Geometric Structures on Lie Groupoids (17w5023)

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1 Overview

Groups typically arise as the symmetries of some given object. The concept of a groupoid allows for more general symmetries, acting on a collection of objects rather than just a single one. Groupoid elements may be pictured as arrows from a source object to a target object, and two such arrows can be composed if and only if the second arrow starts where the first arrow ends. Just as Lie groups (as introduced by Lie around 1900) describe smooth symmetries of an object, Lie groupoids (as introduced by Ehresmann in the late 1950's) describe smooth symmetries of a smooth family of objects. That is, the collection of arrows is a manifold G, the set of objects is a manifold M, and all the structure maps of the groupoid are smooth.

Ehresmann's original work was motivated by applications to differential equations. Since then, Lie groupoids have appeared in many other branches of mathematics and physics. These include:

- *algebraic geometry*: Grothendieck [3] introduced stacks in the late 1960's via fibered categories over a site. Fibered categories can be viewed as a type of sheaf of groupoids. More recently, this has led to the concept of a gerbe. Among stacks, those known as "geometric" play a central role in differential geometry; these are in correspondence with Lie groupoids, up to Morita equivalence.

- *foliation theory*: Haefliger [24] introduced transversal structures to foliations in the 1970's, using the concept of a holonomy groupoid. This approach allows for a systematic study of transversal structures, and has been central to the subsequent development of the subject.

- *noncommutative geometry and index theory*: Lie groupoids made their appearance in noncommutative geometry through the monumental work of Connes [12] in the 1980's. He introduced the tangent groupoid of a space as a central ingredient in his approach to the Atiyah-Singer index theorem. This approach led to a number of refinements of the index theorem, such as the Connes-Skandalis index theory for foliations.

- *Poisson geometry*: motivated by quantization problems, Karasev and Weinstein [36] introduced the symplectic groupoid of a Poisson manifold in the late 1980's, as a way to "untwist" the complicated behavior of the symplectic foliation underlying the Poisson manifold.

Nowadays, one can find many other applications of Lie groupoids, such as in geometric mechanics, equivariant differential geometry, gauge theory, orbifold theory, exterior differential systems, Ricci flows, and generalized complex geometry .

The infinitesimal symmetries corresponding to Lie groupoids are described by Lie algebroids. After contributions by many people over a period of more than 30 years, the precise obstructions for the integration of Lie algebroids to Lie groupoids were finally determined in the work of Crainic and Fernandes [18]. The problem of finding explicit integrations in applications, and the problem of integrating differential-geometric structures on Lie algebroids, remains an important and active area of research.

2 Recent Developments and Open Problems

Over the last two decades, the theory of Lie groupoids and Lie algebroids has undergone several exciting developments. On the foundational side, we highlight the complete solution of the problem of integrability ("Lie's 3rd theorem") for Lie groupoids, an improved understanding of the relationship between Lie algebroid and Lie groupoid cohomology groups ("Van Est theorems"), and deep results relating multiplicative differential forms on groupoids with their infinitesimal counterparts. On the other hand, Lie groupoids were found to be a natural framework for many applications, such as index theory of elliptic operators, Stokes phenomena in complex analysis, generalized complex geometry, geometric flows, and exterior differential systems. Typically, those applications involve additional geometric structures on groupoids:

- *Poisson geometry*: a symplectic form on the space G of arrows of a groupoid is said to be compatible with the groupoid structure if the graph of the groupoid multiplication is Lagrangian. In such a case, the manifold M of objects of the groupoid inherits a unique Poisson structure for which the source map is a Poisson map [36]. In this context, G is called the symplectic groupoid "integrating" the Poisson manifold M. It can be used to obtain information about the underlying Poisson manifold itself. Examples of non-trivial applications include the linearization of Poisson structures around leaves [16, 19] and the linearization of Poisson Lie groups [1].

- *Dirac geometry*: Originally conceived as a framework for Dirac's theory of second class constraints in geometric mechanics, Dirac geometry has emerged as a flexible generalization of Poisson geometry. The global objects integrating Dirac manifolds are the presymplectic groupoids [11]. Lie algebroids and Lie groupoids enter many aspects of the theory, including its connections with the theory of quasi-Hamiltonian spaces, and the construction of Dirac Lie groups [27] and their homogeneous spaces.

- *Generalized complex geometry*: Mirror symmetry suggests interesting relations between complex manifolds with their symplectic 'mirror' manifolds. Generalized complex structures treat symplectic and complex structures on the same footing [25, 23]. The corresponding global objects are Lie groupoids with a multiplicative structure consisting of a symplectic form and a complex structure, satisfying certain compatibility relations [14].

- *Multiplicative structures on Lie groupoids*: generalizing the previous cases, the study of multiplicative structures (differential forms, multivector fields, connections, and so on) and their infinitesimal versions [8, 11, 31] has provided new insights into classical problems of differential geometry. For example, one can recast Cartan's work on Lie pseudogroups in the language of multiplicative forms on Lie groupoids, showing that the classical Spencer operator appears as the linearization data of the Cartan Pfaffian system [20].

- *Riemannian metrics on Lie groupoids*: the notion of a Riemannian metric compatible with a groupoid structure is quite subtle and only recently was fully understood [21]. However, special cases of Riemannian groupoids have been studied before and applied with success, for example, in the study of the long time behavior of the Ricci flow [29]. More generally, one can use Riemannian structures on Lie groupoids to obtain deep linearization (or canonical forms) results [21].

- *Index theory*: Connes' approach to the Atiyah-Singer index theorem [12, 22], using the tangent groupoid of a manifold, has greatly influenced the subject and led to a number of advances, such as the Connes-Skandalis longitudinal index theorem for foliations [35]. Over the last few years, there has been a lot of activity towards extending these results to singular foliations [2], as well as developing a theory of (pseudo)-differential operators on groupoids [26, 34]

3 Presentation Highlights

This workshop brought together mathematicians working on the foundational aspects of Lie groupoids with researchers working on applications. There were present a pool of researchers with distinct backgrounds and research interests, as well as PhD students and postdoctoral researchers. For this reason, each day of the workshop was organized around a different theme and started with a survey lecture. We describe next the highlights in each of these themes.

3.1 Groupoids, Stacks and Higher Structures

Eugene Lerman: Vector fields on stacks form a Lie 2-algebra

Eugene Lerman (University of Illinois at Urbana-Champaign) gave a survey lecture on vector fields on stacks, sketching out the definitions of a stack, a geometric stack, vector field on a stack and of a (Baez-Crans) Lie 2-algebra, which is a categorified version of a Lie algebra. The survey included a new result stating that the category of vector fields on a geometric stack is a Lie 2-algebra.

Rajan Mehta: Constant symplectic 2-groupoids

Rajan Mehta (Smith College) discussed in his lecture one outstanding problem in the theory: the integration of Courant algebroids. Heuristically, it is known that Courant algebroids should "integrate" to symplectic 2-groupoids, but very little of this correspondence has been developed in a precise way. Mehta described in detail the case of a linear 2-groupoid equipped with a constant symplectic form, explaining how these "constant symplectic 2-groupoids" correspond to a certain class of Courant algebroids. The study of constant symplectic 2-groupoids is intended to be a first step toward a more general study of symplectic 2-groupoids. Symplectic 2-groupoids are closely related to the shifted symplectic structures studied by Pantev, et al, although the definition is more "strict" in certain ways. The additional strictness is appropriate for the problem of integrating Courant algebroids.

Matias Luis del Hoyo: The general linear 2-groupoid

When working with Lie groupoids, representations up to homotopy arise naturally, and they are useful, for instance, to make sense of the adjoint representation. Matias Luis del Hoyo (Universidade Federal Fluminense) introduced in his talk the important concept of a general linear 2-groupoid, based in a joint work with D. Stefani. The idea behind it is to use graded vector bundles and allow for non-associativity. By looking at the symmetries of a graded vector bundle, he shows that, in the 2-term case, they can be regarded as a Lie 2-groupoid. The nerve of a Lie 2-groupoid is a simplicial manifold, and del Hoyo uses this construction to realize 2-term representations up to homotopy as pseudo-functors.

João Nuno Mestre : Transverse measures and densities on Lie groupoids

Transverse measures and geometric measures (densities) on Lie groupoids (and their associated stacks) finds many applications in constructions involving groupoids and singular spaces. João Nuno Mestre (University of Coimbra) described his joint work with M. Crainic, extending Haefliger's approach to transverse measures for foliations to general Lie groupoids. This allows one to define and study measures and geometric measures (densities) on differentiable stacks. The abstract theory works for any differentiable stack, but it becomes very concrete for those presented by proper Lie groupoids - for example, when computing the volume associated with a density, ones recover the explicit formulas that were taken as definition by A. Weinstein.

Geoffrey Scott: Deformation of Dirac structures via L_∞ algebras

Geoffrey Scott (University of Toronto) presented his joint work with Marco Gualtieri and Mykola Matviichuk on the deformation of Dirac structures via L_∞-algebras. The deformation theory of a Dirac structure is controlled by a differential graded Lie algebra (dgLa) which depends on the choice of an auxiliary transverse Dirac structure. Scott shows that different choices of transverse Dirac structure may lead to dgLas which are not isomorphic (as dgLas), but which are isomorphic as L_∞-algebras. One application of these results is to the study of the Kodaira-Spencer deformation complex of a complex manifold.

3.2 Geometric Structures on Groupoids

Thiago Drummond: Lie theory of multiplicative structures on Lie groupoids

Thiago Drummond (U F Rio de Janeiro) presented a survey of results regarding multiplicative structures on Lie groupoids (e.g. tensor fields, differential forms with values in representations, foliations). The Lie theory of such structures establishes an infinitesimal vs. global correspondence with relevant data on the Lie algebroid. The main aim was to show how the simple idea of treating tensor fields as functions on Whitney sums of vector bundles allows us to unify various results on the literature as well as extend some of them to more general contexts (e.g. differential forms with values in representations up to homotopy). The talk described recent joint works with Bursztyn, Cabrera, Ortiz and Jotz, as well as some open problems.

Alejandro Cabrera: A construction of local Lie groupoids using Lie algebroid sprays

It is a classical result that every Lie algebroid integrates to a local Lie groupoid. Alejandro Cabrera (U F Rio de Janeiro) gave a talk describing his joint work with I. Marcut and M. Salazar, on a direct, explicit and self-contained construction of a local Lie groupoid integrating a given Lie algebroid, which only depends on the choice of a connection. On the resulting local Lie groupoid, called a spray groupoid, they obtain formulas for integrating infinitesimal multiplicative objects, producing concrete integrations of several geometrical structures: (Nijenhuis-)Poisson, Dirac, Jacobi structures by local symplectic (Nijenhuis), presymplectic, contact groupoids, respectively. They also give a complete account of the local Lie theory based on these explicit constructions.

Dorette Pronk: Structure of mapping objects in the category of orbifolds

Topological orbifolds can be described as proper étale groupoids (sometime called orbigroupoids). Dorette Pronk's (Dalhousie University) lecture described the maps between these groupoids and 2-cells between them, using a bicategory of fractions of the 2-category of orbigroupoids and continuous functors with respect to a subclass of the Morita equivalences. This bicategory of fractions is equivalent to the usual one and renders mapping groupoids that are small. This leads to a very explicit description of the topological groupoid $\mathrm{Map}(G, H)$, encoding the new generalized maps from G to H and equivalence classes of 2-cell diagrams between them, for any orbigroupoids G and H. When G has a compact orbit space she shows that the mapping groupoid is an orbigroupoid and has the appropriate universal properties to be the mapping object. In particular, sheaves on this groupoid form the mapping topos for geometric morphisms between the toposes of sheaves on G and H.

Ivan Struchiner: Lie groupoids which give rise to G-structures

Ivan Struchiner (University of São Paulo) gave a talk on his joint work with R.L. Fernandes on Lie groupoids and G-structures. The infinitesimal data attached to a (finite type class of) G-structures with connections are its structure equations. Such structure equations give rise to Lie algebroids endowed with extra geometric information, deriving from the fact that they come from G-structures. This Lie algebroid comes equipped with an action of G by inner Lie algebroid automorphisms. Conversely, given such a Lie algebroid (called a G-algebroid), a natural question is that of finding G-structures which correspond via differentiation to the Lie G-algebroid. This integration problem is known as 'Cartan's Realization Problem for G-Structures', and it turns out that if a G-algebroid is integrable by a Lie groupoid endowed with an action of G, then each s-fiber of the groupoid can be identified with the total space of a G-structure with connection solving the realization problem.

3.3 Groupoids and Poisson Geometry

Marius Crainic: Poisson manifolds of compact types

One of the main source of new results and examples in Lie groupoid theory is Poisson geometry, since the global objects integrating Poisson manifolds are symplectic groupoids. Marius Crainic (Utrecht University) gave an overview lecture of an important class of Poisson manifolds, called of compact type. These are the analogues in Poisson Geometry of the compact Lie groups from Lie theory. However, within this Poisson geometric context, one finds various parts of geometry coming in and interacting with each together (symplectic, foliations, integrable systems, gerbes). Crainic described is ongoing collaboration with R.L. Fernandes and D. Martinez on Poisson manifolds of compact types, concentrating on Duistermaat-Heckmann-types formulas.

Anton Alekseev : Poisson geometry and non-commutative differential calculus

Anton Alexeev (University of Geneva) introduced a new connection between Poisson geometry and non-commutative algebras. Poisson brackets of special type on n-tuples of $N \times N$ matrices may be encoded by double brackets in the sense of van den Bergh. Interesting examples include constant and linear (KKS) Poisson brackets. In particular, these brackets admit moment maps for the $GL(N)$-action by simultaneous conjugation of matrices in the n-tuple. Surprisingly, there are instances where the theory of double brackets deviates from the standard wisdoms of Poisson geometry. For instance, KKS brackets turn out to be non-degenerate, and under some assumptions a moment map uniquely determines the double bracket. These observations gave rise to a new proof of the theorem by L. Jeffrey on symplectomorphisms between moduli of flat connections and reduced spaces of products of coadjoint orbits. The talk was mostly based on the work by F. Naef, but it sketch also how these results are related to the Kashiwara-Vergne theory and to the Goldman-Turaev Lie bialgebra.

Marco Zambon: Almost regular Poisson structures and their holonomy groupoids

Marco Zambon's (KU Leuven) lecture at the workshop concerned a new class of Poisson manifolds (called "almost regular") that is well-behaved from the point of view of singular foliations, understood as as submodules of vector fields rather than partitions into leaves. They admit a geometric characterization in terms of the symplectic leaves alone, and contain the well studied class of log-symplectic manifolds. Zambon considers the holonomy groupoid integrating the singular foliation of an almost regular Poisson structure, showing that it is a Poisson groupoid, integrating a naturally associated Lie bialgebroid. The Poisson structure on the holonomy groupoid is regular, and as such it provides a desingularization of the original Poisson manifold.

David Iglesias-Ponte : Dirac geometry and the integration of Poisson homogeneous spaces

David Iglesias-Ponte (University of La Laguna) described his joint work with Bursztyn and J-H. Lu concerning the integration problem for Poisson homogeneous spaces. Partial affirmative results to this problem have been obtained by several different authors. Using methods from Dirac geometry they are finally prove that *any* Poisson homogeneous space admits an integration to a symplectic groupoid.

Joel Villatoro: Poisson manifolds and their associated stacks

Joel Villatoro, a PhD student at the University of Illinois at Urbana-Champaign, gave a talk explaining how one can make precise the relationship between Poisson manifolds and stacks, via their symplectic groupoids. He explained how one can interpret a symplectic groupoid and symplectic Morita equivalence as a model for a singular Dirac manifold. To that end, he introduces a site (a category with a topology) whose objects are Dirac manifolds, $DMan$, and shows how to associate a stack over $DMan$ to any symplectic groupoid. Isomorphisms of such stacks are related with symplectic Morita equivalences.

Francis Bischoff: Morita equivalence and the generalized Kahler potential

Francis Bischoff, a PhD student from the University of Toronto, presented a new approach to Generalized Kähler (GK) geometry, in which a GK structure of symplectic type can be described in terms of a holomorphic symplectic Morita equivalence along with a brane bisection. This new approach, which he found in a collaboration with Marco Gualtieri and Maxim Zabzine, can be applied to the problem of describing a GK structure in terms of holomorphic data and a single real-valued function (the generalized Kähler potential).

3.4 Groupoids and Index Theory

Georges Skandalis: Index theory and Lie groupoids (survey)

The classical Atiyah-Singer index theorem has been greatly extended in the realm of Lie groupoids. Georges Skandalis (Universite Paris 7) gave a survey talk on the index theory and Lie groupoids, more precisely giving an overview of the index theory of elliptic pseudo-differential operators on groupoids. Starting from convolution algebras - including the C^*-algebras- of a Lie groupoid, he discussed their K-theory, the construction of the index, the link with the tangent (or adiabatic) groupoid, leading to the Baum-Connes conjecture.

Xiang Tang: An index theorem for proper cocompact actions of Lie groupoids

Xiang Tang (Washington University, St. Louis) discussed the index theorem for proper cocompact actions of Lie groupoids. Given a proper, cocompact action of a Lie groupoid, Tang, in joint work with M. Pflaum and H. Posthuma, introduces a higher index pairing between invariant elliptic differential operators and smooth groupoid cohomology classes, leading to a cohomological index formula computing this pairing, a new version of the index theorem, and new applications.

Markus Pflaum: Inertia Groupoids and their singularity structure or why we need a concept of stratified groupoid

The inertia space of a compact Lie group action, or more generally of a proper Lie groupoid, has an interesting singularity structure. Unlike the quotient space of the group action, respectively the groupoid, the inertia space cannot be stratified by orbit types, in general. In his talk Markus Pflaum (University of Colorado Boulder) discussed this phenomenon and provided a stratification and local description of the inertia space. This leads naturally to the concept of a stratified groupoid which lies in between the one of a Lie groupoid and the one of a topological groupoid. He also shows that de Rham theorem holds for inertia spaces and explains the connection of the inertia space with the non-commutative geometry of the underlying groupoid.

Claire Debord : Blowup and deformation groupoids constructions related to index theory

Claire Debord (Université Blaise Pascal, Clermont-Ferrand) talk was dedicated to several natural constructions of Lie groupoids coming from deformation and blowups, that arose in recent joint work with G. Skandalis. These constructions enable one to recover many known constructions of Lie groupoids involved in index theory and they lead to new index problems.

Robert Yuncken: A groupoid approach to pseudodifferential operators

In the 1980s, Alain Connes gave a new proof of the Atiyah-Singer index theorem by means of the so-called 'tangent groupoid'. The tangent groupoid is a geometric device for connecting a pseudodifferential operator to its principal symbol, via a deformation family. In his talk, Robert Yuncken (University Clermont Auvergne) reported on his work with Erik van Erp, providing a converse: The integral kernels of pseudodifferential operators are precisely those distributions that fit into a family of distributions on the tangent groupoid, with a suitable homogeneity property with respect to the natural R^+-action. Carrying this idea further, van Erp and Yuncken develop pseudodifferential calculi for more general situations, applicable to more general classes of hypo-elliptic operators.

Erik van Erp: The tangent groupoid and hypoelliptic operators

Continuing the theme described in Yuncken's talk, Erik van Erp discussed the role played by groupoids in work on the index problem for hypoelliptic differential operators on contact manifolds. The groupoid perspective proved to be very fruitful in pointing the way to a solution of this problem. A non-trivial hurdle in this case is to construct the correct groupoid, since the standard construction of "adiabatic groupoids" does not give the desired object. The integration theorem for Lie algebroids of Crainic-Fernandes provides a necessary technical tool for this construction. As a result, van Erp was able to obtain a very general index theorem for contact manifolds.

3.5 Other Applications

Lie groupoids, often equipped with geometric structures, have many other applications. The workshop also included some recent advances in the applications of Lie groupoids in several directions.

Alan Weinstein: Hamiltonian Lie algebroids

Alan Weinstein (University of California, Berkeley) discussed applications to general relativity. The constraint manifold for the initial value problem of general relativity is a coistropic subset in the symplectic manifold P of 1-jets of lorentzian metrics along a space-like hypersurface. In an attempt to find the appropriate symmetry structure, C. Blohmann, M.C. Fernandes and Weinstein constructed a Lie algebroid over a space of *infinite* jets for which the brackets relations among constant sections exactly matched the bracket relations among constraints, but this was not enough to explain the coisotropic nature of the constraint set. Two unanswered questions remain. (1) What are the appropriate notions of "hamiltonian Lie algebroid" over a symplectic (or Poisson) manifold and associated "momentum map" which make the zero sets of momentum maps coisotropic? (2) Is there a hamiltonian Lie algebroid over some "extended phase space" P' closely related to P in which the constraint functions can be understood as the components of a momentum map? Weinstein reported on joint work with C. Blohmann and M. Schiavina, on some progress in solving this problem, which appears to require a version of the BV-BFV construction currently under investigation by Cattaneo, Mnev, Reshetikhin, and Schiavina.

Nguyen Zung : Deformations and Stability of Dufour Foliations

Morse and Morse-Bott singular foliations, are an important classes of codimension 1 foliations which have been studied my many authors,. A natural question is what are higher-dimension analogs of these foliations. Nguyen Zung (University of Toulouse) in his talk propose an answer to this question, in terms of what he calls "Dufour foliations". These use in an essential way Nambu structures and Zung is able to extend classical results about the deformations and structural stability of foliations to Dufour foliations.

Juan Carlos Marrero: The exact discrete Lagrangian function on a Lie groupoid: theory and applications

Another interesting direction where groupoid theory has proved to be useful is geometric mechanics and optimization. Juan Carlos Marrero (University of La Laguna) presented some recent results on the geometric construction of the exact discrete Lagrangian function associated with a continuous regular Lagrangian function. This Lagrangian function is defined on the Lie algebroid of a Lie groupoid. This theory has applications. e.g., in the error analysis of discrete solutions of the Euler-Lagrange equations derived by a variational integrator, or in relation with the Hamilton-Jacobi theory for the Hamiltonian function associated with a regular Lagrangian function.

4 Outcome of the Meeting

The workshop brought together mathematicians working on the structure of Lie groupoids and on geometric structures on Lie groupoids on the one hand, and on the various applications of Lie groupoids mentioned above on the other. It included some of the most active researchers in the field, with special emphasis on promising young researchers. The workshop included participants from Europe, South America and North America, as well as Asia. It provided an excellent opportunity for the exchange of ideas from experts working on different aspects of the theory, and a stimulating environment for discussion and consultation. We are confident that the meeting will be the starting point for a number of new, fruitful collaborations. We received enthusiastic feedback from participants, praising the BIRS conference center in general and this workshop in particular, and request for a sequel. To quote one email received from a participant:

> *I wanted to thank you for inviting me to the Banff conference this past week. I very much enjoyed it. It was the most inspiring meeting I've been to in a long time.*

References

[1] A. Alekseev and E. Meinrenken, Linearization of Poisson Lie group structures, Journal of Symplectic Geometry. 14 (2016) 227–267.

[2] I. Androulidakis and G. Skandalis, The holonomy groupoid of a singular foliation, J. Reine Angew. Math., 626 (2009), 1–37.

[3] M. Artin, A. Grothendieck, and J.L. Verdier. Seminaire de Geometrie Algebrique du Bois-Marie 1963-1964 (SGA 4). Lecture Notes in Mathematics, Vol. 269.

[4] K. Behrend and P. Xu, Differentiable stacks and gerbes, J. Symplectic Geom.,9 Number 3 (2011), 285–341.

[5] P. Bouwknegt, A. Carey, V. Mathai, M. Murray and D. Stevenson, Twisted K-Theory and K-Theory of Bundle Gerbes, Communications in Mathematical Physics 228 (2002), 17–49.

[6] R.L. Bryant, Bochner-Kaehler Metrics, J. of Amer. Math. Soc. 14 (2001), 623–175.

[7] R.L. Bryant, Notes on Exterior Differential Systems, preprint arXiv:1405.3116.

[8] H. Bursztyn and A. Cabrera, Multiplicative forms at the infinitesimal level, Math. Annalen 353 (2012), 663–705.

[9] H. Bursztyn and M. Crainic, Dirac geometry, quasi-Poisson actions and D/G-valued moment maps, J. Differential Geom. 82 (2009), 501–566.

[10] H. Bursztyn, G. Cavalcanti and M Gualtieri, Reduction of Courant algebroids and generalized complex structures, Adv. Math. 211 (2007), 726–765.

[11] H. Bursztyn, M. Crainic, A. Weinstein and C. Zhu, Integration of twisted Dirac brackets, Duke Math. J. 123 (2004), 549–607.

[12] A. Connes, Noncommutative Geometry, Academic Press, San Diego, 1994.

[13] A.S. Cattaneo and G. Felder,Poisson sigma models and symplectic groupoids Quantization of singular symplectic quotients, Progr. Math., vol. 198, Birkhaeuser, Basel, 2001, pp. 61–93.

[14] M. Crainic, Generalized complex structures and Lie brackets, Bulletin of the Brazilian Mathematical Society, 42 (2011), 559–578.

[15] M. Crainic, Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes, Comment. Math. Helv. 78 (2003), no. 4, 681–721.

[16] M. Crainic and R.L. Fernandes, A geometric approach to Conn's linearization theorem. Annals of Math 173 (2011), 1119–1137.

[17] M. Crainic and R.L. Fernandes, Integrability of Poisson brackets, J. Differential Geom. 66 (2004), no. 1, 71–137.

[18] M. Crainic and R.L. Fernandes, Integrability of Lie brackets, Ann. of Math. 157 (2003), 575–620.

[19] M. Crainic and I. Marcut, A normal form theorem around symplectic leaves. Journal of Differential Geometry 92 (2012), 417–461.

[20] M. Crainic, A. Salazar and I. Struchiner, Multiplicative forms and Spencer operators. Mathematische Zeitschrift 279 (2015), 939–979.

[21] M. del Hoyo and R.L. Fernandes, Riemannian Metrics on Lie Groupoids, preprint arXiv:1404.5989, to appear in: Journal fuer die reine und angewandte Mathematik (Crelle).

[22] C. Debord and J.-M. Lescure : Index theory and groupoids. In: Geometric and topological methods for quantum field theory, 86-158, Cambridge Univ. Press, Cambridge, 2010

[23] M. Gualtieri, Generalized complex geometry, Annals of Math. 174 (2011), 75–123.

[24] A. Haefliger, Homotopy and integrability, Lecture Notes in Mathematics 197 (1971), 133–163.

[25] N. Hitchin, Generalized Calabi-Yau manifolds, Quart. J.Math. 54 (2003), 281-308.

[26] J.-M. Lescure, D. Manchon, S. Vassout, About the convolution of distributions on groupoids, preprint arXiv:1502.02002.

[27] D. Li-Bland, E. Meinrenken, Dirac Lie groups, Asian Journal of Mathematics 18 (2014), 779–816

[28] D. Li-Bland, E. Meinrenken, On the Van Est homomorphism for Lie groupoids, L'Enseignement Mathematique (2) 61 (2015), 93–137.

[29] J. Lott, Dimensional reduction and the long-time behavior of Ricci flow, Comment. Math. Helv. 85 (2010), 485–534.

[30] K. Mackenzie, Lie groupoids and Lie algebroids in differential geometry Cambridge Univ. Press, 2005.

[31] K. Mackenzie and P. Xu, Classical lifting processes and multiplicative vector gields, Quart. J. Math. 49 (1998), 59–85.

[32] E. Martinez, Lagrangian Mechanics on Lie Algebroids, Acta Appl. Math. 67 (2001), 295–320.

[33] I. Moerdijk. Orbifolds as groupoids: an introduction. In: Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math. 310 (2002), 205-222.

[34] V. Nistor, A. Weinstein, and P. Xu, Pseudodifferential operators on differential groupoids, Pacific Journal of Mathematics 189 (1999), 117–152.

[35] M. Pflaum, H. Posthuma, X. Tang, The localized longitudinal index theorem for Lie groupoids and the van Est map, Adv. Math. 270 (2015), 223–262.

[36] A. Weinstein, Symplectic groupoids and Poisson manifolds. Bull. Amer. Math. Soc. 16 (1987), no. 1, 101–104.