Bounds for Restrictions of Laplace Eigenfunctions

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1 Overview of the Field

Let \((M,g)\) be a compact, smooth, Riemannian surface with no boundary, and consider a sequence \(\{(\phi_h)\}\) of \(L^2\)-normalized Laplace eigenfunctions

\[-h^2\Delta_g \phi_h = \phi_h.
\]

The behavior of \(\phi_h, h \to 0\) and its connection with the geodesic flow on \(M\) is the has been the subject of intense study. Recently, there has been an attempt to understand small scale properties of \(\phi_h\). One way of doing this is to understand properties of the restriction of \(\phi_h\) to various submanifolds. Concentration rates of such restrictions as measured by \(L^p\) norms [BGT, HT, Tac], weak limits of \(|\phi_h|^{2}\) [DZ, TZ2], and size and distribution of nodal sets \(\{\phi_h = 0\}\) [ET, Jun] have all been extensively studied.

2 Recent Developments and Open Problems

The problem of obtaining lower bounds for \(\|\phi_h\|_{L^2(H)}\) is quite challenging and has only been attempted in very specific settings. It has profound applications to the study of the nodal sets of high energy eigenfunctions. In [TZ] where the authors show that if \(\Omega \subset \mathbb{R}^2\) is a bounded domain with piece-wise real analytic boundary and \(\{(\phi_h)\}\) is a sequence of Neumann eigenfunctions, then there exists \(C > 0\) for which \(\|\phi_h\|_{L^2(\partial\Omega)} \geq e^{-C/h}\) as \(h \to +\infty\). Curves satisfying such exponential lower bound (such as \(H = \partial\Omega\)), are said to be good. This is a concept that arises frequently when bounding from above the number of zeros of \(\phi_h\) along \(H\). Indeed, on real analytic compact surfaces, the goodness condition on \(H\) is needed to prove the sharp upper bound

\[\#\{\phi_h^{-1}(0) \cap H\} = O(h),\]

see [CT]. For the purpose of obtaining upper bounds on \(\#\{\phi_h^{-1}(0) \cap H\}\), it is proved in [Jun] that horocycles lying inside compact hyperbolic surfaces are good curves. With the same goal, the authors in [ET] proved that if \(\Omega \subset \mathbb{R}^2\) is a bounded convex domain with piecewise real analytic boundary and \(\{(\phi_h)\}\) is a sequence of quantum ergodic (QE) Neumann eigenfunctions, then any real analytic closed curve with strictly positive geodesic curvature is good. One can view the restriction lower bounds in this proposal as a natural closed manifold analogue of the lower bounds in Theorem 1.3 of [ET]. However, the restriction lower bounds for curved \(H\)'s and QE eigenfunction sequences \(\{(\phi_h)\}\) give \(\|\phi_h\|_{L^2(H)} \geq C\) whereas Theorem 1.3 in [ET] only implies the much weaker goodness estimate \(\|\phi_h\|_{L^2(H)} \geq Ce^{-C/h}\) for some \(C = C(H, \Omega) > 0\).

The exponential lower bound was improved in [GRS] in the case in which \(H\) is a closed horocycle lying inside an arithmetic surface and the eigenfunctions \(\{(\phi_h)\}\) are even Maass cusps forms. In this case the authors prove that for every \(\epsilon > 0\) there exists \(C_\epsilon\) so that \(\|\phi_h\|_{L^2(H)} \geq C_\epsilon h^{-\epsilon}\) as \(h \to +\infty\). An even stronger lower bound was obtained in [BR] for the flat torus. In [BR] the authors prove that for any sequence \(\{(\phi_h)\}\) of Laplace eigenfunctions there exists a constant \(C > 0\) for which \(\|\phi_h\|_{L^2(H)} \geq C\) as \(h \to +\infty\), provided \(H\) has non-vanishing geodesic curvature.

As for upper bounds, the universal estimates in [BGT] give \(\|\phi_h\|_{L^2(H)} = O(h^{1/4})\) when \(H \subset M\) is any curve, and \(\|\phi_h\|_{L^2(H)} = O(h^{1/6})\) when \(H\) has non-vanishing curvature. These upper bounds where slightly improved by a \(\log(h)^{-1}\) factor in [Che] for negatively curved surfaces. In some specific cases, the improvement is polynomial in \(\lambda\). For example, on flat tori in [BR], it is shown that \(\|\phi_h\|_{L^2(H)} = O(1)\) when \(H\) has non-vanishing geodesic curvature. In [GRS], the authors prove that for any \(\epsilon > 0\) there exists...
\( C_\epsilon > 0 \) for which \( \| \phi_h \|_{L^2(H)} \leq C_\epsilon h^\epsilon \) when \( H \) is a closed horocycle inside an arithmetic surface and the eigenfunctions \( (\phi_h) \) are even Maass cusps forms.

One expects to obtain uniform lower and upper bounds for \( \| \phi_h \|_{L^2(H)} \) in cases where the eigenfunctions are equidistributed along \( H \). Indeed, it follows from [TZ, DZ] that if \( (M, g) \) has ergodic geodesic flow and \( H \) has a 'zero measure of microlocal symmetry', then there exists a density one subsequence \( (\phi_{h_j}) \) of the set of Laplace eigenfunctions for which \( \| \phi_{h_j} \|_{L^2(H)} \to C \) as \( j \to \infty \). The microlocal asymmetry assumption on \( H \) in [TZ] is generic; but it is quite difficult to check and has only been established for geodesic circles, closed geodesics and closed horocycles inside certain hyperbolic surfaces [TZ]. The existence of a limit for \( \| \phi_{h_j} \|_{L^2(H)} \) hinges on the assumption that the geodesic flow is ergodic. In particular, this assumption gives the existence of a quantum ergodic sequence from which \( (\phi_{h_j}) \) is built.

### 3 Scientific Progress Made

We discussed several approaches to the problem of obtaining lower bounds on the restriction of Laplace eigenfunctions to hypersurfaces \( H \) inside a compact Riemannian manifold \( (M, g) \). That is we wish to estimate

\[
\| \phi_h \|_{L^2(H)} \geq c > 0
\]

when

\[
(-h^2 \Delta_g - 1) \phi_h = 0, \quad \| \phi_h \|_{L^2} = 1.
\]

While we considered several, two such ideas came to the fore. First, the idea of using boundary integral operators together with the semiclassical FIO and Airy calculi and second, that of understanding the geometry of persistent nodal surfaces through propagation of defect measures.

During the discussion, we realized that, in order to make the layer potential approach work, we need control on concentration of eigenfunctions in shrinking (with respect to the spectral parameter \( h \)) neighborhoods of \( H \). Unfortunately, such control is not known except in very special circumstances. We plan to investigate such concentration in neighborhoods of tangential rays. In such regions, it is possible that we will be able to obtain estimates using the Airy calculus for boundary layer operators. Calculations with the Friedlander model indicate that we need control on \( h^{2/3} \) scales.

The second approach proved more fruitful. By thinking of the surface \( H \) as a transparent boundary, we are able to show that defect measures associated to sequences of eigenfunctions, \( \phi_h \), such that

\[
\| \phi_{h_j} \|_{L^2(H)} \to 0
\]

are jointly invariant under the billiard flow on \( M \) with respect to \( H \) and the geodesic flow of \( M \). This invariance has many implications on the geometry of \( H \) when \( \phi_h \) carries mass in neighborhoods of \( H \). In particular, when the defect measure associated to \( \phi_h \) carries mass transversal to \( H \), this implies that the measure has certain symmetries with respect to \( H \). While we expect this behavior to continue uniformly through tangential rays, we are currently unable to obtain this information. Indeed, it seems that defect measures are not an appropriate tool to probe behavior in this set.

### 4 Outcome of the Meeting

We plan to write a paper based on our observations of defect measures in the presence of nodal persistence and will continue to investigate when the phenomenon of nodal persistence may occur.

**References**


