Model Theory and Operator Algebras

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1 Overview of the Field

Operator algebra has developed into a field of its own since the time of von Neumann. C^*-algebras and von Neumann algebras can be represented as algebras of bounded operators on a Hilbert space and although the relevant topology is different in the study of these algebras (the operator norm for C^*-algebras and the weak-$*$ topology for von Neumann algebras), both classes have interesting ultraproduct constructions. The class of C^*-algebras is closed under the usual norm ultraproduct while the class of II_1 factors is closed under the tracial ultraproduct. Although it took 40 years to notice, the existence of these ultraproduct constructions highlights that model theory has a role to play in the subject. There are many interesting current directions to pursue but we concentrated on three general themes:

- Interaction with the Elliott classification programme
- Relationship with free probability
- Model theoretic considerations

There are several things that model theory brings to the table in this endeavour. First of all, the *theory* of an algebra is an invariant which is complementary to many of the operator algebraic invariants on offer. The utility and consequences of recognizing when two algebras do not have the same theory will be highlighted below. Second, model theory provides methods of constructing examples which are different from those in operator algebra. The primary example is model theoretic forcing which plays a prominent role in [8]. Although to date the examples constructed have been modest, refocusing attention on the construction of specific examples with this technique in mind could pay dividends. Third, the notion of an elementary class in continuous logic is a common generalization of both the class of C^*-algebras and II_1 factors. Model theory can be used for both clarifying concepts and identifying good questions to ask. The key is to identify at an abstract level what role the language is playing and what model theoretic properties are present. Two examples of this appear in [9] where model theoretic properties of strongly self-absorbing algebras are discussed and [10] where isomorphism classes of ultrapowers are considered.

The Elliott classification programme

Elliott had conjectured that the category of separable, simple, nuclear C^*-algebras is equivalent to the category of certain K-theoretic invariants. The original conjecture had generated an impressive body of work, some of which (notably [25] and [28]), necessitated its reformulation. After a succession of spectacular technical breakthroughs, (notably [20], [19], [26], [27], [4], and [6]), a restricted version of Elliott's conjecture has been confirmed. The regularity properties that distinguish 'classifiable' and 'non-classifiable' C^*-algebras are only to a limited extent detectable by K-theoretic invariants or by any well-behaved functorial invariants. Interestingly all known counter-examples to classification by K-theoretic invariants are distinguished by their continuous first order theories. $\mathcal{Z}$-stablity (perhaps the most prominent regularity property of nuclear C^*-algebras) is an elementary property ([8]), and the existence of a nuclear C^*-algebra without the UCT (perhaps the most mysterious regularity property of nuclear C^*-algebras) is equivalent to the existence of a particular first-order theory of C*-algebras omitting certain types. One of the highlights of [8] is the insight that many regularity properties related to the classification programme are equivalent to a particularly nice form of omitting types in continuous logic. The properties include nuclearity, nuclear dimension n, decomposition rank n and many more; conspicuous by its absence is exactness. This raised the intriguing possibility that model theory could play a role in the resolution of the Toms-Winter conjecture and a complete resolution of the quasi-diagonality conjecture (although substantial progress on this has already been made; see [27]). More specifically, the technique of model theoretic forcing could be used to construct examples with given regularity properties and provide counter-examples to certain conjectures. In the positive direction, in [17] it was proved that weaker versions of the UCT problem and the quasi-diagonality problem imply that all nuclear, stably finite, C*-algebras are quasidiagonal; this purely C*-algebraic result was proved using model theoretic forcing.

Free probability and model theory

Free probability theory provides a probabilistic framework for understanding free groups factors. Some of its most powerful tools come from the connection to random matrix theory, which can be naturally placed into the context of ultraproducts of matrix algebras and the quantitative study of matricial approximations. Among interesting free probability approaches is one called linearization, where a matrix trick replaces non-commutative rational functions of n-tuples of variables by corners of matrices whose entries are linear in the variables. It would be instructive to explore this in connection with model-theoretic ideas.

For a free group F_n on n generators, $L(F_n)$ is the corresponding group von Neumann algebra. The celebrated free group factor problem asks if all non-abelian free group factors on finitely many generators are isomorphic. It is known that they are all either isomorphic or all not isomorphic. A model theoretic variant of this problem is to ask if all of the $L(F_n)$'s for $n > 1$ have the same theory as II$_1$ factors. Of course, if they don't then one has a very strong answer to the free group factor problem. On the other hand, if they do have the same theory then this provides a partial explanation for why the free group factor problem is difficult. There are no known examples of explicitly given, elementarily equivalent, C*-algebras or II$_1$ factors which are not isomorphic.

A related problem asks whether $L(F_n)$ is pseudo-finite for $n > 1$. That is, is $L(F_n)$ elementarily equivalent to an ultraproduct of matrix algebras viewed as von Neumann algebras. All of the free group factors and ultraproducts of matrix algebras have the same universal theory (they are R^ω-embeddable). Property Γ is an elementary property ([11]) and all of these examples fail to have Γ so they agree on at least one highly interesting sentence with two quantifiers. Little else is known. It is also not known if the theories of ultraproducts of matrix algebras are all the same. It is known that there is either 1 such theory of ultraproducts or continuum many. Understanding the answer to this question requires an understanding of the asymptotic behaviour of formulas in large matrix algebras. A potential strategy for resolving the model theoretic version of the free group factor problem would be to show that every free group factor is pseudo-finite and that all ultraproducts of matrix algebras have the same theory.

Model theoretic considerations

In [10], the model theory of operator algebras resolved the McDuff problem regarding the isomorphism types of ultrapowers of separable II$_1$ factors and their relative commutants. The question was (slightly rephrased): given a separable II$_1$ factor M, are all ultrapowers of M by non-principal ultrafilters on $\mathbb{N}$ necessarily isomorphic and also are the relative commutants isomorphic. The answer is no but the reason is very interesting. One shows that the relevant concept is the model theoretic notion of stability and when one transports this notion to the class of C*-algebras and asks the corresponding question, one gets exactly the same answer although of course for different local reasons. Since [10], the model theory of operator algebras has matured and it became a subject in its own right.

Much work in classification has gone into analyzing Connes' work [5] on injective factors and transferring the techniques into the study of C*-algebras. With the long term goal of feeding into this programme, a model theoretic study of the hyperfinite II$_1$ factor and strongly self-absorbing algebras was initiated.

On the II$_1$ factor side of the equation, much is known about the model theory of the hyperfinite II$_1$ factor R. R is a prime model of its theory (it embeds elementarily into any other model of its theory; interestingly it embeds into any II$_1$ factor and if that factor is a model of the theory of R, then the embedding is automatically elementary; [3]). The theory of R is not model complete ([15],[11]). Although it is possible that the theory of R is $\forall\exists$-axiomatizable there is speculation that this theory could be undecidable [13].

On the C*-algebra side, the corresponding algebras are the strongly self-absorbing algebras. We say that a separable and unital A is strongly self-absorbing if $A \cong A \otimes A$ and any two unital *-homomorphisms from A to $A \otimes A$ are approximately unitarily conjugate. There are only a handful of known strongly self-absorbing algebras: $\mathcal{Z}$, UHF$^\infty$ (any UHF algebra of infinite type), $\mathcal{O}_\infty$, $\mathcal{O}_\infty \otimes$ UHF$^\infty$ and $\mathcal{O}_2$. It is an open question if this list is exhaustive; potentially, if there is a missing strongly self-absorbing algebra, model theoretic forcing could help to find one. In any case, the model theory of strongly self-absorbing algebras is well understood ([9]). As with the hyperfinite II$_1$ factor, they are the prime models of their theory in the language of C*-algebras. They again have the automatic elementarity mentioned above. Little is known about the theories of strongly self-absorbing algebras except that two non-isomorphic strongly self-absorbing algebras have distinct theories.

One intriguing issue that arises in the study of strongly self-absorbing algebras in either context is the role played by the relative commutant inside an ultrapower or the central sequence algebra. That is, for a separable algebra A, one considers $A' \cap A^U$ for a non-principal ultrafilter U on $\mathbb{N}$. Model theoretically, the role of the ultrapower is clear: it is a countably saturated model. This saturation implies that the central sequence algebra is always quantifier-free saturated and in important instances is actually an elementary submodel (as it is in the case that A is strongly self-absorbing). There is no general model theoretic construct similar to the relative commutant and it would be extremely interesting to investigate the exact formal properties which makes this such an important tool in the study of operator algebras.

The role of the notion of saturation in the general model theoretic study of operator algebras cannot be overstated. As mentioned above, full saturation and quantifier-free saturation play their usual roles as in any model theoretic study. In [7], a very weak form of saturation (countably degree-1 saturated) is discussed. This level of saturation is shared by all coronas of separable C^*-algebras like the Calkin algebra which is known not to be quantifier-free saturated. This provides simplified proofs of Kasparov's technical lemma among other things. This theme was also picked up in [29] where it is shown that certain C^*-algebras, rough analogues of the Calkin algebra, are also countably degree-1 saturated. This restricted form of saturation should play a key role in the construction of a K-theory reversing automorphism of the Calkin algebra.

There is value in continuing the study both of general elementary properties of C^*-algebras and II_1 factors. It was widely believed amongst those working in the area that there should be continuum many distinct theories of II_1 factors. This suspicion was confirmed in [2] where it was shown that McDuff's original family of continuum many pairwise non-isomorphic separable II_1 factors were indeed not elementarily equivalent. The proof proceeded by showing that any two ultrapowers of distinct members of the family were nonisomorphic. It would be interesting to isolate particular sentences that distinguish these factors. In [14], Ehrenfeucht-Fraïssé games were used to at least give upper bounds for the quantifier complexity of sentences distinguishing these factors. A related problem would be the study of the model-theoretic fundamental group of a II_1 factor. In particular, finding an example of a II_1 factor with proper first-order fundamental group would give another proof of the existence of continuum many non-elementarily equivalent II_1 factors.

2 Recent Developments and Open Problems

The following list of problems highlights some of the aspects of the interaction between model theory and operator algebras.

Ilijas Farah

Let $\mathcal{Q}$ be the universal UHF algebra and let $\mathcal{R}$ be the hyperfinite II_1 factor. Are $\mathcal{Q}$ and $\mathcal{R}$ elementarily equivalent *in the language of* C^**-algebras*? In other words, do $\mathcal{Q}$ and $\mathcal{R}$ have isomorphic ultrapowers (again, as C^*-algebras)? Note that a positive answer implies that $\mathcal{R}$ is an MF-algebra.

Related questions are: Is there a unital map from the Jiang-Su algebra $\mathcal{Z}$ to the (norm)central sequence algebra of $\mathcal{R}$? Does the theory of $\mathcal{R}$ in the language of C^*-algebras have a nuclear model?

A positive answer to this question would imply that $\mathcal{R}$ is quasidiagonal, answering a prominent open problem. More precisely, the quasidiagonality of $\mathcal{R}$ is equivalent to $\mathcal{R}$ and $\mathcal{Q}$ having the same universal theory.

Another related question, asked by Chris Schafhauser, is whether R has a theory of a $\mathcal{Z}$-stable C^*-algebra? Although R itself cannot absorb $\mathcal{Z}$ tensorially, a positive answer is equivalent to every separable elementary submodel of R absorbing $\mathcal{Z}$ tensorially.

Ben Hayes

Call a tracial C^*-algebra (A, τ) *Hayesian* if there is a trace-preserving embedding $A \hookrightarrow \prod_{\mathcal{U}} M_n(\mathbb{C})$, where the latter ultraproduct is the C^*-algebra ultraproduct equipped with the trace obtained by taking the $\mathcal{U}$-ultralimit of the normalized traces on the $M_n(\mathbb{C})$. Call a discrete group Γ *Hayesian* if the tracial C^*-algebra $(C^*_r(\Gamma), \tau_\Gamma)$ is Hayesian, where τ_Γ is the canonical trace. Which groups are Hayesian? By standard arguments, this is equivalent to an assertion about the universal theory of $(\mathrm{C}^*_r(\Gamma), \tau_\Gamma)$ ([11]). Here are some facts about Hayesian groups:

- Amenable groups are Hayesian. ([27]).
- $\mathbb{F}_2$ is Hayesian. ([18]).
- Free products of Hayesian groups are Hayesian (Reference?)

- Direct products of exact Hayesian groups.

Are there any non-Hayesian groups? Is the amalgamated free product of Hayesian groups over an amenable amalgam once again Hayesian?

N. Christopher Phillips

Fix $p \in (1, \infty)$. A *unital L^p-operator algebra* is a Banach algebra $\mathcal{A}$ such that there is an L^p-space $L^p(X, \mu)$ and an isometric unital Banach algebra homomorphism $\mathcal{A} \hookrightarrow \mathcal{B}(L^p(X, \mu))$. They appear to be closed under ultraproducts and are clearly closed under ultraroots (in fact substructures), so form an axiomatizable class in the language of unital Banach algebras. What are natural axioms?

Alessandro Vignati

A result of K.P. Hart implies that if X and Y are two nontrivial continua, then $C(X)$ embeds into an ultrapower of $C(Y)$. It is also known that there is no metrizable continuum X such that $C(Y)$ embeds into $C(X)$ for all other metrizable continua Y. In particular, this implies that for every metrizable continuum X, there is a metrizable continuum Y such that $C(X) \equiv C(Y)$ but $X \ncong Y$. For specific X, find examples of such Y. For example, find Y such that $C([0, 1]) \equiv C(Y)$ but $[0, 1] \ncong Y$.

In another direction, suppose that X and Y are locally compact spaces such that $C(\beta X \setminus X) \equiv C(\beta Y \setminus Y)$. What can we say about $C_0(X)$ vs. $C_0(Y)$. Also, under the same assumption, if one assumes CH, do we know that in fact $C(\beta X \setminus X) \cong C(\beta Y \setminus Y)$?

Isaac Goldbring

Call a McDuff II_1 factor *strongly McDuff* if it is isomorphic to one of the form $M \otimes \mathcal{R}$ for M a non-Gamma II_1 factor. Can an existentially closed (e.c.) II_1 factor ever be strongly McDuff? As partial progress, if the non-Gamma factor M is *bc-good* (to be defined shortly), then $M \otimes \mathcal{R}$ is not e.c. Here, M is bc-good if it has a w-spectral gap subfactor N (meaning that $N' \cap M^{\mathcal{U}} = (N' \cap M)^{\mathcal{U}}$ for which $(N' \cap M)' \cap M \neq N$. This leads to the question: is every non-Gamma factor bc-good?

Wilhelm Winter

By [1], a separable, nuclear, C*-algebra satisfies the UCT if and only if it has a Cartan masa. A major open question is whether or not all strongly self-absorbing (ssa) algebras satisfy the universal coefficient theorem (UCT). Towards that goal, here are some intermediate questions. View the set of ssa algebras as a category whose morphisms are unital $*$-homomorphisms up to approximate unitarily equivalence. This category has an initial object, namely the Jiang-Su $\mathcal{Z}$, which has a Cartan masa. Can you prove that the initial object has a Cartan masa without actually knowing that it is $\mathcal{Z}$? Also, one can ask the same question for the category of ssa algebras of the form $A \otimes M_{2^\infty}$ where A is ssa. This also has an initial object, M_{2^∞}, which also has a Cartan masa.

Questions on opposite algebras

It is a major open problem whether every simple, separable, nuclear C*-algebra A is isomorphic to its opposite algebra, A^{op}. (The answer is positive if simplicity is dropped, if nuclearity is relaxed to exactness, and nonseparable examples exist.) Every C*-algebra that is classifiable by its Elliott invariant is isomorphic to its opposite algebra. The following two questions are about C*-algebras not isomorphic to their opposites.

Ilan Hirshberg

Is there a C*-algebra A such that $A \not\equiv A^{\mathrm{op}}$? (If yes, how 'nice' can A be? Can it be unital, simple, ...?) Nonseparable examples given in [12] are elementarily equivalent to Elliott-classifiable C*-algebras, and therefore elementarily equivalent to their opposites.

N. Christopher Phillips

Suppose that A is a unital, simple, purely infinite C*-algebra. Is there a state φ on A which can be distinguished up to unitary equivalence in the sense that for every automorphism α of A there is a unitary u in A such that $\varphi \circ \alpha = \varphi \circ \mathrm{ad}_u$? (If A is a unital C*-algebra with a unique tracial state τ, then one has $\tau \circ \alpha = \tau$ for every automorphism α of A.)

The motivation for this question comes from the argument in [23], where it was proved that if M is a II_1 factor not isomorphic to its opposite and A is a separable, weakly dense, elementary submodel of M (considered as a C*-algebra), then A is not isomorphic to A^{op}. A positive answer would provide a purely infinite, simple, C*-algebra not isomorphic to its opposite.

Ilan Hirshberg

Is there any natural model-theoretic meaning to looking at structures that resemble ultrapowers except one uses βX for X an arbitrary locally compact space (e.g. $\mathbb{R}_+$, which shows up in practice) rather than just βI for I a discrete set? Is there a corresponding logic for which this is well-behaved? Are there parallels to usual model-theoretic facts about ordinary ultrapowers? What uses does this construction have?

2.1 Other problems

One of the most prominent open problems in the theory of operator algebras is whether all II_1 factors associated with nonabelian free groups (the so-called 'free group factors') are isomorphic. A variant of this question is whether all free group factors are elementarily equivalent. (Or equivalently, whether they have isomorphic ultrapowers.) The answer to this question is positive if and only if the free group factors $L(F_2)$ and $L(F_3)$ (or any other $L(F_m)$ and $L(F_n)$, for $m \neq n$) are elementarily equivalent.

The C^*-variant of this question has a negative answer. The reduced group C^*-algebra $\mathrm{C}^*_r(F_m)$ has the K_1-group equal to $\mathbf{Z}^m$, and therefore these algebras are pairwise nonisomorphic. But the following is open.

Dimitri Shlyakhtenko

Are all reduced group C^*-algebras associated with finitely generated free nonabelian groups elementarily equivalent?

One approach to giving a negative answer to this question would be to show that the K_1-groups of the ultrapowers of free group C^*-algebras are nonisomorphic. Since each $\mathrm{C}^*_r(F_m)$ has stable rank 1, its K_1-group is equal to $\mathcal{U}(\mathrm{C}^*_r(F_m)/U_0(\mathrm{C}^*_r(F_m))$. Thus the question reduces to the followig:

Is the homotopy relation on the unitary group of $\mathrm{C}^*_r(F_m)$ definable?

Or even more specific: Is there $n \in \mathbf{N}$ such that every unitary in $\mathrm{C}^*_r(F_m)$ homotopic to 1 can be approximated in norm up to < 2 by a product of n exponentials of self-adjoints, each of norm at most π? A positive answer to this question would imply that the K_1 group of $\mathrm{C}^*_r(F_m)$ belongs to the eq of this algebra ([8]), and in turn give a negative answer to the above question.

3 Presentation Highlights

This workshop brought together experts in operator algebras and model theory. The first two days of the meeting were dominated by three tutorials that provided a fresh look at the background material.

Gabor Szabo, Introduction to C^*-algebras

I will give an introduction to the theory of C^*-algebras. Starting from the basics, we will treat spectral theory in some detail, culminating in the Gelfand-Naimark theorem. We will cover the GNS construction, with highlight being that every abstract C^*-algebras can be realized as a C^*-algebra of bounded operators on a Hilbert space. We will then discuss other constructions/examples such as certain universal C^*-algebras or inductive limits. If there is time, I will give a rough outline of the Elliott classification program.

Adrian Ioana, Tutorial on von Neumann algebras

In the first lecture, I will review basic notions and constructions of von Neumann algebras. The second lecture will be devoted to property Gamma and McDuffs property for II_1 factors. In the third lecture, I will discuss the isomorphism problem for ultrapowers of II_1 factors.

Lecture 1: Define vN algebras and state the bicommutant theorem. Introduce tracial vN algebras and the hyperfinite II_1 factor. Group and group measure space vN algebras.

Lecture 2: Define property Gamma and discuss the connection with inner amenability of groups. Define McDuffs property. Examples of II_1 factors that are Gamma but not McDuff.

Lecture 3: The ultrapower construction for tracial vN algebras. Discuss dependence on the choice of the ultrafilter and examples of II_1 factors with non-isomorphic ultrapowers.

Martino Lupini, Tutorial on model theory

Lecture 1: Structures, ultraproducts, and formulas

I will introduce the fundamental notions of logic for metric structures, such as formulas and ultraproducts. I will then explain how C*-algebras and von Neumann algebras fit into this framework.

Lecture 2: Axiomatizability and definability

I will present the crucial model-theoretic concepts of axiomatizability and definability, and then provide many examples from the theory of operator algebras.

Lecture 3: Nuclearity and omitting types

I will discuss how nuclearity can be captured model-theoretically, and how this opens up the possibility to use constructions from model theory to produce interesting new examples of nuclear C*-algebras.

Leonel Robert, C^*-algebras of stable rank one and their Cuntz semigroups

I will talk about recent joint work with Antoine, Perera, and Thiel. We have shown that the Cuntz semigroup of a separable C*-algebra of stable rank one is inf-semilattice ordered, i.e., has finite infima and addition is distributive over infima. We use this to gain new insights into the structure of these Cuntz semigroups and to answer a number of questions on C*-algebras of stable rank one. We are able to remove the assumption of separability in some of these applications using model theoretic tools.

Christopher Schafhauser, On the classification of simple, nuclear C^*-algebras

I will discuss some recent joint work with José Carrión, Jamie Gabe, Aaron Tikuisis, and Stuart White, which provides a new abstract approach to the classification of simple, nuclear C*-algebras.

Aaron Tikuisis, The Toms-Winter conjecture and complemented partitions of unity

The Toms-Winter conjecture postulates that three very different-looking regularity-type conditions coincide for separable simple infinite-dimensional amenable unital C*-algebras. The different conditions are (i) finite nuclear dimension, (ii) Z-stability, and (iii) strict comparison of positive elements. In the first half of my talk, I will say some things about this conjecture and its important connections to the classification of C*-algebras.

In the second half of the talk, I will discuss a key concept used in the proof that (ii) implies (i), called complemented partitions of unity (CPoU). As I will explain, this concept provides a method for gluing local witnesses of open types in tracial GNS representations to global witnesses (satisfying the type uniformly over all traces). I will explain complemented partitions of unity in the context of this local-to-global type satisfaction device.

This is joint work with Jorge Castillejos, Sam Evington, Stuart White, and Wilhelm Winter.

Stefaan Vaes, Classification of regular subalgebras of the hyperfinite II_1 factor

I present a joint work with Sorin Popa and Dimitri Shlyakhtenko. We prove that under a natural condition, the regular von Neumann subalgebras B of the hyperfinite II_1 factor R are completely classified (up to conjugacy by an automorphism of R) by the associated discrete measured groupoid. The key step in proving this result is the vanishing of the 2-cohomology for cocycle actions of amenable discrete measured groupoids and the approximate vanishing of the 1-cohomology. This leads us to a new notion of treeability for equivalence relations. I also discuss (non-)classification results for amenable discrete measured groupoids.

Bradd Hart, Correspondences and model theory

In joint work with Goldbring and Sinclair, for tracial von Neumann algebras M and N, we show how to capture the notion of an M-N correspondence model theoretically. We use this correspondence framework to study σ-finite von Neumann algebras and the uniform 2-norm on a C*-algebra with respect to a collection of states. The role of the ultraproduct will be highlighted for its guidance in determining the correct languages in these cases.

Eusebio Gardella, Equivariant model theory and applications to C^*-dynamics

The use of (central) sequence algebras in the theory of operator algebras has a long history, dating back to McDuffs characterization of those factors which absorb the hyperfinite II_1-factor. Applications in the context of C*-algebras are both abundant and far-reaching, and they often appear in connection with classification of C*-algebras. Central sequence algebras are fundamental tools in the study of strongly self-absorbing C*-algebras, which themselves have tight connections with the Elliott classification programme. This has prompted a deeper study of ultrapowers and (central) sequence algebras, where model-theoretic methods have become predominant. Ultrapowers and relative commutants have also been a crucial tool in the study of group actions on operator algebras, dating back to the classification of amenable group actions on the hyperfinite II$_1$-factor. A more recent instance of their use in the equivariant setting is the study of strongly self-absorbing actions. As such, equivariant (central) sequence algebras are interesting objects whose systematic study is justified by their wide application in the literature. In this talk, we report on joint work with Lupini, where we consider actions of a compact group on a C*-algebra as a structure in the framework of continuous model theory. The realization that the continuous part of the ultrapower of a G-C*-algebra is just its ultrapower as a structure in the new equivariant language, allows us to establish interesting properties, including saturation and Łoś's theorem. We give various applications to C*-dynamics, including to strongly self-absorbing actions as well as to Rokhlin dimension.

Mikael Rørdam, Non-closure of quantum correlation matrices and certain factorizable maps, traces on free product C^*-algebras, and Connes Embedding Problem

We show that the convex set of factorizable quantum channels on a fixed matrix algebra of size at least 11 which factor through finite dimensional C*-algebras is non-closed, and that there exist factorizable quantum channels on matrix algebras that require an ancilla of type II$_1$. We also give a new and simplified proof of the result by Dykema, Paulsen and Prakash that the set of synchronous quantum correlations $C_q^s(5,2)$ is non-closed. One can describe factorizable quantum channels on a given matrix algebra in terms of traces on the unital free product of that matrix algebra with itself. We give a description of which of these traces correspond to factorizable maps that can be approximated by ones with finite dimensional ancilla, and we relate this to the Connes Embedding Problem.

This is a joint work with Magdalena Musat.

Ilan Hirshberg, On C^*-algebras not isomorphic to their opposites.

For each C*-algebra A, one can construct its opposite A^{op}, which is the same as a Banach space, only with the order of multiplication is reversed. It is a long-standing and difficult open problem whether there exists a simple separable nuclear C*-algebra which is not isomorphic to its opposite. I will survey some of the known results and techniques, focusing on the nuclear case, and discuss a joint paper with Ilijas Farah ([12]) in which we construct a simple nuclear non-separable example.

N. C. Phillips, The Continuum Hypothesis implies existence of outer isometric automorphisms of the l^p Calkin algebra

Let $p \in (1, \infty)$. We show that the Continuum Hypothesis implies that the l^p Calkin algebra $Q(\ell^p) = L(\ell^p(\mathbb{Z}))/K(\ell^p(\mathbb{Z}))$ has isometric automorphisms which are not given by conjugation by invertible isometries in $Q(\ell^p)$. Depending on what is done between now and the time of the talk, we will describe progress towards proving that it is consistent with ZFC that there are no such isometric automorphisms.

This is joint with with Andrey Blinov.

4 Scientific Progress Made

During the breaks and evenings, the participants worked in smaller groups. Only time will tell what progress resulted from this meeting, and we will list only the progress that we are aware of.

Inspired by Robert's talk and recent Thiel's construction of ultraproducts of Cuntz semigroups, several participants gave a rough outline of a formal model-theoretic framework for Cuntz semigroups. Giving a model-theoretic framework for the study of Cuntz semigroup is a major challenge (see [**?**]).

A group of participants worked on the early 1970's problem of Brown, Douglas, and Fillmore, asking whether the Calkin algebra can have a K-theory reversing automorphism. By a 2011 work of Farah, it is known that forcing axioms imply a negative answer. A new alley of attack on this prominent open problem, combining KK-theory with some set-theoretic considerations, has been outlined.

The recent major progress in Elliott's classification programme, reported in talks by Schafhauser and Tikuisis, inspired many conversations. In particular, the principle CPoU (Central Partitions of Unity) can be restated as a transfer principle from the theory of type II$_1$ von Neumann algebras to the theory of the so-called strict closures of C*-algebras. (The latter comes from a generalization of the seminal work of Ozawa on W*-bundles, [22].) The independently developed model theory of correspondences, reported in Hart's talk ([16]), appears to be tailor-made for studying strict closures of C*-algebras.

Vaes reported a major breakthrough in our understanding of regular subalgebras of the hyperfinite II$_1$ factor $\mathcal{R}$ ([24]). The Connes Embedding Problem, one of the most famous problems in the theory of operator algebras, was the subject of Rørdam's talk (see [21]).

Connections to set theory were addressed on the last day of the meeting, in the talks by Hirshberg and Phillips.

5 Outcome of the Meeting

The talks at this meeting were of an exceptionally high quality. All speakers were given a full hour, and the majority of the talks were given in the old-fashioned way, using chalk and blackboard. This encouraged lively discussions that always continued well past the end of the talk and throughout the meeting.

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