BIRS Workshop 16w5047
Geometric and Analytic Inequalities

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1 Overview of the Field

The "Isoperimetric Inequality" is probably one of the best known and most intriguing result in mathematics and of course its fame spreads outside the selected community of mathematicians, as it is giving physical insight in many natural phenomena. Its classical statement in the plane (already familiar to ancient Greeks, especially in its variant known as "the Dido's problem") sounds as follows: among all sets with given area, the disk has the smallest perimeter or, equivalently, among all closed plant curves with given length, the circle encloses the biggest area. This property generalizes to all dimensions: among sets with given Lebesgue measure, balls have the smallest boundary $(n-1)$-dimensional measure. At a first glance it looks a result with a genuinely geometric flavor, but a deeper investigation immediately reveals that its real nature relies in a perfect equilibrium between geometry and analysis. Then the isoperimetric inequality is an enlightening example of what we mean by a geometric-analytic inequality: a geometric-looking inequality which has also a deep analytic nature. The latter is often clearly revealed by the existence of a functional companion: the functional companion of the isoperimetric inequality is the Sobolev inequality, a fundamental corner-stone in modern analysis.

Another example is provided by the Brunn-Minkowski inequality, the starting point of the Brunn-Minkowski theory of convex sets, a beautiful and powerful apparatus to conquer all sorts of problems involving metric quantities such as volume, surface area, mean width, etc. (by the way, the Brunn-Minkowski inequality permits to prove the isoperimetric inequality for convex sets in a few lines). The functional counterpart of the Brunn-Minkowski inequlaity is the Prekópa-Leindler inequality (generalized by the Borell-Brascamp-Lieb inequalities), and in the last 30 years strict relations with many other important analytic inequalities have emerged.

Many other examples come from that part of analysis known as calculus of variations, as for instance the Faber-Krahn inequality, where the interplay between analysis and geometry is even more apparent.

The subject area of the proposed workshop also includes all the geometric variational problems and shape optimization problems.

Most of these inequalities and related problems play a fundamental role in the mathematical modeling of Nature, and in particular, in our quantitative and qualitative understanding of equilibrium states of physical systems. They also often play a pivotal role in other area of Mathematics, in par- ticular Probability Theory.

Moreover many inequalities are naturally linked to problems involving partial differential equations and this interplay with PDEs was precisely at the core of this workshop.

2 Recent Developments and Open Problems

The activity in this area is presently very intense and broad. In recent years new and sharp quantitative versions of many geometric-analytic inequalities have been investigated by the combined use of classical and ”ad hoc” symmetrization methods, new tools coming from mass transportation theory, deep geometric measure tools and theory of PDEs, and the interplay with calculus of variations and PDEs stimulated the research of new inequalities, like Brunn-Minkowski and Urysohn’s type inequalities for many variational functionals.

It has also been recognized the importance of characterizing, from a geometric viewpoint, the equality case of many inequalities and to understand if it implies some kind of symmetry of the optimal sets or functions. This usually leads to some shape optimization problem, a field currently under deep investigation, especially for problems involving optimization of eigenvalues of elliptic operators.

Moreover recent important researches focused on sharpness, rigidity and stability of classical inequalities, like for instance the isoperimetrci inequality, the Brunn-Minkowski inequlity, Sobolev inequalities, etc.

3 Presentation Highlights

Here after we give some hilights about the talks given during the workshop and the related discussions.

3.1 Refined isoperimetric type inequalities for domains in R^n

Chiara Bianchini presented a joint work with G. Croce and A. Henrot [3], titled *On the quantitative isoperimetric inequality in the plane*. Their investigation based on an idea to consider the shape functional $F(\Omega) = \frac{\delta(\Omega)}{\lambda^2(\Omega)}$ and to study optimality conditions a minimizer has to satisfies. Here $\delta(\Omega)$ is the isoperimetric deficit of Ω, that is $P(\Omega)/P(B) - 1$, where B is a ball with $|B| = |\Omega|$, while

$$\lambda(\Omega) = \min_{x \in R^N} \{ \frac{|\Omega \Delta B_x|}{|B_x|},\ |B_x| = \Omega \}.$$

It is proved the existence of a set Ω, different from a ball, which minimizes the functional F. The classical isoperimetric inequality in the space R^N had been already proven in [36]; the present method offers a new proof of the quantitative isoperimetric inequality for $N = 2$. In particular Bianchini analyzes the properties of an optimal domain, recovering some known properties (see [26, 27]) and proving some new ones. In particular she shows that the number of optimal balls which realize the Fraenkel asimmetry is larger than 2, although it is conjectured that they must be exactly 2.

Lorenzo Brasco lectured on *Bounds for Poincaré constants on convex sets*.
Given $1 < p < \infty$ and an open bounded set $\Omega \subset \mathbf{R}^N$, consider the sharp constant in the Poincaré inequality

$$C \left(\min_{t \in \mathbf{R}} \int_\Omega |u - t|^p \, dx \right) \le \int_\Omega |\nabla u|^p \, dx, \qquad \text{for } u \in W^{1,p}(\Omega),$$

which is equivalent to

$$C \int_\Omega |u|^p \, dx \le \int_\Omega |\nabla u|^p \, dx, \qquad \text{for } u \in W^{1,p}(\Omega) \text{ such that } \int_\Omega |u|^{p-2} \, u \, dx = 0$$

He presented some optimal bounds for the sharp constant in this inequality, i. e.

$$\mu_p(\Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, dx \ : \ \int_\Omega |u|^p \, dx = 1,\ \int_\Omega |u|^{p-2} \, u \, dx = 0 \right\},$$

when Ω is convex. It is shown that in this class, $\mu_p(\Omega)$ is equivalent to the diameter of Ω, i.e. there exists two constants $\alpha = \alpha(p) > 0$ and $\beta = \beta(N,p) > 0$ such that

$$\frac{\alpha}{\operatorname{diam}(\Omega)^p} < \mu_p(\Omega) < \frac{\beta}{\operatorname{diam}(\Omega)^p}.$$

Both inequalities are strict, but they are sharp in the following sense: for each of them, there exists a sequence of bounded open convex sets degenerating to a segment for which equality is asymptotically attained.

The lower estimate is a classical result by Payne & Weinberger for $p = 2$ (see [54]), generalized to every p by Ferone, Nitsch & Trombetti in [33]. The upper bound has been recently proved by Nitsch, Trombetti & the speaker in [8].

It is also presented a more general interpolation-type inequality for general convex sets (not necessarily bounded), with explicit constant. This result has been obtained in collaboration with Santambrogio in [9], by means of Optimal Transport techniques. Some consequences of this result have been discussed and shown that this implies again the lower bound on $\mu_p(\Omega)$ in terms of the diameter, with a non-optimal constant.

Andrea Cianchi gave an exciting talk on *Korn type inequalities in Orlicz spaces.*
A standard form of the Korn inequality amounts to an estimate for the L^p norm ($1 < p < \infty$) of the full gradient of a vector-valued function in terms of the same norm of just its symmetric part. It is well known that a result of this kind may fail if the L^p norm is replaced by a more general Orlicz norm L^A associated with a Young function A. He proved that a Korn type inequality in Orlicz spaces can be restored if possibly different norms L^A and L^B are allowed on the two sides of the inequality, provided that the Young functions A and B satisfy suitable, necessary and sufficient balance conditions. Related inequalities for trace-free symmetric gradients, for the Bogovskii operator, and for negative Orlicz-Sobolev norms will also be discussed. Part of this talk is based on collaborations with D.Breit and L.Diening.

Vincenzo Ferone discussed *the minimizers of trace inequalities in BV.*
It is well known that, for any given bounded domain $\Omega \subset \mathbf{R}^n$ with a "nice" boundary, $BV(\Omega)$ embeds in $L^1(\partial\Omega)$, in the sense that the total variation of a function u bounds the L^1 norm of $(u-c)$ through a constants K which depends on Ω. About c various choices can be made. In [24] they consider the cases where c is the median or the mean value of the trace of u over the boundary of Ω. They prove that balls achieve the least embedding constant K in both inequalities. Uniqueness of such minimizers is also discussed in details. Some of the tools used in the proof are: modified Cauchy area formula, characterization of sets of constant brightness, characterization of sets of constant projection.

He considered the nonstandard case of Poincaré trace inequalities for functions which are subject to either a vanishing mean value condition, or a vanishing median condition in the whole of Ω, instead of just on $\partial\Omega$. In [25] we have considered the special case where $\Omega = \mathbf{B}^n$ is the unit ball of the n-dimensional Euclidean space $\mathbf{R}^n$. The extremals in question take a different form, depending on the constraint imposed. In particular, under the latter constraint, unusually shaped extremal functions appear. A key step in their approach is a characterization of the sharp constant in the relevant trace inequalities in any admissible domain $\Omega \subset \mathbf{R}^n$, in terms of an isoperimetric inequality for subsets of Ω.

Nunzia Gavitone's talk is title *Optimizing the first eigenvalue of some quasilinear operators with respect to boundary conditions.*
The problem of optimizing first eigenvalues of certain differential operators is well-known from the literature mainly in connection with the so-called shape optimization. The latter means that one looks for a domain which minimizes (or maximizes) the first eigenvalue under some geometrical constraint, typically keeping the volume fixed. The answer in the case of the Laplace operator, for instance, is given by the well-known Faber-Krahn inequality which states that the minimum is achieved by a ball with the prescribed volume.

She presented a joint work with Hynek Kovařík (Brescia) and Francesco Della Pietra (Napoli Federico II), they analyze a different optimization problem; and keep a bounded domain $\Omega \subset \mathbf{R}^n$ fixed and vary the boundary conditions. They consider the variational problem

$$\inf_{u \in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx + \int_{\partial\Omega} \sigma \, |u|^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |u|^p dx}, \qquad p > 1, \tag{1}$$

and ask which function $\sigma : \partial\Omega \to [0, \infty[$ minimizes or maximizes (1) under the condition

$$\int_{\partial\Omega} \sigma \, d\mathcal{H}^{n-1} = m, \tag{2}$$

where m is a positive constant. Under certain regularity conditions on Ω the infimum in (1) is a minimum and the corresponding minimizer solves an eigenvalue equation for the $p-$Laplace operator with Robin-type boundary conditions.
The main results can be summarized as follows. They show that for sufficiently regular Ω the maximizing σ always exists and is unique.
As for the minimum, they find that as soon as $n > 1$ there is no σ which minimizes (1) in the class of nonnegative functions satisfying (2). Moreover, if $p \leq n$, then the infimum of (1) over σ belonging to this class is zero. However, if $p > n$, then this infimum is positive, and is achieved in the class of Dirac measures on $\partial\Omega$ of total mass m. In other words, it is achieved if σ in (1) is replaced by a Dirac measure concentrated at a point of the boundary. The position of this point, which might not be unique, depends of course on m, but it is possible to describe its asymptotic behavior as $m \to \infty$.

Antoine Henrot talked about *the elastic energy and inradius.*
The elastic energy of a planar regular set is defined by $E(\Omega) = \frac{1}{2} \int_{\partial\Omega} k^2(s) \, ds$ where $k(s)$ is the curvature of the boundary. He reviewed several minimization problems of $E(\Omega)$ with different geometric constraints on Ω. In particular, he considers the minimization of $E(\Omega)$ among convex domains Ω with a constraint on the inradius of Ω. By contrast with all the other minimization problems involving this elastic energy (with a perimeter, area, diameter or circumradius constraints, see [4], [19], [32], [37]) for which the solution is always the disk, he proved that the solution of this minimization problem is not the disk and therefore completely characterize it in terms of elementary functions.

Anna Mercaldo lectured on some new isoperimetric inequalities on R^N with respect to weights $|x|^\alpha$.
A class of isoperimetric problems on R^N with respect to weights that are powers of the distance to the origin is presented. For instance it is shown that, if $k \in [0; 1]$, then among all smooth sets in R^N with fixed Lebesgue measure, $\int_{R^N} |x|^\alpha H_{N-1}(dx)$ achieves its minimum for a ball centered at the origin. These results also imply a weighted Polya-Szego principle. In turn, radiality of optimizers in some Caffarelli-Kohn-Nirenberg inequalities is established, and sharp bounds for eigenvalues of some nonlinear problems is obtained.

Lubos Pick discussed *traces of Sobolev functions.*
he gave a survey of recent results on optimal trace embeddings for functions from Sobolev spaces built upon rearrangement-invariant spaces. He described the optimal function space for which every function from the corresponding Sobolev space admits a trace and established certain reduction principle which will enable us to obtain a necessary and sufficient condition for a trace embedding in terms of an action of a one-dimensional integral operator. He gave a reasonably explicit characterization of the optimal function space in a trace embedding, and he further discussed the applicability of techniques based on interpolation and iteration for trace problems.

Lenka Slavikova discussed *the necessity of Bump Conditions for the two-weighted maximal inequality.*
Given $p \in (1, \infty)$, she considers the problem of characterizing those couples (w, v) of weights for which the Hardy-Littlewood maximal operator M is bounded from $L^p(v)$ into $L^p(w)$, namely,

$$\int_{R^n} (Mf)^p w \leq C \int_{R^n} |f|^p v \tag{3}$$

for every measurable function f. The focus is on the approach via so called “bump conditions”. These conditions, studied, e.g., in [**?**, 55, 56], are strengthenings of the Muckenhoupt A_p-condition and are known to be sufficient for the inequality 3. She showed that they are in general not necessary for this inequality to be true [61].

Constantin Vernicos presented *a centro-projective inequality.*
With Berck and Bernig, they introduced an invariant associated to a pointed convex set which is the closest projective analogue of a valuation. Its similarity with the centro-affine area lead us to call it centro-projective area. With Deane Yang, they recently proved that this invariant is bounded from above by its value on a

euclidean sphere, with equality if and only if the convex set is an ellipsoid. The links with the so-called Hilbert geometries were explained.

Yi Wang presented some interesting results *on fully nonlinear Sobolev trace inequality.*
The k-Hessian operator σ_k is the k-th elementary symmetric function of the eigenvalues of the Hessian. It is known that the k-Hessian equation $\sigma_k(D^2u) = f$ with Dirichlet boundary condition $u = 0$ is variational; indeed, this problem can be studied by means of the k-Hessian energy $-\int u\sigma_k(D^2u)$. She constructed a natural boundary functional which, when added to the k-Hessian energy, yields as its critical points solutions of k-Hessian equations with general non-vanishing boundary data. As a consequence, she proved a sharp Sobolev trace inequality for k-admissible functions u which estimates the k-Hessian energy in terms of the boundary values of u. In the special case when $k = 1$, this gives the standard Sobolev trace inequality.

$$-\int_X u\Delta u\, dx + \oint_M f u_n d\mu \geq \oint_M f(u_f)_n d\mu \tag{4}$$

for all $f \in C^\infty(M)$ and all $u \in C^\infty(\overline{X})$ such that $u|_M = f$, and u_f is the harmonic function in X such that $u_f|_M = f$.

3.2 Inequalities and eigenvalues.

Dorin Bucur presented a joint work with B. Bogosel and A. Giacomini on *optimal shapes maximizing the Steklov eigenvalues.*
Let $\Omega \subseteq R^d$ be a bounded, open, Lipschitz set. For $k \in N$, the $k-th$ eigenvalue of the Steklov problem is defined by

$$\sigma_k(\Omega) = \min_{S\in S_{k+1}} \max_{u\in S\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 dx}{\displaystyle\int_{\partial\Omega} |u|^2 d\mathcal{H}^{d-1}},$$

S_{k+1} being the family of $k+1$ dimensional subspaces in $H^1(\Omega)$. Then

$$0 = \sigma_0(\Omega) \leq \sigma_1(\Omega) \leq \ldots \leq \sigma_k(\Omega) \leq \ldots \to +\infty.$$

This definition is suitably extended to sets Ω which are less regular, such the measurable sets with finite perimeter or arbitrary open sets.
Given $m > 0$, we consider the shape optimization problems

$$\max\{\sigma_k(\Omega) : \Omega \subseteq R^d, |\Omega| = m\}$$

or

$$\max\{F(\sigma_1(\Omega), \ldots, \sigma_k(\Omega)) : \Omega \subseteq R^d, |\Omega| = m\}.$$

Theorem 1. Let $F : R^k \to R$ be an upper semicontinuous function, non decreasing in each variable.
Then, problem

$$\max\{F(\sigma_1(\Omega), \ldots, \sigma_k(\Omega)) : \Omega \subseteq R^d, |\Omega| = m\},$$

has a solution

- in the class of measurable sets of R^d with finite perimeter. The maximizer is a bounded set and both its perimeter and diameter are controlled.
- in R^2 in the class of open sets. The maximizer is union of at most k disjoint, bounded, Jordan domains, with topological boundary of finite length and controlled diameter.

A key step which plays a crucial role in the result above is the following isodiametric control of the Steklov spectrum.
Theorem 2. There exists a constant $C(d)$ such that for every $k \in N$, and every measurable set with finite perimeter we have either

$$\sigma_k(\Omega) diam(\Omega) \leq C(d) k^{\frac{2}{d}+1}$$

or Ω is disconnected, more precisely it is (non trivially) contained in two disjoint, concentric annuli lying at positive distance.

Francesco Chiacchio discussed *an inverse spectral problem for the Hermite operator.*
Let Ω be a convex, possibly unbounded, domain of $\mathbf{R}^N$ and let $\mathbf{n}$ be the outward normal to $\partial\Omega$, denote by $\mu_1(\Omega)$ the first nontrivial Neumann eigenvalue of the Hermite operator in Ω:

$$\begin{cases} -\Delta u + x \cdot \nabla u = \mu u & \text{in} \quad \Omega \\ \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

They proved that

$$\mu_1(\Omega) \geq 1. \tag{5}$$

The estimate is sharp since equality sign holds if Ω is a N -dimensional strip. they observed that (5) can be viewed as an optimal Poincaré-Wirtinger inequality for functions belonging to the weighted Sobolev space $H^1(\Omega, d\gamma_N)$, where γ_N is the N -dimensional Gaussian measure.
When $N = 2$, under some additional assumptions on Ω, they study the equality case and we show that $\mu_1(\Omega) = 1$ if and only if Ω is any 2-dimensional strip. The study of the equality case requires, among other things, an asymptotic analysis of the eigenvalues of the Hermite operator in thin domains.
He also discussed an inequality à la Szegö-Weinberger for $\mu_1(\Omega)$.

Dario Mazzoleni lectured on *regularity of optimal sets for spectral functionals.*
He considers the variational problem

$$\min \big\{ \lambda_1(\Omega) + \ldots + \lambda_k(\Omega) \ : \ \Omega \subset R^d, \ |\Omega| = 1 \big\}, \tag{6}$$

where the variable is the domain Ω, $|\cdot|$ denotes the Lebesgue measure and the cost functional is the sum of the first k Dirichlet eigenvalues on Ω.
This is one of the most important problems in shape optimization and it was studied from many points of view in the last years. In particular it is possible to prove existence of minimizers in the class of quasi-open sets starting from a result by Buttazzo and Dal Maso, then generalized by Bucur, Mazzoleni and Pratelli. The next major and difficult issue is to study the regularity of optimal sets.
A first result was provided by [20], where it was proved that every optimal set Ω^* for problem (6) has the first k eigenfunctions $u_1, \ldots, u_k$ that are Lipschitz continuous in R^N and so it is actually at least an open set.
In the new work with S. Terracini and B. Velichkov [51], they investigate further the regularity of optimal sets. In particular they proved that there is $C^{1,\alpha}$ regular boundary up to a set of zero $\mathcal{H}^{d-1}$–measure. This is strongly related to the regularity of the free boundary $\partial\{|U| > 0\}$ of the local minima of the functional

$$H^1_{loc}(R^d, R^k) \ni W \mapsto \int |\nabla W|^2 + |\{|W| > 0\}|,$$

on which we will focus most of our attention. This free boundary approach is inspired by the well–known work by Alt and Caffarelli [2] and by similar regularity results in the field of optimal partitions for eigenvalues of the Dirichlet-Laplacian [58].

3.3 Inequalities and PDEs

Yuxin Ge presented some new results jointly with E. Sandier et P. Zhang related *from Ginzburg-Landau Equations to n-harmonic maps.*
These are results on the critical points to the generalized Ginzburg-Landau equations in dimensions $n \geq 3$ which satisfy a suitable energy bound, but are not necessarily energy-minimizers. When the parameter in the equations tend to zero, such solutions are shown to converge to singular n-harmonic maps into spheres which are conformally invariant, and the convergence is strong away from a finite set consisting 1) of the infinite energy singularities of the limiting map, and 2) of points where bubbling off of finite energy n-harmonic maps takes place. The latter case is specific to dimensions greater than 2.

Siyuan Lu gave a talk *on Weyl's embedding problem in Riemannian manifold.*
he considers a priori estimate of the Weyl's isometric embedding problem of (S^2, g) in general 3-dimensional Riemannian manifold $(N^3, \bar{g})$. He establishes the mean curvature estimate for the embedding under natural geometric assumption. He also reproves Pogorelov's isometric embedding into $\mathbf{H}^3$ under the condition that $g \in C^{2,\alpha}$. Together with a recent work by Li-Wang, they were able to obtain an isometric embedding of (S^2, g) in Riemannian manifold.

Xinan Ma discussed the joint work with Qiu Guohuan on *the Neumann boundary value problem for Hessian on convex domain in R^n.*
For the Dirichlet problem on the k- Hessian equation, Caffarelli-Nirenberg-Spruck (1986) obtained the existence of the admissible classical solution when the smooth domain is strictly $(k-1)$-convex in R^n. In his talk, they prove the existence of a classical admissible solution to a class of Neumann boundary value problems for k Hessian equations in strictly convex domain in R^n, so answering to a question of N. Trudinger in 1987. The methods depends upon the establishment of a priori estimates of the second order derivatives.

Michele Marini talked on *stationary isothermic surfaces of the solutions of the anisotropic diffusion equation.*
The study of the solutions of certain evolution equations with one or more time-invariant equipotential surfaces is motivated by a conjecture by Klamkin named *Matzoh Ball Soup Problem* and solved by Alessandrini in [1].

Later on the Matzoh Ball Soup Problem has been re-considered and extended by several authors.

He considers the solution of the non linear evolution equation $u_t = \Delta_K u$ in $\Omega \times (0, +\infty)$, $u = 0$ initially and $u = 1$ on the boundary.
Here $\Delta_K u(x) = \operatorname{div}(h_K(Du) Dh_K(Du))$ is the *Finsler Laplacian* associated with the support function of a convex body, h_K.

When K is the Euclidean ball it has been shown in [52] that a solution u has a time-invariant level surface, Γ, only if Ω is a ball (under suitable assumptions on both Ω and Γ).
A crucial step, in order to obtain the symmetry of the domain, is to show that Γ is parallel to the boundary of Ω; in such a way the two unknowns of the problem turn out to be linked by a geometric constraint.

To show this it is used the fact that, if u is a solution of the heat equation (with homogeneous Dirichlet boundary conditions), than $-4t \ln u(x,t)$ converges, as $t \to 0^+$, to $\operatorname{dist}^2(x, \partial\Omega)$, uniformly on Ω (see [65]).

He showed that, for the solutions of the non linear evolution equation taken into account as well, a time-invariant level surface has to be (anisotropic) parallel to the boundary of the domain and in particular we extend the above result by Varadhan, by replacing $\operatorname{dist}(x, \partial\Omega)$ with a suitable concept of anisotropic distance.

Guohuan Qiu discussed *applications of the Neumann problems for Hessian equations to Alexandrov-Fenchel inequalities.*
The classic Neumann problem for laplace equation has many geometric applications. For example, Reilly [60] used its solution to give a new proof of Minkowski inequality and Cabre [21] used it to give a very simple proof of isoperimetric inequality. Recently, Xinan Ma and Guohuan Qiu [49], have proved the existence of the Neumann problems for Hessian equations in uniformly convex domain in $\mathbf{R}^n$. Motivated from Reilly [60] and Ma-Qiu's [49] work, Guohuan Qiu and Chao Xia also find geometric applications about classical Neumann problems for Hessian equations. They prove the existence of classical Neumann problems under the uniformly convex domain. Then they use the solution of the classical Neumann problem to give a new proof of a family of Alexandrov-Fenchel inequalities arising from convex geometry.

Guofang Wang lectured on *the transversal Yamabe problem*: On a Riemannian foliation is there a basic conformal metric with constant transversal scalar curvature? Related to this problem there is an optimal transversal Sobolev inequality.

Gaoyong Zhang presented lecture on *the logarithmic Brunn-Minkowski inequality and Minkowski problem.*

The Brunn-Minkowski inequality and the Minkowski problem are centerpieces of the classical Brunn-Minkowski theory of convex bodies. The logarithmic Brunn-Minkowski inequality and the logarithmic Minkowski problem are recent proposed objects of study. The logarithmic Brunn-Minkowski inequality is stronger than the classical Brunn-Minkowski inequality, and the logarithmic Minkowski problem requires solving a more difficult Monge-Ampere equation — one which requires certain measure concentration for the existence of solutions.

The geometric mean $K^{1-t} \cdot L^t$, $0 \leq t \leq 1$, of convex bodies K, L in $\mathbf{R}^n$ is defined as the largest convex body whose support function is smaller than $h_K^{1-t} h_L^t$, that is,

$$K^{1-t} \cdot L^t = \{x \in \mathbf{R}^n : x \cdot u \leq h_K^{1-t}(u) h_L^t(u),\ u \in S^{n-1}\},$$

where h_K, h_L are the support functions of K, L.

The arithmetic mean $(1-t)K + tL$ is the usual Minkowski sum or vector sum. It is the convex body whose support function is $(1-t)h_K + th_L$. There is the inclusion,

$$K^{1-t} \cdot L^t \subset (1-t)K + tL.$$

The logarithmic Brunn-Minkowski inequality. For origin-symmetric convex bodies K, L, prove the inequality,

$$V(K^{1-t} \cdot L^t) \geq V(K)^{1-t} V(L)^t, \quad 0 < t < 1.$$

This conjectured inequality (see [13]) is stronger than the classical Brunn-Minkowski inequality:

$$V((1-t)K + tL) \geq V(K^{1-t} \cdot L^t) \geq V(K)^{1-t} V(L)^t,$$

and is proved only in $\mathbf{R}^2$ and special cases in higher dimensions.

If K is a convex body in $\mathbf{R}^n$ that contains the origin in its interior, then the cone-volume measure, V_K, of K is a Borel measure on the unit sphere S^{n-1} defined for a Borel set $\omega \subset S^{n-1}$, by

$$V_K(\omega) = \frac{1}{n} \int_{x \in \nu_K^{-1}(\omega)} x \cdot \nu_K(x)\, d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of points of ∂K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

The logarithmic Minkowski problem. Find necessary and sufficient conditions on a finite Borel measure μ on the unit sphere S^{n-1} so that μ is the cone-volume measure of a convex body K in $\mathbf{R}^n$, that is,

$$V_K = \mu.$$

When the measure μ has a density f, the associated partial differential equation in local coordinates for the logarithmic Minkowski problem is the following Monge-Ampere type equation on S^{n-1},

$$h \det(h_{ij} + h\delta_{ij}) = f,$$

where h_{ij} is the covariant derivative of the unknown function h with respect to an orthonormal frame on S^{n-1} and δ_{ij} is the Kronecker delta.

The existence of solution to the logarithmic Minkowski problem involves measure concentration.

Subspace concentration condition. A finite Borel measure μ on S^{n-1} is said to satisfy the *subspace concentration condition* if, for every m-dimensional subspace ξ of $\mathbf{R}^n$, $0 < m < n$,

$$\frac{\mu(\xi \cap S^{n-1})}{\mu(S^{n-1})} \leq \frac{m}{n},$$

with equality only if μ is concentrated on some ξ and its complementary subspace ξ'.

The solution to the symmetric logarithmic Minkowski problem is given by the following theorem (see [14]).

Theorem. A non-zero finite even Borel measure on the unit sphere S^{n-1} is the cone-volume measure of an origin-symmetric convex body in $\mathbf{R}^n$ if and only if it satisfies the subspace concentration condition.

3.4 Isoperimetric type inequalities on manifolds

Keomkyo Seo presented a joint work [52] with Sung-Hong Min on *isoperimetric inequalities for complete proper minimal submanifolds in hyperbolic space.*
Let Σ be a k-dimensional complete proper minimal submanifold in the Poincaré ball model B^n of hyperbolic geometry. If consider Σ as a subset of the unit ball B^n in Euclidean space, one can measure the Euclidean volumes of the given minimal submanifold Σ and the ideal boundary $\partial_\infty\Sigma$. Using this concept, he proves an optimal linear isoperimetric inequality which gives the classical isoperimetric inequality under geometric assumption. By proving the monotonicity theorem for such Σ, he further obtains a sharp lower bound for the Euclidean volume, which can be regarded as an extension of Fraser-Schoen and Brendle's recent results [10, 34] to hyperbolic space. Moreover he introduces the Möbius volume of Σ in B^n to prove an isoperimetric inequality via the Möbius volume for Σ.

Jie Wu gave a lecture on her joint work with Yuxin Ge and Guofang Wang on *geometric inequalities for hypersurface in H^n.*
The Alexandrov-Fenchel inequality, as a generalization of the isoperimetric inequality, is a classical inequality in the Euclidean space, but new in the hyperbolic space. They apply the method of using various geometric flows to derive such kind of inequalities. Precisely, by using the inverse curvature flow we[38] first establish an optimal Sobolev type inequality for hypersurfaces in H^n. As an application, they obtain Alexandrov-Fenchel inequalities for curvature integrals. In the hyperbolic space, another kind of Alexandrov-Fenchel inequalities with weight, which is related to the recent study of the Penrose inequality[39] for various mass, appears naturally in the hyperbolic space. They use a conformal flow, which is used first by Brendle for generalized Heintze-Karcher inequality, to derive optimal Alexandrov-Fenchel inequalities[40] for horoconvex hypersurfaces in the hyperbolic space.

Chao Xia discussed his joint works with Guohuan Qiu, and separately with Junfang Li on *generalized Reilly type formula and applications on geometric inequalities.*
Reillys formula is the integral version of Bochner's formula for manifolds with boundary. It has numerous applications when Ricci curvature is nonnegative. In this talk, he presented two kinds of generalized Reilly type formulas for manifolds with boundary which are applicable for manifolds satisfying either a sectional curvature lower bound or a sub-static condition.
By using the generalized Reilly type formulas, they are able to prove i) A Heintze-Karcher type inequality for hypersurfaces in manifolds with sectional curvature bounded below by -1; ii) A Minkowski type inequality for hypersurfaces in hyperbolic space; iii) A Heintze-Karcher type inequality for hypersurfaces in sub-static manifolds and a new proof of Alexandrov type rigidity theorem for constant mean curvature hypersurface in sub-static warped product manifolds which was due to Brendle; iv) An Alexandrov-Fenchel type inequality for hypersurfaces in hyperbolic space.
It is their hope to find more interesting applications for such formulas.

3.5 Applications

Giuseppe Buttazzo discussed an interesting problem of *symmetry breaking for a problem in optimal insulation.*
He considers the problem of optimally insulating a given domain Ω of $\mathbf{R}^d$; this amounts to solve a nonlinear variational problem, where the optimal thickness of the insulator is obtained as the boundary trace of the solution. Two different criteria of optimization were discussed: the first one consists in the minimization of the total energy of the system, while the second one involves the first eigenvalue of the related differential operator. Surprisingly, the second optimization problem presents a symmetry breaking in the sense that for a ball the optimal thickness is nonsymmetric when the total amount of insulator is small enough. He also discussed the shape optimization problem in which Ω is allowed to vary too.

4 Outcome of the Meeting

The workshop brought together leading experts and emerging young mathematicians on the subject in order to discuss recent developments, open problems and future lines of research. It is perhaps remarkable that major contributions to the field came from researchers from all over the world. Thus the workshop provided a great opportunity for a direct contact between different groups which would normally reside in several distinct continents. Indeed there were 35 participants coming from several different countries (Canada, China, Czech Republic, France, Germany, Italy, Korea, USA) and, in order to stimulate the discussion, many of them gave a talk about their recent results and connected open problems (see above for a description). The atmosphere was very stimulating and collaborative and there were many interactions between the participants, especially between the youngest ones. Although it is of course early for evaluating the outcome of these interactions, we can easily trust that new collaborations were born during the time of the meeting.

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