Homological Mirror Geometry

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1 Overview of the Field

Introduced in 1994 by Maxim Kontsevich [K95], Homological Mirror Symmetry (HMS) proposes a means of describing the mirror symmetry phenomenon originating in string theory by the identification of appropriate categorical structures. Roughly, this means identifying the (derived category of) coherent sheaves on an algebraic variety with a Fukaya category of Lagrangians on a "mirror" symplectic manifold or, in notation, $D^b\text{coh}(X) \cong \text{Fuk}(X^\vee)$. In addition to giving a rigorous mathematical interpretation of ideas emerging from physics, proving the conjecture has required substantial developments in algebraic geometry, symplectic topology, homological algebra, and other disparate fields of mathematics. The aim of this workshop was to investigate cutting edge developments in both fields which are inspired by interactions with HMS. In some cases, this involves direct proofs of cases of HMS, but the workshop also focused on mathematics which has been inspired by the structures appearing in HMS. Let us briefly introduce the themes which were the focus topics of the workshop. A more detailed discussion of the content of the presentations and the surrounding discussions and conversations will follow in the later sections.

Autoequivalences of Derived Categories Inspired by Mirror Symmetry: this topic was represented in the talks by Rina Anno, Paul Aspinwall, Will Donovan, and Timothy Logvinenko. If one has a HMS equivalence of the form $D^b\text{coh}(X) \cong \text{Fuk}(X^\vee)$, then one in particular has an identification of the respective groups of autoequivalences of the categories. Early in the history of HMS, it was recognized by Seidel [S99] that there are natural autoequivalences of the Fukaya categories of symplectic manifolds coming from Dehn twists around Lagrangian spheres. The mirror autoequivalences acting on the derived category of coherent sheaves on a mirror variety were first studied by Seidel-Thomas [ST01], yet still remain somewhat mysterious and do not always have an obvious interpretation in terms of classical projective geometry (i.e. do not correspond to a geometrically induced automorphism of $D^b\text{coh}(X)$). There has been substantial recent interest in generalizing the Seidel-Thomas auotequivalences as a means of enhance our understanding of the structure of derived categories. These studies largely center around Rina Anno's notion of a "spherical functor", whose foundations have recently been formalized [AL13]. In the reverse direction, birational equivalences such as flops often induce autoequivalences of $D^b\text{coh}(X)$, e.g. via "flop-flop" functors [BO95]. The corresponding mirror theory and associated symplectic operations remains largely unexplored..

Grade Restriction Windows and Geometric Invariant Theory: this topic was represented in the talks of Gabriel Kerr, Daniel Halpern-Leistner and Kentaro Hori. Physical arguments of Herbst, Hori, and Page in 2008 [HHP08] suggested a method of understanding the derived categories of geometric invariant theory

(GIT) quotients motivated via their study of the mirror geometry. Roughly, they considered transitions which mathematicians would identify as variations of wall-crossings in GIT and, by studying the behavior of these objects under parallel transport in the mirror symplectic (or Kähler) moduli space, developed rules which identify which objects in the equivariant derived category of a space descend to objects in the equivariant derived category of a GIT quotient. Implementation of this theory in concrete mathematical terms has been an active area of research, with notable contributions from several workshop participants. First studied in the mathematical literature by Segal [S09] and developed further by Donovan-Segal [DS15a] the theory was interpreted by Halpern-Leistner as a categorification of the classical notion of Kirwan surjectivity in equivariant cohomology [H-L15]. Parallel work by Ballard, Favero, and Katzarkov [BFK15] exhibited the theory as a powerful method for constructing preferred decompositions of derived categories (e.g. exceptional collections). The theory remains very much an active area with many applications still under construction. A particular topic of interest among several workshop participants is further applications to wall-crossing equivalences. For example, the "flop-flop" functors mentioned above are not yet completely implemented in the theory, despite their intrinsic geometric importance, although important developments have been obtained by Donovan and Segal [DS15a] and Halpern-Leistner and Shipman [H-LS13], and a fairly complete understanding for toric wall-crossings was given by Herbst and Walcher [HW09].

Geometry of the Gauged Linear Sigma Model (GLSM): this topic was represented in the talks of Nick Addington, Shinobu Hosono, and Paul Aspinwall. Introduced by Witten, the gauged linear sigma model is a supersymmetric quantum field theory originally introduced to understand the "Landau-Ginzburg/Calabi-Yau" correspondence. The correspondence has several deep manifestations in the current mathematical literature; one avatar of particular relevance to the workshop is Orlov's interpretation [O09] (developed further in independent work of Isik [I1] and Shipman [S12]) which described the derived category of a hypersurface or complete intersection in terms of graded matrix factorizations. This identification can be understood as a phase transition in the corresponding GLSM. Indeed, the work of Herbst, Hori, Page mentioned above [HHP08] also included a framework for understanding the LG/CY correspondence via grade restriction rules - this was confirmed in the mathematical literature both in the proof of Shipman and the work of Ballard, Favero, Katzarkov. Beyond the LG/CY correspondence, the underlying geometry of these phase transitions has been the source of some truly fascinating developments. Indeed, many of the topics mentioned above (flop-flop equivalences, spherical functors, grade restriction windows, ...) all fit into the framework of the GLSM. A crucial theme of this workshop was extracting yet more geometry from phase transitions in the the GLSM.

Bridgeland Stability: this topic was represented in the talks of Arend Beyer, Ludmil Katzarkov, and Yokinobu Toda. The Bridgeland stability manifold is a deep construction originally designed to construct mathematically the "stringy Kähler moduli space" of a triangulated category. Roughly, the Bridgeland stability manifold encodes the non-commutative deformations of a category, and thus a huge amount of geometry, and should control many aspects of the deformation theory of the categories involved in HMS. It's initial motivation follows heavily from homological mirror symmetry, as it is expected to be the "correct" (from a physical perspective) mirror moduli space of symplectic (A-model) mirrors. Explicit constructions of even individual points or open subsets of a stability manifold are notoriously difficult, although there as been dramatic progress in the last few years.

2 Recent Developments and Open Problems

The major recent developments and open problems presented in the workshop revolved around the connection between birational geometry and symplectic automorphisms in mirror symmetry and derived categories.

Provide a mathematical framework for Hori's generalized Grade Restriction Rules: In an extremely fundamental development for derived categories [O09], Orlov provided a deep correspondence between B-branes on LG-models and on algebraic varieties. This mathematical development was later explained in the physics literature by Herbst-Hori-Page [HHP08] by introducing the concept of a "Grade Restriction Rule". As a consequence of this idea from high-energy theoretical physics, in independent work Ballard-Favero-

Katzarkov, and Halpern-Leistner, developed a unifying relationship between derived categories and variation of geometric invariant theory quotients. However, the relationship was limited to the types of Grade Restriction Rule in the Herbst-Hori-Page framework and has failed to explain, for example, the famous relationship between Pfaffian and Grassmanian Calabi-Yau 3-folds. In his talk, Hori explained a new, mathematically precise, Grade Restriction Rule, motivated physically. An open problem is to explain and expand the mathematical nature of this rule and use it to extend the work of Ballard-Favero-Katzarkov and Halpern-Leistner.

Find an action of the fundamental group of the Kähler moduli space or space of Bridgeland Stability Conditions of X on the derived category: HMS predicts that symplectic automorphisms should manifest in the derived category of mirror algebraic varieties. More precisely, given an algebraic variety X, let $X^\vee$ be the mirror. Symplectic automorphisms of $X^\vee$ give autoequivalences of $D^b(X)$. One way to produce symplectic automorphisms of $\hat{X}$ is take the monodromy around loops in the complex structure moduli space. This should produce a faithful action of the fundamental group of this moduli space on $D(X)$. A lot of progress in this direction was presented during this workshop by Daniel Halpern-Leistner, Paul Aspinwall, Rina Anno, Timothy Logvinenko, and Will Donovan. A lot of progress has been made in this direction, however, it remains open how to do this more generally, especially using Bridgeland's model for this moduli space.

Use tilting to construct new examples of Bridgeland Stability Conditions in all dimensions: Bridgeland stability is a very important concept in derived categories and mirror symmetry, serving as a mathematically well-defined notion of the stringy Kähler moduli space which is discussed in high-energy theoretical physic circles. Bayer-Macri-Toda have recently made many breakthroughs in constructing Bridgeland stability conditions which was presented by Bayer during the workshop. In [BMT14], they have also presented a conjectural program for constructing these stability conditions more generally. The program involves generalizing the classical Bogomolov-Gieseker inequality and using tilts of classical stability conditions to get ones on derived categories. This is very exciting, as, for one, it shows that the moduli space of stability conditions is non-empty. In particular, Bridgeland's notion of a stringy Kähler moduli space is not trivial. Participants Benjamin Schmidt, Barbara Bologenese, Arend Bayer, and Yukinobu Toda are working in this direction.

Develop semi-orthogonal decompositions as a phenomenon in homological mirror symmetry, especially their relationship with mirror constructions to variation of geometric invariant theory quotients: Semi-orthogonal decompositions are a way to simplify derived categories into more managable pieces or at least establish interesting relationships between different derived categories. In many ways, they are predicted by the structure of the mirror and Kontsevich's homological mirror symmetry conjecture. The appearance of semi-orthgonal decompositions in algebraic and symplectic geometry can, therefore, be viewed as an additional structure one should attach to the homological mirror symmetry conjecture. This conjectural enhancement of homological mirror symmetry was fleshed out in [BDFKK15]. Developments in this direction were discussed by Gabriel Kerr and Nick Addington.

3 Presentation Highlights

· Monday's talks focused on grade restriction windows and autoequivalences induced by birational transformations and wall-crossings.

Kentaro Hori (IPMU): *Grade Restriction Rules from Hemisphere Partition Functions*
The workshop began with a talk by Prof. Kentaro Hori. He spoke on in progress work with R. Eager, J. Knapp, and M. Romo describing the underlying physics of grade restrictions rules. As discussed above, Hori's ongoing contributions to this topic have drawn substantial interest from mathematicians working in derived categories, as such his talk was crucial for setting the tone for many of the subsequent talks. Hori began with a review of the GLSM and the data which goes into defining the particular (2,2) supersymmetric

quantum field theory under study, which roughly consists of a representation of a gauge group G and a G-invariant superpotential W. Hori reviewed his classical work with Tong [HT07] describing the structures behind the Rødland (or Pfaffian-Grassmannian) model [R00], historically the first example of a non-birational derived equivalence. The talk then emphasized the played by the Hemisphere partition function which is roughly the exact partition function of the GLSM, and how its convergence dictates the grade restriction rules. The talk concluding with some explicit examples where full exceptional collections can be extracted by these physical methods, which appeared to be beyond the scope of the current state of rigorous mathematical proofs.

Daniel Halpern-Leistner (Columbia): *Magic windows and representations of generalized braid groups on the derived category of a GIT quotient*

Prof. Halpern-Leistner spoke on recent work with Steven Sam which studies certain cases where the behavior the derived category under a GIT wall-crossing can be understood combinatorially. The particular setup involves considering symmetric representations, which are roughly linear representations of a reductive group whose associated GIT chamber structure has a zonotopal structure. This corresponds to the “local” case of a very large class of wall-crossings. In this situation all wall-crossings are equivalences, and the talk explained how the corresponding derived categories are generated by certain strong exceptional collections of line bundles (“magic windows”); these sets of line bundles have appeared implicitly in many recent works, for example those of workshop participants W. Donovan and E. Segal concerning Grassmannian flops [DS15a]. The zonotopal GIT chamber determines a complexified hyperplane arrangement whose fundamental groupoid should, assuming homological mirror symmetry, act faithfully on these derived categories, as it constitutes the mirror Kähler moduli space. The talk showed how this representation can be constructed explicitly (without invoking HMS).

Official Abstract: One consequence of the homological mirror symmetry conjecture predicts that many varieties will have “hidden symmetries” in the form of autoequivalences of their derived categories of coherent sheaves which do not correspond to any automorphism of the underlying variety. In fact the fundamental groupoid of a certain ”complexified Kaehler moduli space” conjecturally acts on the derived category. When the space in question is the cotangent bundle of a flag variety, actions of this kind have been studied intensely in the context of geometric representation theory and Kahzdan-Lusztig theory. We establish the conjectured group action on the derived category of any variety which arises as a symplectic or hyperkaehler reduction of a linear representation of a compact Lie group. Our methods are quite explicit and essentially combinatorial – leading to explicit generators for the derived category and an explicit description of the complexified Kaehler moduli space. The method generalizes the work of Donovan, Segal, Hori, Herbst, and Page which studies grade restriction rules in specific examples associated to “magic windows.” Based on joint work with Steven Sam.

Nick Addington (University of Oregon): *Complete intersections of unequal degrees*

A fundamental theorem of Orlov [O09] asserts that if X_d is a hypersurface in $\mathbb{P}^d$ with degree $d < n+1$ (i.e. the Fano case), then the derived category $D^b(X)$ admits a semiorthogonal decomposition into exceptional objects (coming from twists of $\mathcal{O}(1)$) and the orthogonal category given by graded matrix factorizations of the defining polynomial f of X_d. A prototypical example due to Kuznetsov is the derived category of a cubic fourfold in which the orthogonal component can be viewed as the derived category of K3 category [K10]. A similar result can be described for complete intersections of equal degrees, with the orthogonal component consisting of, roughly, a family or bundle of matrix factorization categories. For example, the intersection of two quadrics in $\mathbb{P}^5$ is a Calabi-Yau threefold and such a description of its derived category was given at least implicitly in classic work of Bondal-Orlov [BO95]. Prof. Addington’s talk reported on joint work with conference participant Paul Aspinwall investigating the case of complete intersections of unequal degrees. The main moral of the talk was that is profitable to view the component orthogonal to the exceptional collection as a noncommutative resolution in the sense of van den Bergh [VDB05]. The main theorem announced stated that if $f_1, \ldots, f_k$ are homogenous polynomials with degrees d_i and subject to the restriction $d_1 > \max\{d_2, \ldots, d_k\}$, then the orthogonal component is a categorical resolution of the category of graded matrix factorizations of a singular hypersurface of degree d_1 (in an appropriate weighted projective space). The talk concluded by showing how this analysis can be applied to the study of extremal transitions of Calabi-Yau manifolds.

Official Abstract: For a Fano hypersurface in $\mathbb{P}^n$, the derived category decomposes into an exceptional collection and a category of matrix factorizations. For a complete intersection of k hypersurfaces of degree d, it decomposes into an exceptional collection and a sort of bundle of categories of matrix factorizations over $\mathbb{P}^{k-1}$. What about a complete intersection of hypersurfaces of unequal degrees $d_1 \dots d_k$? Do we get a similar bundle over weighted $\mathbb{P}^{k-1}$, with weights $d_1 \dots d_k$? Not really: it is better to view it as a categorical resolution of the category of matrix factorizations of some higher-dimensional, singular hypersurface. The prototypical example is Kuznetsov's degree-6 K3 surface resolving the category of matrix factorizations of a nodal cubic 4-fold. We will discuss several other examples and state some general results. This is joint work with Paul Aspinwall.

Will Donovan (IPMU): *Twists and braids for general threefold flops*

The pioneering example in the study of the relationships between birational geometry and derived categories is Bondal-Orlov's study of the Atiyah flop (essentially the simplest threefold flop) [BO95], and it associated autoequivalence (which is not trivial). Prof. Donovan spoke on joint work with Michael Wemyss studying the flop-flop equivalences for more general classes of threefold flops. As in the talk of Halpern-Leistner, one expects that these autoequivalences should correspond to monodromy transformations on a mirror symplectic manifold; thus, if one one studies multiple flops HMS predicts that this should correspond to an action of the fundamental group(oid) of an associated hyperplane or discriminantal arrangement. A basic example is if one flops two $(-1,-1)$ curves meeting along a point one obtains an action of the pure braid group PB_3 on the derived category. For a more general threefold flop, if one assumes Gorenstein singularities near the exceptional locus, it was observed by Reid that, upon taking a resolution of a generic hyperplane section, one may associate a marked Dynkin diagram to the flop. It is a folklore theorem from 1980's birational geometry that if one associates the root hyperplane arrangement to this Dynkin configuration, then the group of connected components of its complement is in bijection with the relative minimal models (roughly, all associated flops). The main theorem announced by Prof. Donovan was indeed a representation of this group of connected components on the autoequivalence group of the original threefold X, thus suggesting a very strong relationship between flops and derived autoequivalences, consistent with the conjectures of Bondal-Orlov. HMS predicts that the representation constructed by Donovan-Wemyss should be faithful, although this remains open.

Official abstract: When a 3-fold contains a floppable curve, there is an associated equivalence between the derived categories of the 3-fold and its flop. If the curve is reducible, there may exist multiple such flop functors, one for each irreducible component. I will explain joint work with Michael Wemyss, showing how this leads to new actions of braid-type groups on the derived category, and give an update on related results.

· Day 2 of the workshop focused more directly on mirror symmetry (both homological and "classical").

Charles Doran (University of Alberta and University of Maryland): *Mirror Symmetry, Tyurin Degenerations, and Fibrations on Calabi-Yau Manifolds*

Prof. Doran spoke on mirror constructions for Calabi-Yau threefolds equipped with a fibration structure in codimension one (for example, a Calabi-Yau threefold fibered in K3 surfaces). The principal observation in the talk was that the mirror to such fibration structures should be a particular type of degeneration which roughly consists of two copies of Fano varieties glued along a codimension one subscheme ("Tyurin degenerations"). Prof. Doran discussed motivations for this principle based on his previous work (with many collaborators) on classifying the types of variations of mixed Hodge structures which can occur for mirrors to Calabi-Yau threefolds, and sketched proofs for many classes of examples that the appropriately defined Hodge numbers agree with the mirror symmetry expectations. The talk included discussions of how this proposal fits with Batyrev-Borisov's combinatorial mirror symmetry for Calabi-Yau's which are complete intersections in toric varieties, as well as interplay with the compactification of the moduli space of K3 surfaces. The talk generated a substantial amount of audience discussion based on possible interpretations of

Prof. Doran's results in homological mirror symmetry.

Official abstract: We present a new construction of mirror pairs of Calabi-Yau manifolds. On one side of the mirror correspondence are Calabi-Yau manifolds fibered in codimension one by Calabi-Yau submanifolds, for example elliptic fibered K3 surfaces or K3 surface fibered Calabi-Yau threefolds. On the other side are so-called Tyurin degenerations, i.e., smoothings of pairs of quasi-Fano varieties whose common intersection Calabi-Yaus are mirror to the fibers; these correspond to Type II Kulikov degenerations in the K3 surface case and Kawamata-Namikawa smoothings in the case of Calabi-Yau threefolds. Evidence that the construction produces mirror pairs comes from several directions: The fibered Calabi-Yaus are constructed by gluing the pair of Landau-Ginzburg models mirror to the pair of quasi-Fano varieties, and we establish mirror symmetry of Euler and Hodge numbers. Our construction is compatible with the Batyrev-Borisov mirror construction, wherein a bipartite nef partition produces the structures on both sides and the singular fibers of the fibration encode properties of the Landau-Ginzburg models mirror to the two quasi-Fano varieties. In the case of elliptic fibered K3 surfaces, the KSBA compactification of moduli of pairs suggests a broad correspondence between Type II degenerations of a lattice-polarized K3 surface and elliptic fibrations on its Dolgachev-Nikulin mirror. A complete classification of Calabi-Yau threefolds fibered by mirror quartic K3 surfaces leads to explicit constructions of candidate mirror threefolds and their Tyurin degenerations, showing that our construction is not limited to threefolds constructed as toric complete intersections. Finally, we show that in the context of homological mirror symmetry, non-commutative K3 fibrations should be mirror to Tyurin degenerations along loci in moduli disjoint from points of maximal unipotent monodromy.

Helge Ruddat (Mainz): *Tropical descendent Gromov-Witten invariants*
Prof. Ruddat spoke on joint work with Travis Mandel studying a tropical formulation of descendent Gromov-Witten invariants. Recall that a main theme in tropical geometry is, very roughly, to provide a calculus for doing enumerative calculations with curves in terms of piecewise linear "tropical" analogues - a process which often makes the underlying combinatorics more transparent. A landmark result in the field was Mikhalkin's tropical calculation of the genus zero Gromov-witten theory of $\mathbb{P}^2$, which was later generalized to arbitrary toric surfaces by Nishinou-Siebert. However, classical mirror symmetry in a strong form works at the level of "big" quantum cohomology, whose associated enumerative geometry is that of descendent invariants (roughly: the intersection theory of ψ-classes on the Moduli space of curves). Tropical descendent invariants have previously been studied by Markwig and Rau, who verified the equivalence of the classical and the tropical description in the case of $\mathbb{P}^2$. Prof. Ruddat announced a comparison of this tropical calculus with the rapidly emerging field of log Gromov-Witten invariants. The talk discussed the applications of this result to mirror symmetry, with log Gromov-Witten invariants playing a crucial role in the construction of Gross-Siebert mirrors.

Official abstract: Descendent Gromov-Witten invariants play a central role in canonical deformations of Landau-Ginzburg models as well as the multiplication rules of generalized theta functions, both relevant for (homological) mirror symmetry. In a joint work with Travis Mandel, I prove that tropical Gromov-Witten invariants with psi class conditions coincide with descendent log Gromov Witten invariants for smooth toric varieties whenever non-superabundance is given. We use toric degenerations a la facon de Siebert-Nishinou and we expect that our approach will be generalizable to Mumford or Gross-Siebert type degenerations.

Ludmil Katzarkov (University of Miami and University of Vienna): *A categorical Donaldson-Uhlenbeck-Yau correspondence*
Prof. Katzarkov's talk began with a departure from the title and abstract, wherein he offered a proposal to a question raised during Prof. Doran's earlier talk: namely how to understand the role of Tyurin degenerations in homological mirror symmetry. Katzarkov proposed interpreting the question in terms of Lagrangian skeleta (themselves a fairly recent proposal in mirror symmetry under heavy recent investigation by Nadler and Zaslow and collaborators in particular). In particular, Katzarkov proposed some elementary examples of Tyurin degenerations which could be understood combinatorially, and in principle which should give categorical equivalences. The main portion of Katzarkov's talk then concentrated on joint work with F. Haiden, M. Kontsevich, and P. Pandit which explores categorifications of the classical theorem(s) of

Donaldson-Uhlenbeck-Yau which identify semistable holomorphic vector bundles on Kähler manifolds with Hermitian-Einstein connections. More precisely, Katzarkov proposed a definition of "DUY structure" on a category which conjecturally would identify the moduli space of semistable bundles with a Bridgeland moduli space of semistable objects. Indeed, this suggestion appears to be more in line with the original physical construction of the Bridgeland stability manifold as first constructed by Michael Douglas and conference participant Paul Aspinwall [AD04].

Official abstract: In this talk we will introduce the notion of Donaldson Uhlenbeck Yau correspondence. A connection with sheaves of categories will be discussed.

Shinobu Hosono (Gakushuin University): *Conifold transitions in mirror symmetry of CICYs*
Prof. Hosono spoke on his joint works with Hosono-Takagi which study (non-birational) derived equivalences generalizing the Rødland (Pfaffian-Grassmannian) model [R00]. Their recent works have been a fruitful testing ground for geometric techniques for proving derived equivalences. After a brief review of the Batyrev-Borisov mirror construction and Landau-Ginzburg models, Hosono introduced a particular model known as the Reye congruence which, roughly, is the set of lines in $\mathbb{P}^4$ which lie in a two dimensional family of quadrics. This set X is known to be a Calabi-Yau threefold, and admits a Landau-Ginzburg description via toric geometry. Hosono discussed the derived equivalent spaces (Fourier-Mukai partners) and various phase transitions which can be obtained from VGIT. Via mirror symmetry, there is a mirror family, and in this particular example the entire Kähler moduli space (including its discriminant locus) can be computed explicitly, and the large complex structure monodromy can also be computed. The calculations given in the talk agree with predictions from homological mirror symmetry, as well as allow from explicit calculation in "classical" mirror symmetry relating Gromov-WItten invariants with period integrals.

Official abstract: In a series of collaborations with Hiromichi Takagi, I have been studying certain complete intersection Calabi-Yau spaces, which are nicely related to determinantal varieties in projective spaces. After summarizing relations to the linear duality (due to Kuznetov), I will focus on the mirror symmetry of these Calabi-Yau spaces. In particular, I will describe conifold transitions explicitly for the case of mirror family obtained in our CMP paper (2014, vol.329, 1171–1218).

· Day 3 of the workshop focused on connections with physics.

Paul Aspinwall (Duke University): *Mirror symmetry and discriminants*
Prof. Aspinwall began his talk with a historical overview of mirror symmetry, with a focus on the developments in physics in the mid 90's, and in particular the works carried out by Aspinwall in various joint works with B. Greene, D. Morrison, and R. Plessar. The focus was on the (complexified) Kähler moduli space of a Calabi-Yau complete intersection, which roughly encodes the symplectic deformations of a mirror partner, and whose maximal degeneration points should correspond to different birational models under mirror symmetry. This overall structure has already appeared implicitly in the prior talks of W. Donovan, D. Halpern-Leistner and especially that of S. Hosono. Aspinwall drew attention to the discrimant locus of singular models which appear in the Kähler moduli space which, in the context of the GLSM, can be described at least implicitly via the techniques of Gelfand-Kapranov-Zelevinsky. Monodromy around a large complex structure limit induces a symplectomorphism (and thus an autoequivalence of the Fukaya category) which is mirror to a spherical functor on the correponding derived category of a given birational model (or matrix factorization category of the corresponding phase of the GLSM). This portion of the story is mostly well known to experts, and the philosophy is implicit in many of the talks at this workshop. However, Aspinwall made some conjectures which propose studying degenerations away from the large complex structure limit and deeper in the discriminant locus, which appear to be beyond the scope of the current mathematical literature; although in principle the general theory of spherical functors (cf. R. Anno's talk on Friday) should be able to accomodate such a theory. In particular, Aspinwall proposed an explicit subcategory given by the underlying toric geometry which should be the subcategory about which one "twists".

Official abstract: We analyze singularities in the parameter space of the gauged linear sigma model and *show how they coincide with the GKZ A-determinant in the noncompact case. We show that this requires logarithmic coordinates to work correctly. The same analysis gives a natural picture for generic monodromy in the derived category around components of the discriminant in terms of specific spherical functors.*

Eric Sharpe (Virginia Tech): *Heterotic mirror symmetry*

Prof. Sharpe spoke on the physics behind a more general version of mirror symmetry than that usually considered by mathematicians. Roughly, the mathematics literature focuses on models which come from quantum field theories with (2,2) supersymmetry, although physically only (0,2) supersymmetry is required to discuss mirror phenomena. Recall that "classical" mirror symmetry identified the complex deformations of a Calabi-Yau manifold X with the Kähler deformations of its mirror $X^\vee$, and thus implies an equivalence of Hodge numbers $h^{p,q}(X) = h^{n-p,q}(X^\vee)$. In (0,2) mirror symmetry the initial data consists of not just a Calabi-Yau manifold X, but the data of a stable vector bundle E on it, and the corresponding identification of bundle moduli implies an isomorphism of sheaf cohomologies $h^p(X, \wedge^q, E^*) = h^p(X^\circ, \wedge^q E^\vee)$. Explicit mathematical verification of even simple examples of this phenomenon is still under development. A crucial role has been the development of "quantum sheaf cohomology" which is a (0,2) analogue of ordinary quantum cohomology and which should encode the corresponding equivalence of complex and Kähler bundle moduli between a (0,2) mirror pair. Sharpe lectured on several classes of examples where one takes E to be the tangent bundle; two cases of particular interest were where $X = \mathbb{P}^1 \times \mathbb{P}^1$ and where X is a Grassmannian. The lecture generated a substantial amount of audience discussion on a hypothetical "(0,2) homological mirror symmetry".

Official abstract: In this talk we will describe progress towards a generalization of mirror symmetry pertinent for heterotic strings. Whereas ordinary mirror symmetry relates, in its simplest incarnations, pairs of Calabi-Yau manifolds, the heterotic generalization relates pairs of holomorphic vector bundles over (typically distinct) Calabi-Yau's, satisfying certain consistency conditions. We will also outline the corresponding analogue of quantum cohomology, known as quantum sheaf cohomology, describing results for deformations of tangent bundles of toric varieties and Grassmannians, and we will discuss (0,2) Landau-Ginzburg Toda-like mirrors to deformations of tangent bundles of products of projective spaces.

· Day 4 of the workshop was focused on stability conditions and notions of wall-crossing beyond those considered in previous talks.

Yokinobu Toda (IPMU): *Wall-crossing formulas of higher rank DT invariants*

Prof. Toda gave a very interesting talk on Donaldson-Thomas invariants. These invariants count curves on Calabi-Yau 3-folds. He described their relationship to Gromov-Witten invariants, another way of counting curves on Calabi-Yau 3-folds related to the classical enumerative mirror symmetry story. From 2008-2010, Bridgeland-Toda studied how DT-invariants change in a certain parameter space (wall-crossing formulas) and proved, for example, the DT/PT correspondence, and rationality of generating functions for these invariants. In this talk, Toda announced his results extending his work with Bridgeland for higher rank DT invariants.

Arend Bayer (Edinburgh: *Stability conditions on surfaces: an update*

Prof. Bayer spoke on recent advances in understanding the Bridgeland stability manifold of algebraic surfaces. Some of the first explicit examples of stability conditions were constructed by, of course, Bridgeland in his study of K3 surfaces, which proceeded by using the Harder-Narisimhan filtration on coherent sheaves on a surface to cosntruct a so-called "torsion pair", from which one can construct an explicit stability condition on $D^b\mathrm{coh}(X)$. Extending these methods to threefolds is a very active problem. However, Bayer spoke on applications to some classical problems on surfaces, demonstrating the applicability of Bridgeland stability to concrete problems. To an algebraic surface, one may consider a Brill-Noether locus, which is roughly a moduli space of one-dimensional sheaves with fixed Euler characteristic and support on a fixed curve. This moduli space contains constructible subsets of sheaves with support on a fixed curve and having a prescribed number of global sections. Bayer showed in particular how a careful analysis of moduli spaces of Bridgeland semistable objects can lead to a new proof of the following classical result of Lazarsfeld: let (X, H) be a polarized K3 surface such that H^2 divides $H \cdot C$ for all curve classes C, if C' is any any smooth curve in the linear system $|H|$. Then the Brill-Noether locus has the expected dimension $g(r+1)(gd+r)$ where g is the genus, d is the degree H^2, and r is the number of sections.

Official abstract: I will give an update on applications of stability conditions for surfaces within algebraic geometry.

Gabriel Kerr (Kansas State): *Mirror symmetry for elementary birational cobordisms*
Prof. Kerr spoke on new results regarding a form of homological mirror symmetry for some toric varieties motivated by VGIT wall-crossings. Let $\mathbb{C}^*$ act on $V = \mathbb{C}^n$, and consider the GIT quotient $V//\mathbb{C}^*$. For example, one obtains (weighted) projective spaces as examples, and by varying the GIT quotient one may also view the standard Atiyah flop and other birational maps such as standard flips within such examples. Mirror symmetry for Fano toric varieties X of dimension n says that the mirror should be a Landau-Ginzburg model $w : (\mathbb{C}^*)^n \to \mathbb{C}$ where w is an explicit Laurent polynomial built from the vertices of the polytope defining X. Homological mirror symmetry says that there should be a derived equivalence $D^b(X) \cong \mathrm{FS}(w)$ where $\mathrm{FS}(w)$ denotes the Fukaya-Seidel category of vanishing cyles of w. In this form, relatively few instances of this theorem are completely proven. Indeed, though, Kerr outlined a proof for all examples of the above form. The principal idea was to make used of the semiorthogonal decompositions of $D^b(X)$ obtained by the grade restriction windows for the VGIT setup. This idea had been exploited in previous work, but only in very specific examples, see, e.g [BDFKK15]. For $\mathbb{C}^*$ acting on V, though, Kerr demonstrating that this structure is all very computable. A very careful analysis of the vanishing cycles of the mirror potential w then exhibits a similar structure. A subtle point is that it is not sufficient to simply match up the corresponding semiorthogonal components, but one must compute all homomorphisms and higher intersection products in order to compute the full dg structure on both sides and thus obtain a full proof of homological mirror symmetry for these examples.

Official abstract: It has been known since Bondal and Orlov's work on semi-orthogonal decompositions that for blow-ups, projective bundles and certain flips $f : X \dashrightarrow Y$, one may decompose the derived category of $D^b(X) = \langle D(Y), C\rangle$. In this talk I will describe the mirror LG model to C when f is a birational cobordism with trivial center. Diemer-Katzarkov-K. conjectured that this was a Fukaya-Seidel category FS (W) of a potential W from a higher dimensional pair of pants to the punctured plane. I will explain a recent proof of this conjecture. The classical version of HMS for weighted projective spaces of arbitrary dimension then will be observed as a corollary.

· Day 5 of the workshop focused on spherical functors.

Rina Anno (MIT): *DG enhancements of derived categories of sheaves (Part I).*
Timothy Logvinenko (Cardiff): *P-functors (Part II).*
Prof. Anno and Prof. Logvinenko gave a two part lecture series introducing the notion of P-functors. P-functors provide a beautiful way to categorify autoequivalences of the derived category appearing naturally in symplectic and hyperkähler geometry. They are an extension of the work of Seidel-Thomas [ST01] and Huybrechts-Thomas [HT06]. They explained the necessity of the DG category language in defining P-functors, their appearance in algebraic and symplectic geometry, and, how they lead to an action of a "Braid category" (which they defined) on the 2-category of derived categories of flag manifolds. Anno and Logveniko's work builded off previous work of Addington. Their lecture series on P-functors also tied in beautifully with the talks by Halpern-Leistner and Donovan who described similar phenomena, namely, braid group actions on derived categories and/or the 2-category of derived categories.

Official abstract: This talk is based on a joint work with T. Logvinenko, and gives some background for his talk on "P-functors". One of the major problems of working with triangulated categories is that the cone of a map between functors is not well defined, and in constructions such as that of P-twists, we need not just a cone, but a convolution of a three-term complex. In this talk, we will discuss Bondal and Kapranov's pretriangulated categories, where such convolutions exist naturally. We are also going to introduce the *construction of a twisted line bundle over a DG category, a version of which is going to be instrumental in the definition of P-functors.*

Official Abstract: $\mathbb{P}^n$ *objects are a class of objects in derived categories of algebraic varieties first studied by Huybrechts and Thomas. They were shown to give rise to derived autoequivalences in a similar fashion to Seidel-Thomas spherical objects. It was also shown that they could sometimes be produced out of spherical objects by taking a hyperplane section of the ambient variety. In this talk, based on work in progress with Rina Anno, we will first recall the basics on spherical and* $\mathbb{P}^n$ *objects, and then explain how to generalise the latter to the notion of P-functors between (enhanced) triangulated categories. We'll also discuss a closely related notion of a non-commutative line bundle over such category, inspired by a construction of Ed Segal.*

4 Scientific Progress Made

The meeting collected a broad group of experts and early career mathematicians to focus on areas surrounding Homological Mirror Symmetry. The organizers made a concerted effort to invite smaller groups of existing collaborators and provide planned spaces in the schedule for these groups to focus on their work. Some particular groups:

- Rina Anno and Timothy Logvinenko
- Charles Doran, Tyler Kelly, and Ursula Whitcher
- Matthew Ballard and Tyler Kelly
- Will Donovan, and Ed Segal
- Yijia Liu and Andrew Harder
- Barbara Bolognese and Benjamin Schmidt
- Ludmil Katzarkov and Paul Horja
- Alessio Corti and Alexander Kaspryzk
- Nicholas Addington and Paul Aspinwall
- Gabriel Kerr and Ilia Zharkov

In addition, upon communication with participants after the completion of the conference, several people reported progress on new collaborations or emerging discussions which evolved during the week:

- Nick Addington: "I had a nice chat with Ed Segal about cubic 4-folds of discriminant 38."
- Will Donovan: "Toda and I discussed examples of rational curves with interesting non-commutative deformations, arising from work of Thomas (arXiv:math/9903034)."
- Paul Horja: "During the workshop, I worked on a joint project with Gabriel Kerr and Ludmil Katzarkov on Perverse Sheaves of Categories and Mirror Symmetry. I also had very fruitful discussions with Ed Segal and Will Donovan on the connection between global windows in VGIT, zonotopes and non-commutative resolutions."
- Tyler Kelly: "The workshop led to me having a conversation that helped find a bridge amongst symplectic geometry and algebraic geometry. Having people from both sides of mirror symmetry led to fruitful discussions between G. Kerr and myself, inspired by T. Logvinenkos talk."
- Helge Ruddat: "I had useful scientific exchange with Patrick Clarke on Frobenius manifolds and with Matt Ballard on log derived categories."

- Ed Segal: “This was a very productive workshop. I collaborated with Dan Halpern-Leistner developing some of our ongoing projects relating to derived categories and variation-of-GIT, I also had useful discussions with Kentaro Hori about some of his recent work in physics and how it relates to my mathematical work.”

We are already aware of one paper whose creation owes substantial debt to the collaborative environment afforded by the workshop:

- Timothy Logvinenko: “I have worked with long-time collaborator Rina Anno on our ongoing project concerning P-functors, and more generally on the categorification of generalised braids, the project which the work on P-functors fits into and which we spoke on during the BIRS workshop.We have a pre-print titled ”P-functors”, shortly to appear on arXiv, and a large chunk of work on it was done during the BIRS workshop.”

Based on informal conversations with participants, we expect more such papers to appear imminently as many people are developing and finishing up projects over the summer.

5 Outcome of the Meeting

The workshop had 37 participants, ranging from Professors at Imperial and Duke to earliest career mathematicians (10 postdoctoral researchers and 2 graduate students). Among these were 5 female mathematicians. The scientific program consisted of 15 research lectures. Five talks were given by early career mathematicians. The participant list was international (UK, Japan, etc.) along with 6 Canadians. The schedule provided ample time for active research and structured mathematical discussion. Activity, in this regard, was high and organizers received compliments from participants for creating a productive atmosphere. The marvelous support and environment fostered by BIRS freed participants from day-to-day concerns and allowed for greater focus and productivity.

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