Diophantine Approximation and Algebraic Curves

Michael Bennett (University of British Columbia),
Aaron Levin (Michigan State University),
Jeff Thunder (Northern Illinois University)

July 2–7, 2017

1 Introduction

The first topic of the workshop, Diophantine approximation, has at its core the study of rational numbers which closely approximate a given real number. This topic has an ancient history, going back at least to the first rational approximations for π. The adjective Diophantine comes from the third century Hellenistic mathematician Diophantus, who wrote an influential text solving various equations in integers and rational numbers (and whose name is now also attached to such "Diophantine equations"). The subject rose to prominence in the last century beginning with the seminal work of Thue, and it has been the source of a large number of deep and influential results, including far-reaching work of Siegel, Roth, Baker, Schmidt, and Faltings, among others. It remains a highly active area at the forefront of number theory and mathematics.

The second topic of the workshop, algebraic curves, has long been implicit and in the background of Diophantine problems. In this direction, the fundamental role of algebraic curves was cemented in the influential 1922 conjecture of Mordell that an algebraic curve of genus at least two possesses only finitely many rational points. The interplay between geometry and arithmetic has increased rapidly since that time, and the use of increasingly advanced tools from algebraic and arithmetic geometry has led to the solution of many outstanding and previously inaccessible problems, including the resolution of Mordell's conjecture (by Faltings).

The workshop centered on the interplay between Diophantine approximation and algebraic curves, with interconnections to a diverse array of topics in algebra, geometry, analysis, and logic, among others.

2 Overview of the Field

A foundational topic in Diophantine approximation is the study of rational numbers which closely approximate a given real number. Naïvely, since the rational numbers are dense in the real numbers, one may approximate a real number arbitrarily well by a rational number. Diophantine approximation studies the closeness of this approximation in terms of the "size" of the rational number (e.g., the denominator). Specifically, if α is a real number, one can study the rational solutions $\frac{p}{q}$ to an inequality of the form

$$\left|\alpha - \frac{p}{q}\right| < \frac{c(\alpha)}{q^{\delta}}, \tag{1}$$

where $c(\alpha)$ and δ are positive real numbers.

If $\delta = 2$, then an elementary result of Dirichlet shows that for any irrational real number α, the set of rational solutions $\frac{p}{q}$ to (1) is infinite when $c(\alpha) = 1$ (or even $c(\alpha) = \frac{1}{\sqrt{5}}$, due to Hurwitz). The theory of continued fractions provides a method for explicitly calculating such good approximations.

In a complementary direction, Liouville (1844) showed that if α is an algebraic number of degree $d \geq 2$, then there exists an explicit positive constant $c(\alpha)$ such that (1) has no solutions for $\delta \geq d$. Liouville's result is already sufficient to show that numbers such as $\sum_{i=1}^{\infty} 10^{-i!}$ are transcendental, giving a first link between Diophantine approximation and transcendence problems. On the other hand, Liouville's inequality is typically too weak to obtain interesting consequences for Diophantine equations.

A breakthrough occurred in 1909 when Thue [53] improved the exponent in Liouville's result to $\delta > \frac{1}{2}d + 1$. As an application, Thue proved the finiteness of integer solutions $x, y \in \mathbb{Z}$ to equations (now called Thue equations) of the form $F(x, y) = a$, where $a \in \mathbb{Z}$ and $F \in \mathbb{Z}[x, y]$ is an irreducible binary form of degree $d \geq 3$.

Siegel [50], in 1921, expanded on Thue's method to prove finiteness for exponents $\delta > 2\sqrt{d}$. The improvement to an exponent δ of order $o(d)$ proved essential when, in 1929, Siegel [52] proved his famous theorem on integral points on curves: if $C \subset \mathbb{A}^n$ is an affine curve defined over a number field k with positive geometric genus or possessing at least 3 (geometric) points at infinity, then the set of points of C with coordinates in the ring of integers of k, $C(\mathcal{O}_k)$, is finite. More generally, in this case finiteness holds for any set of S-integral points $C(\mathcal{O}_{k,S})$, where $\mathcal{O}_{k,S}$ (S finite) is a ring of S-integers of k. Siegel's theorem on integral points provides a basic link between the two topics of the workshop, and more generally, represents an early instance of the deep and fundamental interplay between geometry and arithmetic. In the case of rational affine curves, Siegel's theorem is equivalent to the finiteness of solutions to the so-called S-unit equation

$$au + bv = c, \quad u, v \in \mathcal{O}_{k,S}^*, \tag{2}$$

where $a, b, c \in k^*$ are nonzero constants. This equation had been studied earlier by Siegel (when $\mathcal{O}_{k,S}^* = \mathcal{O}_k^*$) and in the above generality by Mahler. In fact, one can study the S-unit equation (and Siegel's theorem) in far more general contexts (e.g., when u, v lie in a finitely generated subgroup of $\mathbb{C}^*$).

Siegel's result was improved to $\delta > \sqrt{2d}$ by Gelfand and Dyson (1947) (independently), and finally, the optimal result $\delta > 2$ was obtained in 1955 by Roth [43]: If α is an algebraic number, $\epsilon > 0$, and $C > 0$, then there are only finitely many rational numbers $\frac{p}{q} \in \mathbb{Q}$ satisfying

$$\left| \alpha - \frac{p}{q} \right| < \frac{C}{q^{2+\epsilon}}. \tag{3}$$

Beginning with work of Mahler, Roth's theorem was subsequently generalized by Ridout and Lang [33, 42] to an arbitrary fixed number field k (in place of $\mathbb{Q}$) and to allow for finite sets of absolute values (including non-archimedean ones).

In contrast to Liouville's result, Roth's theorem is ineffective. That is, given α an algebraic number, the method of proof does not yield a way, in general, to compute the finitely many solutions to (3). In fact, this defect is already present in all of the results on the inequality (1) beginning with Thue (consequently, Siegel's theorem on integral points is also ineffective). This problem of effectivity remains a major open problem. On the other hand, it is possible to derive an upper bound on the number of solutions to Roth's inequality (3) and related problems. For instance, beginning with work of Evertse [25], many authors have studied the problem of producing bounds for the number of solutions to the S-unit equation (2) that, remarkably, depend only on the cardinality $|S|$. For example, Beukers and Schlickewei [7] have proven the bound $2^{16|S|}$.

In the 1960's, in a series of papers, Alan Baker [3] revolutionized the subject by producing effective lower bounds for linear combinations of logarithms of algebraic numbers. Baker's results found numerous applications, including effective bounds for solutions to the unit equation (2) (from which many other Diophantine equations may be effectively solved). Analogous lower bounds for linear forms in p-adic logarithms were obtained by van der Poorten [40], Yu [59], and others.

In a different direction, Schmidt [48] proved a deep higher-dimensional generalization of Roth's theorem to the setting of hyperplanes in projective space. Schmidt's Subspace Theorem, as it is known, has found numerous applications to Diophantine problems (more precisely, the version typically applied includes subsequent improvements due to Schlickewei [45], analogous to Ridout-Lang's generalization of Roth's theorem).

Among the many applications is the fundamental result of Evertse [24] and van der Poorten and Schlickewei [41] on the n-term unit equation: all but finitely many solutions of the unit equation

$$u_1 + u_2 + \ldots + u_n = 1, \quad u_1, \ldots, u_n \in \mathcal{O}_{k,S}^*, \tag{4}$$

satisfy an equation of the form $\sum_{i \in I} u_i = 0$, where I is a nonempty subset of $\{1, \ldots, n\}$.

In 1983, another leap forward occurred when Faltings [27] proved Mordell's conjecture dating from 1922: for any number field k and any smooth projective curve C of genus at least two, the set of rational points $C(k)$ is finite. Thus, the qualitative behavior of rational points on algebraic curves is completely determined by the crudest geometric invariant of a curve, the genus. Although Faltings' original proof avoided Diophantine approximation, Vojta [55] subsequently found a new and influential proof of the Mordell conjecture based in Diophantine approximation. Building on this work, Faltings [28] proved a conjecture of Lang classifying the subvarieties of an abelian variety which contain a Zariski dense set of rational points, and another conjecture of Lang generalizing Siegel's theorem to affine open subsets of an abelian variety. Even more generally, Vojta [56, 57] extended Faltings' results to subvarieties of semi-abelian varieties.

Generalizing Mordell's conjecture, Bombieri and Lang conjectured that if X is a variety of general type, over a number field k, then the set of rational points $X(k)$ is not Zariski dense in X. In fact, the conjecture can be extended to include integral points and varieties of log general type. In this form, the Bombieri-Lang conjecture unifies all of the aforementioned qualitative results on integral and rational points. In particular, it contains as special cases the results of Siegel and Faltings for curves, and the results of Faltings and Vojta for integral and rational points on subvarieties of semi-abelian varieties.

Returning to Diophantine approximation, and ending the overview, we mention Vojta's conjectures [54], which posit a precise inequality for algebraic points on a variety, at once quantifying the Bombieri-Lang conjecture and generalizing the Diophantine approximation inequalities of Roth and Schmidt. Vojta's conjectures came as the result of the surprising discovery of analogies between Diophantine approximation and Nevanlinna theory, the quantitative theory that grew out of Picard's classical theorem in complex analysis on entire functions omitting two values. This has led to the development of strong ties and analogies between arithmetic and complex geometry, in addition to the more classical analogies between number fields and function fields.

3 Recent Developments and Presentation Highlights

In this section we discuss select recent developments, with a view towards the presentations given at the workshop, and give a description of some of the presentation highlights.

3.1 Schmidt's Subspace Theorem and Its Generalizations

Schmidt's Subspace Theorem remains one of the most powerful Diophantine approximation results available, and constitutes the central tool in a diverse and increasingly large number of applications (e.g., see Bilu's survey article [8]). Generalizing the Subspace Theorem to other contexts began with work of Faltings and Wusthölz [29], and more recently, Corvaja and Zannier [22] and Evertse and Ferretti [26] have proven generalizations to higher-degree hypersurfaces in projective space and to more general projective varieties. Closely related techniques have been used to study integral points on varieties, beginning with a new proof of Siegel's theorem by Corvaja and Zannier [21], and continuing in their work [23] and work of Levin [34], Corvaja, Levin, and Zannier [19], and Autissier [2].

Continuing this line of research, **Paul Vojta** (University of California, Berkeley) spoke on joint work with Min Ru on birational Nevanlinna constants. In earlier work of Ru, a new invariant, the Nevanlinna constant, was introduced to clarify and unify the Diophantine approximation method (and analogues in Nevanlinna theory) developed in the previous work. Vojta spoke on variants of this constant, and corresponding generalizations of the Subspace Theorem, which go further by incorporating the filtrations introduced in Autissier's work, in addition to other novel ideas.

3.2 Unit equations and other related equations

The unit equation (2) (and more generally (4)) is ubiquitous in number theory, and many other famous Diophantine equations (e.g., Thue-Mahler equations, Mordell equations, hyperelliptic equations) can be reduced to it. Such an observation goes back to at least the 1926 work of Siegel [51] (written under the pseudonym X), who used the unit equation to study integral points on affine hyperelliptic curves. The groundbreaking work of Baker led to effective height bounds for solutions to the unit equation (and a host of other Diophantine equations) and the possibility of practical general algorithms. We can (loosely) organize a number of the presentations around these core themes.

3.2.1 Height bounds and algorithmic aspects

The classical approach to producing effective height bounds for solutions to the unit equation (and related Diophantine equations) comes from the theory of linear forms in logarithms (including p-adic and elliptic versions). Since the bounds produced by this method are typically too large to admit a naïve exhaustive search, finding practical algorithms for the resolution of such Diophantine equations has been intensively studied, with key developments going back to Baker and Davenport [4] and the introduction of the LLL-algorithm by de Weger [58].

Building on observations of Frey [30] and the proof of the Taniyama-Shimura conjecture [14], recently "modular" approaches to the unit equation, and related equations, have been developed (over the rational numbers). Work in this direction includes results of Murty and Pasten [37], Bennett and Billerey [5], and **Benjamin Matschke** (University of Bordeaux), in joint work with Rafael von Känel, who spoke on solving S-unit, Mordell, Thue, Thue-Mahler, and generalized Ramanujan-Nagell equations via the Taniyama-Shimura conjecture. There are close relations between all of these classical Diophantine equations, and Matschke focused on the Mordell equation $y^2 = x^3+a$, where $a \in \mathbb{Z}$ is fixed and x and y lie in a ring of S-integers $\mathbb{Z}_S = \mathbb{Z}[1/N_S]$, where S is a given set of rational primes and $N = \prod_{p\in S} p$. Several improvements to the theory and practical resolution of the Mordell equation were given, including improved height bounds for the solutions (using an approach based on the Taniyama-Shimura conjecture, as opposed to linear forms in logarithms) and improvements to the known algorithms for solving the Mordell equation, including novel sieving techniques. Applications to elliptic curve databases were discussed, including the explicit computation of all elliptic curves (up to isomorphism) with good reduction outside a set of of sufficiently small primes.

Beth Malmskog (Villanova University) discussed implementing a solver for S-unit equations in Sage and applications to algebraic curves (joint work with Alejandra Alvarado, Angelos Koutsianas, Christopher Rasmussen, Christelle Vincent, and Mckenzie West). In contrast to Matschke's talk, a key focus of the algorithm was handling S-unit equations over arbitrary number fields. As an application, all 63 $\mathbb{Q}$-isomorphism classes of Picard curves with good reduction away from 3 were computed (by a result of Börner-Bouw-Wewers, all Picard curves over $\mathbb{Q}$ have bad reduction at 3).

Adela Gherga (University of British Columbia) spoke on implementing algorithms to compute elliptic curves over $\mathbb{Q}$. Specifically, she discussed difficulties with implementing the algorithms of de Weger and Tzanakis, particularly with respect to performing p-adic computations. This is an area where many current computer algebra packages are rather lacking.

3.2.2 Closely related inequalities and equations

Jan-Hendrik Evertse (Universiteit Leiden) discussed joint work with Yann Bugeaud and Kálmán Győry on S-parts of values of binary forms and decomposable forms. If $S = \{p_1, \ldots, p_n\}$ is a finite set of primes, $a \in \mathbb{Z}$, $a \neq 0$, and we have a factorization $a = p_1^{i_1} \cdots p_n^{i_n} a'$ with $p_i \nmid a'$, $1 \leq i \leq n$, then the S-part of a is defined to be $[a]_S = p_1^{i_1} \cdots p_n^{i_n}$. For a decomposable form $F \in \mathbb{Z}[X_1, \ldots, X_m]$, let

$$\alpha(F) = \inf\{\alpha \mid \forall S, \exists c > 0 \text{ such that } [F(\mathbf{x})]_S \leq c|F(\mathbf{x})|^\alpha \text{ for all } \mathbf{x} \in \mathbb{Z}^m, \gcd(x_1, \ldots, x_m) = 1, F(\mathbf{x}) \neq 0\}.$$

If F is a binary form ($m = 2$) of degree $n \geq 3$ and nonzero discriminant, then it is proven that $\alpha(F) = \frac{2}{n}$. More generally, they prove results for decomposable forms under certain conditions, and give quantitative results counting the number of solutions to related inequalities.

Yann Bugeaud (University of Strasbourg) discussed the binary representation of smooth numbers. He proved a series of results all with the general theme that a large integer cannot simultaneously have only very small prime divisors and very few nonzero binary digits. The given new results are effective, and rely on estimates for linear forms in complex and p-adic logarithms of algebraic numbers.

Shabnam Akhtari (University of Oregon) discussed joint work with Jeffrey Vaaler on finding height inequalities for units in the ring of S-integers of a number field. More specifically, if $h(\cdot)$ denotes the (absolute) Weil height, then Akhtari discussed the problem of finding estimates for the quantity

$$\min\{h(\beta\gamma) \mid \gamma \in \Gamma\},$$

where Γ is a given subset of a group of S-units $\mathcal{O}_S^*$ of k, and $\beta \neq 0$ is a nonzero S-integer. Aside from intrinsic interest, such height inequalities have applications to Diophantine equations, and in particular, the relevance to solving norm form equations was discussed.

3.2.3 Unit equations and arithmetic dynamics

The application of unit equations to arithmetic dynamics, and in particular as a tool to study periodic and preperiodic points of a rational function, goes back to at least Narkiewicz [38], who used unit equations to study rational periodic points of monic polynomials. In the more general setting of rational functions, recent works of Canci [15], Canci and Paladino [16], and Canci and Vishkautsan [17] have used unit equations as an essential tool to study rational preperiodic points and give partial results towards the Morton-Silverman conjecture in arithmetic dynamics.

Extending this line of results, **Sebastian Troncoso** (Michigan State University) spoke on rational preperiodic points and hypersurfaces in projective space. In the one-dimensional case, if $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is an endomorphism of degree $d \geq 2$ over a number field k, and ϕ has good reduction outside of a set of places S of k (including archimedean places), then Troncoso shows that the number $|\mathrm{PrePer}(\phi, k)|$ of k-rational preperiodic points satisfies

$$|\mathrm{PrePer}(\phi, k)| \leq 5 \cdot 2^{16|S|d^3} + 3.$$

A sharper bound is obtained for strictly preperiodic points. In higher dimensions, quantitative results are obtained for the number of rational preperiodic hypersurfaces satisfying certain conditions. Key ingredients in the proofs are bounds for the number of solutions to S-unit equations and Thue-Mahler equations.

3.3 Runge's method

An old method of Runge [44], dating from 1887, proves the effective finiteness of the set of integral points on certain affine curves. In its most basic form, Runge proved that if $f \in \mathbb{Q}[x, y]$ is an absolutely irreducible polynomial and the leading term of f factors nontrivially over $\mathbb{Q}$ into nonconstant coprime polynomials, then the set of solutions to

$$f(x, y) = 0, \quad x, y \in \mathbb{Z},$$

is finite and can be effectively computed. More generally, Runge's method can be used to prove effective finiteness of S-integral points on an affine curve C when $|S|$ is small compared to the number of rational components of C at infinity. Bombieri [12] proved a uniform version of Runge's theorem, allowing the number field K and set of places S to vary.

More recent developments include a generalization of Runge's method to higher dimensions by Levin [35], a series of papers by Bilu and Parent [9, 10, 11] applying Runge's method to modular curves and solving certain cases of a well-known conjecture of Serre, an application in arithmetic dynamics to integral points in orbits by Corvaja, Sookdeo, Tucker, and Zannier [20], and an application to the Nagell-Ljunggren equation by Bennett and Levin [6].

Further developing these themes, **Samuel Le Fourn** (ENS Lyon) spoke on Runge's method and its application to integral points on Siegel modular varieties. Expanding on Levin's higher-dimensional version of Runge's method, Le Fourn proves a "tubular" version of Runge's method which permits wider applicability to higher-dimensional problems on integral points. As an application, Le Fourn proves an explicit finiteness result for integral points on the Siegel modular variety $A_2(2)$.

3.4 Unlikely intersections

If X and Y are subvarieties in a space of dimension $n > \dim X + \dim Y$, then one usually expects that the intersection $X \cap Y$ will be empty (the intersection is "unlikely"). In fact, if X is fixed and Y runs through an infinite family of suitable subvarieties with $\dim X + \dim Y < n$, then one still expects that X has small intersection with the entire family unless X is of a certain special type related to the family. One may view conjectures of Lang, Manin-Mumford, Zilber, André-Oort, Bombieri-Masser-Zannier, and others in this general setting. The topic has been developing rapidly, with notable contributions including work of Bombieri, Masser, and Zannier [13], Maurin [36], Habegger [31, 32], and Pila [39].

Within this framework, **Laura Capuano** (Oxford University) spoke on joint work with Barroero on unlikely intersections in families of abelian varieties. In particular, the case where X is a curve in a family of abelian varieties was studied, and applications were given to the study of families of polynomial Pell equations.

3.5 Perfectoid spaces and p-adic hyperbolicity

Vojta's dictionary [54] gives a precise way to "translate" between statements in Diophantine approximation and statements in (complex) Nevanlinna theory. This analogy between the two subjects has proved very fruitful and influential, and has led to substantial progress in both subjects. Since at least the 1980's, a theory of p-adic (or non-Archimedean) Nevanlinna theory has been developed, and in particular, various notions of p-adic hyperbolicity were introduced by Cherry [18]. Similar to the classical case, a dictionary between p-adic Nevanlinna theory and Diophantine approximation over the rational numbers was studied by An, Levin, and Wang in [1].

The recent theory of perfectoid spaces, due to Scholze [49], has been quickly recognized as a major advance, and has already found many deep applications in diverse areas. In this context, **Ariyan Javanpeykar** (University of Mainz) discussed the notion of p-adic hyperbolicity and its relation to other common notions of hyperbolicity. Using Scholze's theory of perfectoid spaces, he presented a proof of a strong p-adic hyperbolicity statement for moduli spaces of polarized abelian varieties.

3.6 Transcendence Theory

Michael Coons (University of Newcastle) spoke on joint work with Yohei Tachiya on problems in the transcendence theory of functions of one complex variable. The talk centered on proving generalizations and extensions of a result of Duffin and Schaeffer. A primary result yielded the transcendence, over the field of meromorphic functions, of complex functions with a finite (positive) bounded radius of convergence satisfying certain radial unboundedness conditions. As an application, one obtains that the field $\mathbb{C}(z)(e^z, \sum_{n\geq 0} z^{2^n})$ has transcendence degree two over $\mathbb{C}(z)$, answering a function-theoretic analogue of a question of Nishioka on the algebraic independence of e and the number $\sum_{n\geq 0} 2^{-2^n}$.

Noriko Hirata-Kohno (Nihon University) discussed the independence of values of the polylogarithm function and a generalized polylogarithm function with periodic coefficients. The method of proof relies on the construction of new Padé approximations built on earlier (unpublished) work of Nishimoto.

3.7 Further Presentation Highlights

Richard Guy (University of Calgary) spoke on some relations between number theory and the Euclidean geometry of triangles. He discussed some classical geometric configurations of lines and circles associated to triangles, including configurations arising from the Simson line theorem, and raised some open problems about the structure of a related graph.

Lajos Hajdu (University of Debrecen) discussed the problem of finding well approximating lattices for a finite set of points (joint work with András Hajdu and Robert Tijdeman). After giving some motivation for the problem coming from medical image processing, Hajdu defined an appropriate measure for the approximation of a lattice to a finite set of points. Under this notion, results on finding well-approximating lattices were given in all dimensions, with more complete results in the one-dimensional case. The techniques used include the LLL algorithm and techniques from simultaneous Diophantine approximation.

Fabien Pazuki (University of Copenhagen) spoke on inequalities for the j-invariants of isogenous elliptic curves. Specifically, Pazuki showed that if h denotes the absolute logarithmic Weil height, E_1 and E_2 are elliptic curves over $\overline{\mathbb{Q}}$ with j-invariants j_1 and j_2, respectively, and $\phi : E_1 \to E_2$ is an isogeny, then the following two inequalities hold:

$$|h(j_1) - h(j_2)| \leq 9.204 + 12 \log \deg \phi,$$
$$h(j_1) - h(j_2) \leq 10.68 + 6 \log \deg \phi + 6 \log(1 + h(j_1)).$$

Some applications of the inequalities were discussed, including the problem of bounding the coefficients of modular polynomials.

Cameron Stewart (University of Waterloo) spoke on joint work with Stanley Xiao on the representation of integers by binary forms. Let F be a binary form with integer coefficients, non-zero discriminant, and degree $d \geq 2$. Let $R_F(Z)$ denote the number of integers of absolute value at most Z which are represented by F. Computing $R_F(Z)$ in the case of binary quadratic forms ($d = 2$) is classical, and goes back to work of Fermat, Lagrange, Legendre, and Gauss. The cubic case ($d = 3$) was studied by Hooley, and many authors have studied various higher degree classes of binary forms. For $d \geq 3$, Stewart and Xiao are able to give a precise asymptotic result, showing that there is a positive number C_F such that $R_F(Z)$ is asymptotic to $C_F Z^{2/d}$. The techniques used include Heath-Brown's p-adic determinant method, as refined by Salberger.

Amos Turchet (University of Washington) discussed joint work with Kenneth Ascher on the uniformity of integral points on curves and surfaces. A well-known result of Caporaso-Mazur-Harris shows that, assuming the Bombieri-Lang-Vojta conjecture, there should exist a uniform bound $B(k, g)$ for the number of k-rational points on a smooth projective curve C, where $B(k, g)$ depends only on the number field k and the genus g of C. Higher-dimensional uniformity results on rational points in the same vein were proven by Hassett, Abramovich, and Abramovich-Voloch. Turchet discussed partial results towards a program for proving an integral points analogue of Caporaso-Mazur-Harris's result, extending results of Abramovich for integral points on elliptic curves. Geometrically, this involves extending the previous uniformity results from varieties of general type to varieties of log general type.

4 Outcome of the Meeting

References

[1] T. T. H. An, A. Levin, and J. Wang, *A p-adic Nevanlinna-Diophantine correspondence*, Acta Arith. **146** (2011), no. 4, 379–397.

[2] P. Autissier, *Sur la non-densité des points entiers*, Duke Math. J. **158** (2011), no. 1, 13–27.

[3] A. Baker, *Linear forms in the logarithms of algebraic numbers. I, II, III*, Mathematika 13 (1966), 204-216; ibid. 14 (1967), 102-107; ibid. **14** (1967), 220–228.

[4] A. Baker and H. Davenport, *The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$*, Quart. J. Math. Oxford Ser. (2) **20** (1969), 129–137.

[5] M. A. Bennett and N. Billerey, *Sums of two S-units via Frey-Hellegouarch curves*, Math. Comp. **86** (2017), no. 305, 1375–1401.

[6] M. Bennett and A. Levin, *The Nagell-Ljunggren equation via Runge's method*, Monatsh. Math. **177** (2015), no. 1, 15–31.

[7] F. Beukers and H. P. Schlickewei, *The equation $x + y = 1$ in finitely generated groups*, Acta Arith. **78** (1996), no. 2, 189–199.

[8] Y. Bilu, *The many faces of the subspace theorem [after Adamczewski, Bugeaud, Corvaja, Zannier…]*, Astérisque (2008), no. 317, Exp. No. 967, vii, 1–38, Séminaire Bourbaki. Vol. 2006/2007.

[9] Y. Bilu and P. Parent, *Integral j-invariants and Cartan structures for elliptic curves*, C. R. Math. Acad. Sci. Paris **346** (2008), no. 11-12, 599–602.

[10] Y. Bilu and P. Parent, *Runge's method and modular curves*, Int. Math. Res. Not. IMRN (2011), no. 9, 1997–2027.

[11] Y. Bilu and P. Parent, *Serre's uniformity problem in the split Cartan case*, Ann. of Math. (2) **173** (2011), no. 1, 569–584.

[12] E. Bombieri, *On Weil's "théorème de décomposition"*, Amer. J. Math. **105** (1983), no. 2, 295–308.

[13] E. Bombieri, D. Masser, and U. Zannier, *Intersecting a curve with algebraic subgroups of multiplicative groups*, Internat. Math. Res. Notices (1999), no. 20, 1119–1140.

[14] C. Breuil, B. Conrad, F. Diamond, and R. Taylor, *On the modularity of elliptic curves over* $\mathbf{Q}$*: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), no. 4, 843–939.

[15] J. Canci, *Finite orbits for rational functions*, Indag. Math. (N.S.) **18** (2007), no. 2, 203–214.

[16] J. Canci and L. Paladino, *Preperiodic points for rational functions defined over a global field in terms of good reduction*, Proc. Amer. Math. Soc. **144** (2016), no. 12, 5141–5158.

[17] J. Canci and S. Vishkautsan, *Scarcity of cycles for rational functions over a number field*, Trans. Amer. Math. Soc. Ser. B, to appear.

[18] W. Cherry, *Hyperbolic* p*-adic analytic spaces*, Ann Arbor, MI, 1993, Thesis (Ph.D.)–Yale University.

[19] P. Corvaja, A. Levin, and U. Zannier, *Integral points on threefolds and other varieties*, Tohoku Math. J. (2) **61** (2009), no. 4, 589–601.

[20] P. Corvaja, V. Sookdeo, T. J. Tucker, and U. Zannier, *Integral points in two-parameter orbits*, J. Reine Angew. Math. **706** (2015), 19–33.

[21] P. Corvaja and U. Zannier, *A subspace theorem approach to integral points on curves*, C. R. Math. Acad. Sci. Paris **334** (2002), no. 4, 267–271.

[22] P. Corvaja and U. Zannier, *On a general Thue's equation*, Amer. J. Math. **126** (2004), no. 5, 1033–1055.

[23] P. Corvaja and U. Zannier, *On integral points on surfaces*, Ann. of Math. **160** (2004), no. 2, 705–726.

[24] J.-H. Evertse, *On sums of* S*-units and linear recurrences*, Compositio Math. **53** (1984), no. 2, 225–244.

[25] J.-H. Evertse, *On equations in* S*-units and the Thue-Mahler equation*, Invent. Math. **75** (1984), no. 3, 561–584.

[26] J.-H. Evertse and R. G. Ferretti, *A generalization of the Subspace Theorem with polynomials of higher degree*, Diophantine approximation, Dev. Math., vol. 16, SpringerWienNewYork, Vienna, 2008, pp. 175–198.

[27] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366.

[28] G. Faltings, *The general case of S. Lang's conjecture*, Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), Perspect. Math., vol. 15, Academic Press, San Diego, CA, 1994, pp. 175–182.

[29] G. Faltings and G. Wüstholz, *Diophantine approximations on projective spaces*, Invent. Math. **116** (1994), no. 1-3, 109–138.

[30] G. Frey, *Links between solutions of* $A-B=C$ *and elliptic curves*, Number theory (Ulm, 1987), Lecture Notes in Math., vol. 1380, Springer, New York, 1989, pp. 31–62.

[31] P. Habegger, *Intersecting subvarieties of abelian varieties with algebraic subgroups of complementary dimension*, Invent. Math. **176** (2009), no. 2, 405–447.

[32] P. Habegger, *On the bounded height conjecture*, Int. Math. Res. Not. IMRN (2009), no. 5, 860–886.

[33] S. Lang, *Fundamentals of Diophantine geometry*, Springer-Verlag, New York, 1983.

[34] A. Levin, *Generalizations of Siegel's and Picard's theorems*, Ann. of Math. (2) **170** (2009), no. 2, 609–655.

[35] A. Levin, *Variations on a theme of Runge: effective determination of integral points on certain varieties*, J. Théor. Nombres Bordeaux **20** (2008), no. 2, 385–417.

[36] G. Maurin, *Courbes algébriques et équations multiplicatives*, Math. Ann. **341** (2008), no. 4, 789–824.

[37] M. R. Murty and H. Pasten, *Modular forms and effective Diophantine approximation*, J. Number Theory **133** (2013), no. 11, 3739–3754.

[38] W. Narkiewicz, *Polynomial cycles in algebraic number fields*, Colloq. Math. **58** (1989), no. 1, 151–155.

[39] J. Pila, *O-minimality and the André-Oort conjecture for* $\mathbb{C}^n$, Ann. of Math. (2) **173** (2011), no. 3, 1779–1840.

[40] A. J. van der Poorten, *Linear forms in logarithms in the* p*-adic case*, (1977), 29–57.

[41] A. J. van der Poorten and H. P. Schlickewei, *Additive relations in fields*, J. Austral. Math. Soc. Ser. A **51** (1991), no. 1, 154–170.

[42] D. Ridout, *The* p*-adic generalization of the Thue-Siegel-Roth theorem*, Mathematika **5** (1958), 40–48.

[43] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20; corrigendum, 168.

[44] C. Runge, *Über ganzzahlige Lösungen von Gleichungen zwischen zwei Veränderlichen*, J. Reine Angew. Math. **100** (1887), 425–435.

[45] H. P. Schlickewei, *The* $\mathfrak{p}$*-adic Thue-Siegel-Roth-Schmidt theorem*, Arch. Math. (Basel) **29** (1977), no. 3, 267–270.

[46] W. M. Schmidt,, *Approximation of algebraic numbers*, Diophantine approximation (Cetraro, 2000), Lecture Notes in Math., vol. 1819, Springer, Berlin, 2003, pp. 107–170.

[47] W. M. Schmidt, *Simultaneous approximation to algebraic numbers by rationals*, Acta Math. **125** (1970), 189–201.

[48] W. M. Schmidt, *Norm form equations*, Ann. of Math. (2) **96** (1972), 526–551.

[49] P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313.

[50] C. L. Siegel, *Approximation algebraischer Zahlen*, Math. Z. **10** (1921), no. 3-4, 173–213.

[51] C. L. Siegel, *The Integer Solutions of the Equation* $y^2 = ax^n + bx^{n-1} + \cdots + k$, J. London Math. Soc. **1** (1926), 66–68.

[52] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abhandlungen der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse 1929, Nr. 1.

[53] A. Thue, *über Annäherungswerte algebraischer Zahlen*, J. Reine Angew. Math. **135** (1909), 284–305.

[54] P. Vojta, *Diophantine approximations and value distribution theory*, Lecture Notes in Mathematics, vol. 1239, Springer-Verlag, Berlin, 1987.

[55] P. Vojta, *Siegel's theorem in the compact case*, Ann. of Math. (2) **133** (1991), no. 3, 509–548.

[56] P. Vojta, *Integral points on subvarieties of semiabelian varieties. I*, Invent. Math. **126** (1996), no. 1, 133–181.

[57] P. Vojta, *Integral points on subvarieties of semiabelian varieties. II*, Amer. J. Math. **121** (1999), no. 2, 283–313.

[58] B. M. M. de Weger, *Algorithms for Diophantine equations*, CWI Tract, vol. 65, Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[59] K. Yu, *Linear forms in p-adic logarithms. III*, Compositio Math. **91** (1994), no. 3, 241–276.