Bridging the Gap between Kähler and non-Kähler Complex Geometry

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This workshop brought together mathematicians from all over the world who work on a wide range of different but related topics. The nineteen talks and the many private mathematical discussions sprang mainly from three very active research directions:

- Complex Geometry;
- Symplectic Geometry;
- Mathematical Physics.

Stimulating a dialogue between these vast research areas was at the core of the organisers' and the participants' preoccupations. We believe that new bridges have been built and new contacts have been established. So, we are able to say that this workshop was a success.

One of the unifying themes spanning the three main research directions listed above is **Mirror Symmetry**. This is a conjecture that arose in physics in the 1980s and the 1990s (the work [COGP91] of Candelas-de la Ossa-Green-Parkes being thought of as pioneering) but is still not entirely understood mathematically in spite of significant recent progress. Put in a crude way, the Mirror Symmetry Conjecture predicts that Calabi-Yau manifolds (i.e. compact Kähler manifolds X whose canonical bundle K_X is trivial) come in pairs $(X, \widetilde{X})$ such that the *complex structures* supported by the smooth manifold underlying X (the complex moduli space of X, i.e. the universal family of local deformations of the original complex structure of X, called the *Kuranishi family* $\mathrm{Def}(X)$ of X) bear a certain relationship with the *Kähler structures* supported by the mirror dual $\widetilde{X}$ (in the form of the complexified Kähler cone of $\widetilde{X}$, also called the Kähler moduli space).

A tremendous amount of different mathematics goes into this issue, ranging from complex algebraic geometry (sheaves, vector bundles, toric varieties), complex analytic and differential geometry (deformations of complex structures, connections, special Hermitian metrics), symplectic geometry (Lagrangian fibrations), Hodge Theory (variations of Hodge structure, spectral sequences), mathematical physics and pluripotential theory (Monge-Ampère-type equations and other geometric PDEs). All these research topics were represented in the talks and the participants' discussions in this workshop.

The study of Calabi-Yau manifolds can be seen in the context of the study of compact Riemannian 7-manifolds with holonomy G_2 and that of compact Riemannian 8-manifolds with holonomy $\mathrm{Spin}(7)$. Three possible models for the universe have been proposed as based on these three types of manifolds. Our workshop brought together experts from all these three areas.

Another unifying theme for the three main research directions mentioned above is the link between the existence of **Kähler-Einstein metrics** on certain classes of compact complex manifolds and the various notions of **stability**. Significant progress was accomplished recently by Chen-Donaldson-Sun and Tian and the many geometric flows have found a wide array of applications, some of which were reported during this workshop.

1 Overview of the Field

The classification of compact complex, not necessarily projective or even Kähler, manifolds is a major undertaking in modern mathematics. It involves the interplay between several kinds of structures (complex, metric, cohomological, topological, etc) that such a manifold supports. Hermitian metrics (defined as C^∞, positive definite, differential forms ω of bidegree $(1,\,1)$) exist on any compact complex manifold, but if they are required to satisfy d-closedness or $\partial\bar\partial$-closedness conditions, they need not exist. When they do, they impose a certain type of geometry on the underlying manifold and become a powerful tool in the classification theory.

Kähler metrics ω (i.e. those Hermitian metrics ω such that $d\omega = 0$) exist only rarely on compact complex manifolds X of complex dimension $n \geq 3$. However, related but weaker properties of either a metric nature (e.g. the balanced and the strongly Gauduchon conditions) or a cohomological nature (e.g. the $\partial\bar\partial$-property requiring the validity of the $\partial\bar\partial$-lemma) have emerged recently as powerful criteria for classification. They allow for large classes of possibly non-Kähler manifolds to behave in certains respects as Kähler manifolds and thus enable researchers to build virtually from scratch a classification theory of non-Kähler manifolds that share metric or cohomological properties with their Kähler counterparts.

Here is a brief description of some of the recent advances in this fast growing area.

(A) The metric side of the theory

A spectacular use of Kähler metrics and of possibly singular Hermitian metrics on the fibres of holomorphic line bundles was made by Siu in his proof of the **invariance of the plurigenera** conjecture for projective families of compact complex manifolds.

Theorem (Siu 1998 and 2000) *Let $\pi : \mathcal{X} \longrightarrow \Delta$ be a* **projective** *holomorphic family of compact complex manifolds over the unit disc $\Delta \subset \mathbb{C}$. Then, for every $m \in \mathbb{N}^\star$, the m-genus $\dim_{\mathbb{C}} H^0(X_t,\, mK_{X_t})$ is independent of the fibre $X_t := \pi^{-1}(t)$, $t \in \Delta$.*

By K_{X_t} one means the canonical line bundle of the fibre X_t, mK_{X_t} stands for its m^{th} tensor power (in additive notation in the Picard group), while H^0 denotes the space of global holomorphic sections. This was a major success of transcendental (mainly analytical) methods in algebraic geometry. Except for the case where the fibres X_t are assumed of general type, no algebraic proof of this result is known. The invariance of the plurigenera plays a major role in the **Minimal Model Program (MMP)** where projective (or merely compact Kähler) manifolds are to be classified up to birational equivalence.

The use of transcendental methods (e.g. special Hermitian metrics, non-rational cohomology classes, currents, pluripotential theory, geometric PDEs) becomes inevitable on general compact Kähler and non-Kähler manifolds since most of them contain no other complex submanifolds than the points. Thus, the familiar apparatus of curves, divisors and global holomorphic sections of line bundles used in algebraic geometry is often unavailable.

A major purely transcendental result was the following characterisation of the Kähler cone (= the open convex cone of all Kähler cohomology classes) of an arbitrary compact Kähler manifold.

Theorem(Demailly-Paun 2004) *Let X be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$. The Kähler cone $\mathcal{K}_X$ of X is one of the connected components of the cone $\mathcal{P}_X$ consisting of the real $(1,\,1)$-cohomology classes $\{\alpha\}$ such that $\int_Y \alpha^p > 0$ for every irreducible analytic subset $Y \subset X$ of dimension p and for every $p \in \{1, \dots, n\}$.*

This important result has had a major impact in complex geometry, including in the development of an MMP theory for Kähler 3-folds and in Mirror Symmetry.

Another example where special Hermitian metrics were instrumental in the resolution of a long-standing algebro-geometric problem for which purely algebraic methods had failed is the following **deformation**

closedness property of a class of algebraic manifolds. The **Moishezon manifolds** are the compact complex manifolds that are bimeromorphically equivalent to projective manifolds.

Theorem (Popovici 2013) *Let $\pi : \mathcal{X} \longrightarrow \Delta$ be a holomorphic family of compact complex manifolds over the unit disc $\Delta \subset \mathbb{C}$. Suppose that the fibre X_t is* **projective** *for every $t \in \Delta \setminus \{0\}$ and that the fibre X_0 is a* **strongly Gauduchon** *manifold. Then X_0 is Moishezon.*

A Hermitian metric ω is said to be *strongly Gauduchon* if $\partial\omega^{n-1}$ is $\bar\partial$-exact. Manifolds carrying such a metric are called *strongly Gauduchon manifolds*.

(B) The cohomological side of the theory

The prototype of this endeavour is the notion of Kähler class on a Kähler manifold and the search for a canonical Kähler metric representing it, known as the *Calabi program*.

The notion of *strongly Gauduchon metrics* led to a generalisation in the non-Kähler context and in bidegree $(n-1,\, n-1)$ of the notion of Kähler class and of Yau's celebrated resolution of the Calabi conjecture. The idea of considering a Monge-Ampère-type equation in bidegree $(n-1,\, n-1)$ was first suggested by Demailly in discussions (2009) with Popovici about strongly Gauduchon metrics.

Theorem (Popovici–Tosatti–Weinkove–Szekelyhidi 2013, 2015) *Given an arbitrary* **Gauduchon metric** ω *on X (i.e. such that $\partial\bar\partial\omega^{n-1} = 0$), the following equation*

$$\left[\left(\omega^{n-1} + i\partial\bar\partial\varphi\wedge\omega^{n-2} + \frac{i}{2}\,(\partial\varphi\wedge\bar\partial\omega^{n-2} - \bar\partial\varphi\wedge\partial\omega^{n-2})\right)^{\frac{1}{n-1}}\right]^n = e^{f+c}\,\omega^n \quad (\star)$$

subject to the positivity and normalisation conditions

$$\Omega_\varphi := \omega^{n-1} + i\partial\bar\partial\varphi\wedge\omega^{n-2} + \frac{i}{2}\,(\partial\varphi\wedge\bar\partial\omega^{n-2} - \bar\partial\varphi\wedge\partial\omega^{n-2}) > 0 \quad \text{and} \quad \sup_X \varphi = 0, \qquad (1)$$

has a unique C^∞ solution $\varphi : X \to \mathbb{R}$ for any given C^∞ real-valued function f and a unique constant $c \in \mathbb{R}$ depending on f.

This equation produces, in the **Aeppli cohomology** class $[\omega^{n-1}]_A$ of each Gauduchon metric ω on X, a unique Gauduchon metric ω_φ (the $(n-1)^{st}$ root of Ω_φ) depending on ω such that the volume form of ω_φ has been prescribed. (Recall that the Aeppli cohomology groups of X are defined as $H^{p,\,q}_A(X,\,\mathbb{C}) := \ker(\partial\bar\partial)/\mathrm{Im}\,(\partial\bar\partial)$ in every bidegree $(p,\,q)$, where the operators ∂ and $\bar\partial$ act on either smooth differential forms or currents.) This relates to a natural object on the given compact complex manifold X, the so-called **Gauduchon cone** $\mathcal{G}_X$ (Popovici 2013) consisting of the Aeppli cohomology classes of all the Gauduchon metrics. It is an open convex cone in the Aeppli cohomology space $H^{n-1,\,n-1}_A(X,\,\mathbb{R})$ and is dual to Demailly's pseudo-effective cone of Bott-Chern cohomology classes of all the d-closed positive $(1,\,1)$-currents on X. It plays a growing role as a possible substitute of the classical Kähler cone (even on Kähler manifolds), including in a new approach to Mirror Symmetry, extended to possibly non-Kähler Calabi-Yau (i.e. whose canonical bundle is trivial) manifolds, very recently proposed by Popovici.

We end this brief overview with a reminder of the important class of **$\partial\bar\partial$-manifolds**, a key link between Kähler and non-Kähler geometries. These are the compact complex manifolds X such that for every bidegree $(p,\,q)$ and every d-closed $(p,\,q)$-form (or current) u on X, the following exactness properties are equivalent:

$$u \in \mathrm{Im}\,\partial \iff u \in \mathrm{Im}\,\bar\partial \iff u \in \mathrm{Im}\,d \iff u \in \mathrm{Im}\,(\partial\bar\partial).$$

All compact Kähler manifolds are $\partial\bar\partial$-manifolds, by the classical $\partial\bar\partial$-lemma, but there exist many important classes of $\partial\bar\partial$-manifolds that are not Kähler. Examples of non-Kähler Calabi-Yau $\partial\bar\partial$-manifolds include some 3-dimensional solvmanifolds (Fino-Otal-Ugarte 2014) and the Clemens manifolds (Friedman 2017). The $\partial\bar\partial$-condition implies the existence of a canonical (i.e. depending only on the complex structure) Hodge Decomposition and Symmetry, so $\partial\bar\partial$-manifolds behave cohomologically like Kähler manifolds.

2 Recent Developments and Open Problems

(A) Let us mention at least two alternative and more recent approaches to the **Mirror Symmetry Conjecture**.

• The SYZ approach to Mirror Symmetry

Another approach to Mirror Symmetry centred around the so-called Strominger-Yau-Zaslov (SYZ) conjecture. It predicts the existence of dual special Lagrangian T^n-fibrations in a mirror pair of Calabi-Yau manifolds. The existence of topological T^n-fibrations for non-singular Calabi-Yau hypersurfaces in toric varieties was proved by Zharkov in 1998, while Ruan proved in 1999 the existence of a Lagrangian torus fibration for the quintic threefold.

The underlying idea is to use the SYZ conjecture to construct the mirror manifold of a given Calabi-Yau manifold. This involves dualising and compactifying torus fibrations, i.e. fibrations $\pi : \mathcal{X} \to B$ whose general fibre is a torus. There are cases where it is known that the SYZ conjecture explains mirror symmetry from a topological point of view.

A significant result is the following

Theorem 2.1. *(Gross 1999) The quintic threefold $X \subset \mathbb{P}^4$ has a well-behaved T^3-fibration with semistable fibres $f : X \to B$.*

Moreover, f has a well-behaved dual $\check{f} : \check{X} \to B$ and $\check{X}$ is diffeomorphic to a specific non-singular minimal model of the mirror quintic.

Shortly after the above result, Ruan proved the existence of Lagrangian T^3-fibrations on the quintic and the mirror quintic that are probably dual to each other as can be seen by comparing their monodromies.

• The Gauduchon cone and Frölicher spectral sequence approach to Mirror Symmetry

This approach was proposed recently in [Pop18] in the general context of possibly non-Kähler Calabi-Yau manifolds (i.e. compact complex manifolds X whose canonical bundle K_X is trivial). Thus, it extends the Mirror Symmetry Conjecture to the possibly non-Kähler context.

As is well known, there is an obvious cohomological obstruction to some Kähler C-Y threefolds X having Kähler mirror duals $\widetilde{X}$. The Kuranishi family $(X)_{t\in\Delta}$ of a given Kähler C-Y manifold $X = X_0$ is unobstructed (i.e. its base space Δ is **smooth**, hence can be viewed as an open ball in the classifying space $H^{0,\,1}(X,\, T^{1,\,0}X)$) by the Bogomolov-Tian-Todorov theorem ([Bog78], [Tia87], [Tod89]). The triviality of the canonical bundle K_X implies the isomorphism $H^{0,\,1}(X,\, T^{1,\,0}X) \simeq H^{n-1,\,1}(X,\,\mathbb{C}) = H^{2,\,1}(X,\,\mathbb{C})$, where the last identity follows from the assumption $\dim_{\mathbb{C}} X := n = 3$. On the other hand, the complexified Kähler cone $\widetilde{\mathcal{K}}_{\widetilde{X}}$ of $\widetilde{X}$ is an open subset of $H^{1,\,1}(\widetilde{X},\,\mathbb{C})$. So a necessary condition for X and $\widetilde{X}$ to be mirror dual is that the tangent space to Δ at 0 (i.e. $H^{2,\,1}(X,\,\mathbb{C})$) be isomorphic to the tangent space to the complexified Kähler cone $\widetilde{\mathcal{K}}_{\widetilde{X}}$ at some point (i.e. $H^{1,\,1}(\widetilde{X},\,\mathbb{C})$), and vice-versa. It is thus necessary to have

$$h^{2,\,1}(X) = h^{1,\,1}(\widetilde{X}) \quad \text{and} \quad h^{2,\,1}(\widetilde{X}) = h^{1,\,1}(X).$$

However, there exist Kähler C-Y threefolds X such that $h^{2,\,1}(X) = 0$ (the so-called *rigid* such threefolds, those that do not deform). Consequently, the mirror dual $\widetilde{X}$, if it exists, cannot be Kähler since $h^{1,\,1}(\widetilde{X}) = 0$.

The idea of investigating the possible existence of a mirror symmetry phenomenon beyond the Kähler world was loosely suggested by Reid in 1987 and received attention recently in 2015 in a work by Lau-Tseng-Yau. However, the methods and point of view adopted in [Pop18] are very different from those of Lau-Tseng-Yau.

The standard approach to the study of the Kähler side of the mirror is to use Gromov-Witten invariants attached to pseudo-holomorphic curves and to count rational curves. However, on many non-Kähler compact complex threefolds with trivial canonical bundle, there exist no rational curves.

The work [Pop18] proposed a new approach to mirror symmetry by means of transcendental methods in the general, possibly non-Kähler context of compact complex manifolds whose canonical bundle is trivial. By extension of the classical definition, we shall still call them **Calabi-Yau (C-Y) manifolds**. This new point of view was then tested on the Iwasawa manifold, a well-known non-Kähler compact complex C-Y manifold, where one can take full advantage of the explicit nature of extensive computations for this particular manifold found in the works [Nak75], [Ang11] and [Ang14] of Nakamura and Angella.

We hope that the methods introduced in [Pop18] will apply to larger classes of C-Y manifolds in the future and that this article is the first in a series. One of the new ideas it introduces is the notion of local universal family of *essential deformations*, viewed as a subfamily of the Kuranishi family, of the Iwasawa manifold X. Three equivalent definitions are given: by removing the complex parallelisable small deformations from the Kuranishi family; by selecting the small deformations that have a kind of *polarisation* by the holomorphic non-closed 1-form γ associated with X; and by selecting the vector subspace of the Dolbeault cohomology space $H^{n-1,\,1}(X,\,\mathbb{C})$ (known to parametrise all the small deformations of a C-Y manifold X, while the complex dimension of X is $n=3$ here) that is naturally isomorphic to the vector space $E_2^{n-1,\,1}(X)$ featuring in bidegree $(n-1,\,1)$ on the second page of the Frölicher spectral sequence of X.

Looking ahead beyond the special case of the Iwasawa manifold treated in this paper, we come up against the question of what makes a deformation of a general, possibly non-Kähler, C-Y manifold *essential*. Our hunch is that a definition in terms of the Frölicher spectral sequence, that will yield a replacement for the Hodge decomposition in middle degree n, is the best bet in a general pattern that will hopefully emerge in the future after further examples of C-Y manifolds have been investigated.

One of the main ideas in this work was to overcome the double whammy of a possible non-existence of both Kähler metrics and rational curves by using the Gauduchon cone of the given non-Kähler C-Y manifold X. This furnishes both an alternative to the classical Kähler cone (that is empty on a non-Kähler manifold) and a transcendental substitute for cohomology classes of (currents of integration on) curves (e.g. by virtue of its elements' bidegree $(n-1,\,n-1)$, but also in a far deeper sense). We stress that the Gauduchon cone is relevant even on projective and on Kähler non-projective manifolds where it might be preferable to the Kähler cone in certain circumstances (for example, when it is strictly bigger, allowing for more flexibility).

The compact complex manifolds X on which every Gauduchon metric is strongly Gauduchon were introduced under the name of **sGG manifolds** and studied in a 2014 paper by Popovici and Ugarte. They contain the Iwasawa manifold and all its small deformations.

The main object of study in [Pop18] was the standard *Iwasawa manifold* $X = G/\Gamma$, defined as the quotient of the Heisenberg group

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \;;\; z_1, z_2, z_3 \in \mathbb{C} \right\} \subset GL_3(\mathbb{C})$$

by its discrete subgroup $\Gamma \subset G$ of matrices with entries $z_1, z_2, z_3 \in \mathbb{Z}[i]$.

The map $(z_1, z_2, z_3) \mapsto (z_1, z_2)$ factors through the action of Γ to a (holomorphically locally trivial) proper holomorphic submersion

$$\pi : X \to B,$$

where the base $B = \mathbb{C}^2/\mathbb{Z}[i] \oplus \mathbb{Z}[i] = \mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i]$ is a two-dimensional Abelian variety (the product of two elliptic curves) and where all the fibres are isomorphic to the Gauss elliptic curve $\mathbb{C}/\mathbb{Z}[i]$.

Since G is a connected, simply connected, *nilpotent* complex Lie group, X is a *nilmanifold*. Furthermore, X is a *complex parallelisable* compact complex manifold (i.e. its holomorphic tangent bundle $T^{1,\,0}X$ is trivial) of complex dimension 3. In particular, its canonical bundle K_X is trivial, so X is a Calabi-Yau manifold in our generalised sense.

It is well known that X is not a $\partial\bar\partial$-manifold (in particular, it is not Kähler). In fact, its Frölicher spectral sequence does not even degenerate at E_1, so there is no Hodge decomposition either canonical or non-canonical on X.

However, despite X lacking the $\partial\bar\partial$ property, Nakamura showed in 1975 that the Kuranishi family $(X_t)_{t\in\Delta}$ of $X = X_0$ is *unobstructed*, so its base Δ is smooth and can be identified with an open ball in $H^{0,\,1}(X,\,T^{1,\,0}X) \simeq H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$. It can be easily checked that there is no Hodge decomposition of weight 3 since the Dolbeault cohomology group $H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$ does not inject canonically into $H^3_{DR}(X,\,\mathbb{C})$. In fact, $b_3 = 10$ while $h^{3,\,0} = h^{0,\,3} = 1$ and $h^{2,\,1} = h^{1,\,2} = 6$, so in a sense the vector space $H^{2,\,1}_{\bar\partial}(X,\,\mathbb{C})$ is "too large" to fit into $H^3_{DR}(X, \mathbb{C})$.

The main result of [Pop18] can be loosely stated as follows.

Theorem 2.2. *The Iwasawa manifold is its own mirror dual in the sense that its local universal family of essential deformations corresponds to its complexified Gauduchon cone.*

The meaning of "corresponds" was made precise in the paper. Loosely speaking, it means that there exists a local biholomorphism between the local universal family of essential deformations and the complexified Gauduchon cone of the Iwasawa manifold and there exists an induced C^∞ isomorphism of variations of Hodge structures (VHS) that exist on either side of the mirror. Moreover, this isomorphism is holomorphic at the level of the rank-1 components and anti-holomorphic at the level of the rank-4 components of these VHS.

(B) The study of holomorphic families of compact complex, possibly non-Kähler, manifolds, that plays a key role in Mirror Symmetry, was recently given a different treatment from the point of view of Riemann-Roch-type theorems via a key use of the Bott-Chern cohomology that is often better suited to the non-Kähler context than the more familiar Dolbeault cohomology.

• A Riemann-Roch theorem in Bott-Chern cohomology

J.-M. Bismut, who participated in this workshop and gave a far-reaching talk with implications in complex, algebraic and arithmetic geometry, described a geometric problem on families of elliptic operators. He had solved this problem via a deformation to a family of non self-adjoint Fredholm operators.

Specifically, let $p : M \to S$ be a proper holomorphic projection of complex manifolds and let F be a holomorphic vector bundle on M. It is assumed that the vector spaces $H^{(0,p)}(Xs, F|X_s)$ have locally constant dimension. They are the fibers of a holomorphic vector bundle on S.

The problem that was addressed in this work is the computation of characteristic classes associated with the above vector bundle in a refinement of the ordinary de Rham cohomology of S, its Bott-Chern cohomology, and the proof of a corresponding theorem of Riemann-Roch-Grothendieck. When M is not Kähler, none of the existing techniques to prove such a result using the fiberwise Dolbeault Laplacians can be used. The solution is obtained via a proper deformation of the corresponding Dolbeault Laplacians to a family of **hypoelliptic Laplacians**, for which the corresponding result can be proved. This deformation is made to destroy the geometric obstructions which exist in the elliptic theory, like the fact that the metric is Kähler, namely the fact that the Kähler form is $\bar{\partial}\partial$-closed.

• A conjecture on deformation limits of Moishezon manifolds

In 2019, the resolution of a conjecture dating back to the 1970s about limits under holomorphic deformations of Moishezon manifolds was announced in [Pop19]. A *Moishezon manifold* is a compact complex manifold Y for which there exists a projective manifold $\widetilde{Y}$ and a holomorphic bimeromorphic map $\mu : \widetilde{Y} \longrightarrow Y$. By a classical result of Moishezon (1967), a Moishezon manifold is not Kähler unless it is projective.

The context is the following. We consider a complex analytic (or holomorphic) family of compact complex manifolds. This is a *proper holomorphic submersion* $\pi : \mathcal{X} \to B$ between two complex manifolds $\mathcal{X}$ and B. In particular, the fibres $X_t := \pi^{-1}(t)$ are compact complex manifolds of the same dimension. By a classical theorem of Ehresmann (1947), any such family is locally (hence also globally if the base B is contractible) C^∞ trivial. Thus, all the fibres X_t have the same underlying C^∞ manifold X (hence also the same De Rham cohomology groups $H^k_{DR}(X, \mathbb{C})$ for all $k = 0, \dots, 2n$), but the complex structure J_t of X_t depends, in general, on $t \in B$.

The statement below is a closedness result under deformations of complex structures: any deformation limit of a family of Moishezon manifolds is Moishezon. Indeed, the fibre X_0 can be regarded as the limit of the fibres X_t when $t \in B$ tends to $0 \in B$. We can, of course, suppose that B is an open disc about the origin in $\mathbb{C}$.

Theorem 2.3. *([Pop19]) Let $\pi : \mathcal{X} \to B$ be a complex analytic family of compact complex manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin such that the fibre $X_t := \pi^{-1}(t)$ is a* **Moishezon manifold** *for every $t \in B \setminus \{0\}$. Then $X_0 := \pi^{-1}(0)$ is again a* **Moishezon manifold**.

One of the major open problems in this direction is the transcendental version of the above result about deformation limits of *Fujiki class $\mathcal{C}$ manifolds*. Recall that a *Fujiki class $\mathcal{C}$ manifold* is a compact complex manifold Y for which there exists a compact Kähler manifold $\widetilde{Y}$ and a holomorphic bimeromorphic map $\mu : \widetilde{Y} \longrightarrow Y$.

Conjecture 2.4. *Let $\pi : \mathcal{X} \to B$ be a complex analytic family of compact complex manifolds over an open ball $B \subset \mathbb{C}^N$ about the origin such that the fibre $X_t := \pi^{-1}(t)$ is a* **Fujiki class $\mathcal{C}$ manifold** *for every $t \in B \setminus \{0\}$. Then $X_0 := \pi^{-1}(0)$ is again a* **Fujiki class $\mathcal{C}$ manifold**.

A two-step strategy for tackling this conjecture was outlined in a work by Popovici and Ugarte (2014):

Step 1: *prove that a compact complex manifold X belongs to the class $\mathcal{C}$ if and only if there are "many" closed positive $(1,\,1)$-currents on X.*

Step 2: *prove that there can only be "more" closed positive $(1,\,1)$-currents on X_0 than on the generic fibre X_t.*

The notion of *sGG manifold* was introduced there for this purpose by requiring the Gauduchon cone of the manifold to be as small as possible (i.e. equal to the a priori smaller strongly gauduchon cone). This is a way of saying that the given compact complex manifold X carries "many" closed positive $(1,\,1)$-currents since the closed convex cone of cohomology classes of such currents (called the *pseudo-effective cone*) is dual, under the duality between the Bott-Chern cohomology of bidegree $(1,\,1)$ and the Aeppli cohomology of bidegree $(n-1,\,n-1)$, to the closure of the *Gauduchon cone* introduced by Popovici in 2013 and also used in his new approach to Mirror Symmetry.

(C) Another very active area of research at the moment is the theory of geometric flows. The idea goes back to Hamilton and was used spectacularly by Perelman in his resolution of the Poincaré Conjecture. The Ricci flow, the Kähler-Ricci flow and the much more recent Anomaly flow have been staples of complex geometry and geometric analysis for the past fifteen years. Several interesting talks given in this workshop focused on some of these flows and their geometric applications.

3 Presentation Highlights

We will organise the presentation highlights according to the themes outlined above.

Mirror Symmetry and related issues

• **T. Collins** reported on some substantial progress on the SYZ appproach to Mirror Symmetry in joint work with A. Jacob and Y.-S. Lin. He discussed the existence of special Lagrangian torus fibrations on log Calabi-Yau manifolds constructed from del Pezzo surfaces, and some progress towards establishing SYZ mirror symmetry for these non-compact Calabi-Yau manifolds.

• In a similar but more gauge-theoretical vein, **M. Garcia Fernandez** reported on joint work with Rubio, Tipler, and Shahbazi centred on the *Hull-Strominger system and holomorphic string algebroids*. Specificallly, he gave an overview of a new gauge-theoretical approach to the Hull-Strominger system using holomorphic string algebroids.

In the smooth setup, a string algebroid provides an infinitesimal version of a principal bundle for the string group. The main focus of his talk was on the consequences of this approach for the existence and uniqueness problem, as well as for the moduli space metric. The discussion was illustrated by a key example.

Geometric flows and related issues

• There was a very interesting talk by **T. Fei** lying at the interface between geometric flows and mirror symmetry. It reported on some recent progress on the *anomaly flow* in joint work with Z.-J. Huang, D.H. Phong and S. Picard.

Specifically, the Hull-Strominger system describes the geometry of compactifications of heterotic superstrings with flux, which can be viewed as a generalization of Ricci-flat Kähler metrics on non-Kähler Calabi-Yau manifolds.

To overcome the difficulty of lacking the $\partial\bar{\partial}$-lemma, Phong, Picard and Zhang initiated the *Anomaly Flow* programme in a bid to understand the Hull-Strominger system. It has been proved in many cases that the Anomaly Flow serves as an effective way to investigate the Hull-Strominger system and in general canonical metrics on complex manifolds, such as giving new proofs of the Calabi-Yau theorem and the existence of a Fu-Yau solution.

In this talk, T. Fei presented some new progress on the Anomaly Flow, including its behavior on generalized Calabi-Gray manifolds and a unification of the Anomaly Flow with vanishing slope parameter and the Kähler-Ricci flow, which further allows one to generalise the notion of the Anomaly Flow to arbitrary complex manifolds.

• Another interesting talk was given by **L. Vezzoni** about the many links between the various geometric flows and several kinds pf special Hermitian metrics on compact complex manifolds.

Specifically, his talk centred on a *geometric flow of balanced metrics*. Recall that balanced metrics ω, introduced by Gauduchon in 1977, are a kind of dual of Kähler metrics. They are defined by the requirement that $d\omega^{n-1} = 0$ (where n is the dimension of the compact complex manifold), or equivalently that $d^\star_\omega \omega = 0$ (i.e. ω is required to be *co-closed*). Like Kähler metrics, they need not exist, but they exist in far more general situations than Kähler metrics, including on all Kähler manifolds.

Typical examples of balanced manifolds include modifications of Kähler manifolds, twistor spaces over anti-self-dual oriented Riemannian 4-manifolds and nilmanifolds. In his talk, L. Vezzoni discussed a geometric flow of balanced metrics that he had co-introduced with L. Bedulli in 2017. This flow is a generalisation of the Calabi flow to the balanced context. It preserves the Bott-Chern cohomology class of the initial metric and in the Kähler case reduces to the classical Calabi flow.

It was explained in the talk that this flow is well-posed and that its stability around Ricci flat Kähler metrics was emphasized. The talk also focused on a rencent problem in balanced geometry proposed Fino and Vezzoni.

Complex surfaces

The theory of complex surfaces holds a special place in complex geometry and differs in many respects from the geometry of compact complex manifolds of dimension ≥ 3. Compact complex surfaces were classified by Kodaira long ago, but the class VII in Kodaira's classification is still not entirely understood and a lot of effort has been going since at least the early 1980s into trying to complete Kodaira's classification. A whole array of different methods, coming from complex and algebraic geometry, differential geometry, the topology of 4-manifolds and, more recently, the various theories of geometric flows, have been used.

• An interesting talk that provided a link between this theme and the previous one was given by **J. Streets**. It reported on joint work with Y. Ustinovskiy on *generalised Kähler-Ricci solitons on complex surfaces*.

Specifically, generalised Kähler-Ricci solitons are canonical geometric structures on complex, non-Kähler manifolds. In his talk, J. Streets gave a complete classification of such structures on complex surfaces.

• This talk can be compared with the one given by **G. Dloussky** on *smooth rational deformations of singular contractions of class VII surfaces*.

Specifically, he considered normal compact surfaces Y obtained from a minimal class VII surface X by contraction of a cycle C of r rational curves with $c_2 < 0$. Dloussky's main result states that, if the obtained cusp is smoothable, then Y is globally smoothable. The proof is based on a vanishing theorem for $H^2(Y, \Theta)$, where Θ is the dual the sheaf of Kähler forms.

If $r \leq b_2(X)$ any smooth small deformation of Y is rational, and if $r = b_2(X)$ (i.e. when X is a half-Inoue surface), any smooth small deformation of Y is an Enriques surface.

Locally conformally Kähler (lck) geometry

This is another active area of research that was represented by several internationally recognised experts in this workshop.

• **A. Moroianu** lectured on *locally conformally Kähler manifolds with holomorphic Lee field*.

Specifically, a locally conformally K̈hler (lck) manifold is a compact Hermitian manifold (M, g, J) whose fundamental 2-form $\omega := g(J\cdot, \cdot)$ verifies the condition $d\omega = \theta \wedge \omega$ for a certain closed 1-form θ called the Lee form. This talk, based on joint work with F. Madani, S. Moroianu, L. Ornea and M. Pilca, focused on lck manifolds whose Lee vector field (the metric dual of θ) is holomorphic. It was shown that if its norm is constant or if its divergence vanishes, then the metric is Vaisman, i.e. the Lee form is parallel with respect to the Levi-Civita connection of g.

The talk went on to give examples of non-Vaisman lck manifolds with holomorphic Lee field and to explain how all such structures on manifolds of Vaisman type are classified.

• A link between the theory of complex surfaces and lck manifolds was provided by **A. Otiman** who lectured on *a class of Kato manifolds*.

Specifically, she described Kato manifolds, also known as manifolds with a global spherical shell. She revisited Brunella's proof of the fact that Kato surfaces admit locally conformally Kähler metrics and went on to show that this holds for a large class of higher-dimensional complex manifolds containing a global spherical shell.

On the other hand, she explained the construction of manifolds containing a global spherical shell which admit no locally conformally Kähler metric.

She went on to consider a specific class, which can be seen as a higher-dimensional analogue of the Inoue-Hirzebruch surfaces, and to study several of their analytical properties. In particular, she gave new examples, in any complex dimension $n \geq 3$, of compact non-exact locally conformally Kähler manifolds with algebraic dimension $n - 2$, algebraic reduction bimeromorphic to $\mathbb{CP}^{n-2}$ and admitting non-trivialholomorphic vector fields.

Hodge theory

This is another major research theme that featured prominently in this workshop. Among other things, it is related to Mirror Symmetry.

• **M. Verbitsky** lectured on the *deformation theory of non-Kähler holomorphically symplectic manifolds*, based on joint work with N. Kurnosov.

Specifically, in 1995, D. Guan constructed examples of non-Kähler, simply-connected holomorphically symplectic manifolds. An alternative construction, using the Hilbert scheme of Kodaira-Thurston surface, was given by F. Bogomolov.

Verbitsky and his co-author prove the *Local Torelli Theorem*, showing that holomorphically symplectic deformations of BG-manifolds are unobstructed, and the corresponding period map is locally a diffeomorphism.

Subsequently, using the local Torelli theorem, the authors prove the Fujiki formula for a BG-manifold, showing that there exists a symmetric form q on the second cohomology with the same properties as the Beauville-Bogomolov-Fujiki form for hyperkahler manifolds.

Verbitsky's talk tied in with both Stelzig's talk on a new computation method of various types of cohomology and Rollenske's talk on $\partial\bar{\partial}$ Calabi-Yau manifolds. All these three talks fall into the realm of possibly non-Kähler complex geometry.

• Specifically, **J. Stelzig** lectured on *zigzags and the cohomology of complex manifolds*. Deligne, Griffiths, Morgan and Sullivan famously characterised the $\partial\bar{\partial}$-property of compact complex manifolds by the following property:

The double complex of forms decomposes as a direct sum of two kinds of irreducible subcomplexes: 'squares' and 'dots', where only the latter contribute to cohomology.

In his talk, the author explored the implications of the following folklore generalisation of this:

Every (suitably bounded) double complex decomposes into irreducible complexes and these are 'squares' and 'zigzags', with a dot being a zigzag of length 1.

This affords insight into the structure of and relation between the various cohomology groups. When applied to complex manifolds, this yields, among other things, the analogue of the Serre duality for all the pages of the Frölicher spectral sequence, a three-space decomposition on the middle cohomology and new bimeromorphic invariants.

The talk ended with several open questions.

• **S. Rollenske**'s talk, based on joint work with B. Anthes, A. Cattaneo and A. Tomassini, was about *$\partial\bar{\partial}$-complex symplectic and Calabi–Yau manifolds: Albanese map, deformations and period maps*.

Specifically, let X be a compact complex $\partial\bar{\partial}$- manifold with trivial canonical bundle. The $\partial\bar{\partial}$-assumption means that the $\partial\bar{\partial}$-lemma is satisfied by X.

If X is Kähler then, up to a finite cover, X is the product of a simply connected manifold with its Albanese Torus $Alb(X)$, by the Beauville-Bogomolov decomposition theorem. The authors showed that in their more general setting, the Albanese map is still a holomorphic submersion but, in general, does not split after finite pullback.

The authors also showed that the Kuranishi space of X is a smooth universal deformation and that small deformations enjoy the same properties as X. If, in addition, X admits a complex symplectic form, then the local Torelli theorem holds and we obtain some information about the period map. It is in this last part of the talk that a direct link was established to Verbitsky's talk.

4 Conclusions

This workshop was rich in stimulating discussions and enlightening presentations. This report only scratches the surface of what was discussed. We greatly appreciated this opportunity of bringing together mathematicians from diverse backgrounds and the new ideas that arose from their discussions and presentations.

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