it some more.

(Simpson) The stacks conference was a great occasion to meet a lot of people whose work I knew of, but whom I hadn't met in person before; and also to see some colleagues whom I hadn't seen for many years, as well as some who I meet more regularly of course. I learned a lot about several subjects, perhaps foremost among them, the relationship between stacks and different aspects of mirror symmetry. This includes a very interesting discussion with Jim Bryan about hypergeometric functions and Givental's theorem. Other new things include different aspects of Brauer groups, and also Kapranov's talk on higher Segal conditions which might well bear some analogies with a seemingly unrelated subject of 2-metric spaces that I have been thinking about. I also hope to have been able to provide some helpful replies to people's queries about higher categories, parabolic bundles, Higgs bundles and fundamental groups. While none of my students came, I did make contact with someone at the place where my student will be going for a post-doc next year, so that should be really helpful.

The setting was really wonderful (this was my first visit to Banff), the food was awesome, and everything was extremely well organized.

(Vakil) A number of conversations helped me both in my ongoing work, and in understanding that will lead to future work. Most substantively, (i) discussions with Angelo Vistoli on Chow groups of infinite symmetric products will substantively help a paper I'm writing; (ii) discussions with Donu Arapura answered some important questions I had in this paper as well, and will lead to him visiting me in the fall; (iii) a series of discussions with Matt Satriano led to an ongoing project (we'll see how it goes). But I had a number of other mathematical significant discussions, and from experience I know that some of them will have more impact on my work than the ones I've already listed; but it's hard to know in advance which!

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Final List of participants

Total number of participants including organizers: 33

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9	Dhillon, Ajneet	University of Western Ontario
10	Edidin, Dan	University of Missouri
11	Geraschenko, Anton	California Institute of Technology
12	Jarvis, Tyler	Brigham Young University
13	Joshua, Roy	Ohio State University
14	Kapranov, Mikhail	Yale University
15	Kimura, Takashi	Boston University
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27	Sheshmani, Artan	University of British Columbia
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