Stability Indices for Nonlinear Waves and Patterns in Many Space Dimensions

Margaret Beck (Boston University)
Graham Cox (Memorial University)
Christopher Jones (UNC Chapel Hill)
Yuri Latushkin (Missouri)

June 17–24, 2018

1 Overview of the Field

Assessing the stability of distinguished states of a nonlinear partial differential equation is a key step in understanding the behavior of the physical system modeled by the equation. Stability here means the robustness of the dynamics to perturbations in initial conditions from a particular state. The distinguished state may be a nonlinear wave, pattern or coherent structure arising in applications such as optics, fluids, chemical reactions, neuroscience, and ecology. Its stability indicates its physical realizability, while any instability suggests more complex dynamics, and understanding the nature of such instabilities can be used as a jumping off point for understanding the organization of the nonlinear dynamics away from the unstable state.

For instance, consider a system of reaction-diffusion equations

$$u_t = \Delta u + F(u)$$

on a bounded domain $\Omega \subset \mathbb{R}^n$, where $u(x,t) \in \mathbb{R}^N$ and $F\colon \mathbb{R}^N \to \mathbb{R}^N$. It is well known that the stability of a stationary solution $\hat{u}$ is determined by the spectrum of the linearization

$$Lv = \Delta v + \nabla F(\hat{u})v.$$

The case of a scalar ODE, where $n = N = 1$, is well understood. In this case Sturm–Liouville theory relates the stability of $\hat{u}$ to its geometric properties (e.g. number of critical points). The Maslov index generalizes this theory to systems of ODEs in the special case that the operator L is selfadjoint; see [3] for a recent application.

The Maslov index has also been used to obtain stability criteria for multi-dimensional problems, but in general is very difficult to compute. Other methods that exist in multiple dimensions, such as the Evans function on channel domains, and variational methods for Hamiltonian systems, apply only in special cases. For the above reasons, our workshop emphasized the following generalizations of Sturm's oscillation theory:

1. non-symmetric systems of ODEs—we focused on extending the definition of the Maslov index to non-Hamiltonian systems, which allows us to study non-selfadjoint eigenvalue problems $Lv = \lambda v$.

2. PDEs (selfadjoint or otherwise)—here we focused on developing new tools for multi-dimensional stability, as well as finding new tools for efficiently computing the already developed Maslov index.

2 Recent Developments and Open Problems

The following sections summarize the major topics discussed during our program, including recent developments, progress made during the course of the workshop, and topics of future research.

2.1 The fat Lagrangian Grassmannian and the generalized Maslov index

Let (V, ω) be a symplectic vector space of dimension $2n$. The Lagrangian Grassmannian is the submanifold of the Grassmannian of n-planes $\Lambda \subseteq Gr_n(V)$ consisting of n-planes that are isotropic with respect to ω. It is a homogeneous space diffeomorphic to $U(n)/O(n)$ and has fundamental group $\pi_1(\Lambda) \cong \mathbb{Z}$. Therefore continuous loops $f\colon S^1 \to \Lambda$ are classified up to homotopy by an integer invariant, called the Maslov class.

Fix a $P \in \Lambda$ and define $Z := \{W \in \Lambda \mid W \cap P \neq \{0\}\}$ to be the set of Lagrangian n-planes which intersect P non-trivially. We call Z the "train" or the "Maslov cycle." It represents a homology class which is Poincaré dual to a generator of $H^1(\Lambda, \mathbb{Z}) \cong \mathbb{Z}$. Consequently, if a loop $f\colon S^1 \to \Lambda$ intersects Z in a sufficiently generic way, the Maslov index of f is equal to the geometric intersection number of f with Z (see [1, 12]).

An open problem is to construct a "Fat Lagrangian Grassmannian"; that is, a subspace $F \subseteq Gr_n(V)$ strictly larger than Λ which satisfies

i) $H^1(F; \mathbb{Z}) \cong \mathbb{Z}$ so that one can assign a Maslov index to loops in F, which

ii) equals with the geometric intersection number with the train $Z_F := \{W \in F | W \cap P \neq \{0\}\}$.

In the case $n = 2$, such an F has recently been constructed; this was presented by Tom Baird at the workshop. Graham Cox presented some work he had done relating the generalized Maslov index to the Turing instability phenomenon.

During discussions, it was observed that F in $Gr_2(V)$ is contains the complement of the Lagrangian Grassmanian of a different symplectic structure. This works as follows: Identify $V = \mathbb{H}$ with a one dimension quaternionic vector space. In particular, $\mathbb{H}$ admits three orthogonal complex structures I, J, K satisfying the quaternionic relations. Each of these determines a symplectic form $\omega_I, \omega_J, \omega_K$ in the standard way. Suppose that $P \subset V$ is Lagrangian with respect to both ω_I and ω_J. Then the fat Lagrangian Grassmannian F_I corresponding to I contains the complement of the Lagrangian Grassmannian Λ_J determined by J. This observation implies that Hamiltonian flow with respect to ω_J will preserve a dense open subset of F_I. Furthermore, given appropriate initial conditions, it should be possible to violate the Hamiltonian property to some degree while still remaining inside of F_I. This hopefully will allow Maslov index methods to be applied in non-Hamiltonian situations.

2.2 Sturm oscillation theory for non-local operators

Non-local operators naturally appear in fluid mechanics, mathematical physics, mathematical finance, and in probability theory (see, for example, [2, 4, 5, 9]). The spectral properties of such operators play an important role in the study of stability of solitary waves for several nonlinear dispersive models, e.g., generalized Benjamin–Ono equation, Benjamin–Bona–Mahony equation, and the fractional non-linear Schrödinger equation. One of the main open problems in the spectral theory of the associated linearized operators is finding sharp estimates on the number of nodal domains of the eigenfunctions.

Several recent results (cf., e.g., [2, 5, 9]) rely on an elegant reformulation of the problem in terms of the Dirichlet-to-Neumann maps for higher-dimensional local operators, the Courant nodal domain Theorem, and the combinatorics of noncrossing partitions. The obtained bounds do not match their counterparts from the standard (i.e., local) Sturm oscillation theory and have not been shown to be sharp.

Since most of the classical tools (such as Sturm comparison results, shooting argument) do not seem to be applicable in the study of nodal domains for non-local operators, it is natural to approach this topic using the recently discovered symplectic properties of the operators in question. We therefore propose to investigate the "Maslov-index" program for the fractional Laplacian.

Problem 1. Reformulate the eigenvalue problem for the non-local operators in terms of the intersection of certain Lagrangian planes (i.e., generalize [10, Theorem 3.2] to the case of fractional Laplacian).

Problem 2. Establish a relation between the eigenvalue counting function and the Maslov index of the Lagrangian planes (i.e., extend [10, Theorem 3.3] to the case of fractional Laplacian).

Problem 3. Investigate monotonicity of the crossings/conjugate points for a fixed spectral parameter.

Problem 4. Estimate the Maslov index to obtain a bound on the eigenvalue counting function in terms the number of zeros of the eigenfunctions.

2.3 The Souriau map and the Evans function

One of the main achievements of the workshop was realization of the fact that Maslov index computations are closely related to a construction of the infinite dimensional Evans function. The Evans function is a Wronskian-type determinant which is an analytic function of the spectral parameter λ. It is equal to zero at the points of the spectrum of the differential operator under consideration. The Evans function is usually constructed by means of certain subspaces parametrized by the spectral parameter. The intersection of these subspaces is nontrivial if and only if the respective value of the spectral parameter λ is a zero of the Evans function. To detect if these subspaces indeed have a nonzero intersection we propose to employ the widely used in the Maslov index theory Souriau map $W(\lambda)$ (cf. [7, 8]). This is a unitary operator whose spectral flow through -1 is a count, including both multiplicity and direction, of the number of times these subspaces intersect. We obtained some preliminary results indicating that -1 belongs to the spectrum of $W(\lambda)$ if and only if certain perturbation determinant, $E(\lambda)$, constructed by means of $W(\lambda)$, is equal to zero. Thus, $E(\lambda)$ may serve as an infinite dimensional Evans function. We stress that one of many possible definitions of the Maslov index is given in terms of the spectral flow of eigenvalues of the Souriau map through -1.

2.4 Geometric phase

An alternative approach to counting the eigenvalues inside a simple closed curve is given by a geometric phase defined in the Hopf Bundle and its generalizations. The idea is that a stability index can then be determined by taking a curve that would be known a priori to contain any potential unstable eigenvalues. The theory was developed in a paper by Grudzien, Bridges and Jones [6] in the context of traveling waves. There is an alternative formulation of phase in terms of Stiefel bundles and one topic discussed was whether this phase contains any extra information. The structure group of the bundle will be $U(n)$ in this case and not simply $\mathbb{S}^1$. There is therefore the potential for extra information. The geometric phase also offers the possibility of an alternative index for stability in problems on multi-dimensional domains. This is the main focus of our effort for generalizing the geometric phase. It is analogous to defining the Evans Function for a multi-dimensional domain, but offers the possibility of circumventing some of the technical issues in defining the full Evans Function. The formulation of a geometric phase will then be based on infinite-dimensional bundles. We considered the work of Quillen [11] and others as a basis for this formulation. While we did not resolve the problem and, indeed it looks challenging, we were able to lay out a clear plan for moving forward.

2.5 The Spatial Evolutionary System (SES)

Another major topic of discussion was the spatial evolutionary system—a reformulation of a semilinear elliptic PDE as a first-order infinite-dimensional dynamical system. In its simplest form it says that the PDE $\Delta u = F(u)$ on $\mathbb{R}^n$ is equivalent to the system

$$\frac{df}{dt} = g, \quad \frac{dg}{dt} = F(f) - \frac{1}{t^2}\Delta_{\mathbb{S}^{n-1}} f - \frac{n-1}{t} g,$$

where f and g are functions on $\mathbb{S}^{n-1}$, parameterized by t. More generally, it describes the evolution of the Cauchy data

$$f(t) = u\big|_{\partial\Omega_t}, \quad g(t) = \frac{\partial u}{\partial n}\bigg|_{\partial\Omega_t}$$

on a one-parameter family of hypersurfaces, where $\{\Omega_t\}$ is a family of domains that shrinks to a point as $t \to 0$.

So far we have been able to prove the equivalence of the SES to the original PDE in a wide variety of settings. Moreover, we have shown that the linearized equation admits an exponential dichotomy. This means at each time t_0, the space of data $(f(t_0), g(t_0))$ can be split into two complementary subspaces: one consisting of solutions that grow exponentially as $t \to -\infty$, and one of solutions that decay exponentially in the same limit. The latter of these two spaces, the so called *unstable subspace*, arises in the construction of the multi-dimensional Maslov index and Evans function. Thus the SES gives a new dynamical interpretation of this object, which we believe will be useful in linear stability computations.

3 Outcome of the Meeting

At the meeting we made significant progress on ongoing projects (the SES and the generalized Maslov index) and new projects (the multi-dimensional Evans function), and started discussing new stability tools as well as new applications of our existing machinery (the geometric phase, oscillation theory for non-local operators). Overall the meeting was very productive, and led to many new ideas with the potential for significant impact on the study of dynamical stability in multiple spatial dimensions.

References

[1] V.I. Arnol'd, Characteristic class entering in quantization conditions, *Functional Analysis and its applications* **1.1** (1967), 1-13.

[2] R. Banuelos, T. Kulczycki, The Cauchy process and the Steklov problem, *J. Func. Anal.* **211** (2004), 355–423.

[3] M. Beck, G. Cox, C. Jones, Y. Latushkin, K. McQuighan and A. Sukhtayev, Instability of pulses in symmetric reaction–diffusion systems: a symplectic approach, *Philos. Trans. Roy. Soc. A* **376** (2018).

[4] R. Frank, L. Geisinger, Refined semiclassical asymptotics for fractional powers of the Laplace operator, *J. Reine Angew. Math* **712** (2016), 1–37.

[5] R. Frank, E. Lenzmann, Uniqueness of non-linear ground states for fractional Laplacian in $\mathbb{R}$, *Acta Math.* **210** (2013), 261–318.

[6] C. J. Grudzien, T. J. Bridges, C.K.R.T. Jones, Geometric phase in the Hopf bundle and the stability of non-linear waves, *Phys. D* **334** (2016), 4–18.

[7] P. Howard, Y. Latushkin, and A. Sukhtayev, The Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$, *J. Math. Anal. Appl.* **451** (2017), 794–821.

[8] P. Howard and A. Sukhtayev, The Maslov and Morse indices for Schrödinger operators on $[0, 1]$, *J. Differential Equations* **260** (2016), 4499–4559.

[9] V. Hur, M. Johnson, J. Martin, Oscillation Estimates of Eigenfunctions Via the Combinatorics of Noncrossing Partitions, *Discrete Analysis* **13** (2017).

[10] Y. Latushkin, S. Sukhtaiev, The Maslov index and the spectra of second order differential operators, *Adv. Math.* **329** (2018), 422–486.

[11] D. Quillen, Determinants of Cauchy-Riemann operators over a Riemann surface, *Funk. Anal. i ego Prilozhenya* **19** (1985), 37–41.

[12] J. Robbin and D. Salamon, The Maslov index for paths, *Topology* **32.4** (1993), 827–844.