Locally Conformal Symplectic Manifolds: Interactions and Applications

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November 7 - 12, 2021

1 Overview of the Field

A *locally conformal symplectic structure* (LCS for short) on a smooth manifold M is a generalisation of a symplectic structure. Locally, a LCS manifold is equivalent to a symplectic manifold, but the local symplectic structure is only well-defined up to homotheties. This gives rise to a global non-degenerate 2-form ω on M which, instead of being closed, satisfies the weaker relation

$$d\omega = \theta \wedge \omega \text{ for a closed 1-form } \theta.$$

The de Rham class $[\theta] \in H^1_{dR}(M)$ determined by the closed 1-form θ is called the *the Lee class* of ω and is the obstruction for the LCS structure to be symplectic. LCS structures with non-trivial Lee class will be referred to as *strict* LCS structures.

They were introduced in the 1970-80's by I. Vaisman [5, 6] and also appeared in the work of Guedira–Lichnerowicz [3]. They were later studied both from the point of view of the symplectic geometry and in the framework of *locally conformally Kähler* (LCK) complex manifolds.

2 Context

2.1 The symplectic side of the theory

The following general existence result strongly motivates the study of LCS structures from the point of view of symplectic topology.

Theorem 1. *[2] Any compact almost complex manifold with non zero first Betti number admits a LCS structure.*

It is thus natural to try to extend various techniques from symplectic geometry to the much more general LCS framework. There is a natural framework of extending the notion of a Hamiltonian group action on a symplectic manifold to the LCS case, and a number of key constructions in the symplectic case generalise accordingly.

2.2 The complex side of the theory

The relevance of the theory of LCS structures to classification problems in complex geometry was amplified by the following general existence result

Theorem 2. *[1] Any compact complex surface (M, J) with odd first Betti number admits a LCS structure ω taming the almost complex structure J.*

On the other hand, in complex dimension higher than 2, the existence of compatible LCK structures on a given complex manifold is much more restrictive and a number of rigidity results have been obtained. For instance, if the LCK metric admits a certain global Kähler potential on its universal cover, we have

Theorem 3. *[4] A compact LCK manifold with proper potential of complex dimension n at least 3, can be holomorphically embedded in a linear Hopf manifold $S^1 \times S^{2n-1}$.*

2.3 Aim of the workshop.

The above mentioned achievements definitively place the theory of locally conformal symplectic manifolds at a busy crossroad of different branches of mathematics: Kähler and complex geometry, Riemannian geometry, symplectic geometry and topology. At the same time, we feel that a number of modern methods in these domains remain to be adapted to the LCS context. Our workshop thus aimed at bringing together established and young mathematicians working in complex analytic, differential and symplectic geometry, as well as in geometric PDE's, in an attempt to produce a road-map for future research in the domain of LCS manifolds.

3 Organisation of the workshop.

The first day of the conference was devoted to introductory talks on the subject. Then the next four days were organised with two talks in the morning and afternoon devoted to discussions and open problems session. During those problem session people would meet on the main virtual room to discussion questions and problems related to the previous talks. Smaller groups would form and discuss in parallel room. The last discussion session was devoted to talk about open problem and questions that aroused during the week and how one question on one side of the field could be interpreted on the other side.

4 Presentation Highlights

The first day of the conference was devoted to introductory talks on the subject:

Kevin Sackel (Stony Brooks) introduced the symplectic side of locally conformal symplectic manifolds. Mélanie Bertelson (Brussels) introduced some of the tools used to prove existence results of such structures (notably flexible methods coming from h-principle). Liviu Ornea gave an overview of locally conformally Kähler metrics and the state of the art of properties of manifolds admitting such structures. For all those talks a slot of one hour and a half were booked to leave room for discussion and questions.

The three next days where organised with three one hour talks a day and some question/discussion session. The last day consisted of two talks and an open problem session. This schedule and organisation allowed enough flexibility for participant to come despite the online format and the fact that they were all spread around the globe (with a good concentration of people in Europe). There was an average number of participant around 25 at every talk and discussion sessions, it is our opinion that this is a good number considering the schedule and the number of online events that have been taken place in the past two years.

The talks were aimed to highlight the dynamism of and the diversity of mathematical tools used in the field, here is a brief summary of what each talks consisted of.

Alexandra Otiman described some special hermitian metric on some non-Kähler Oeljeklaus-Toma manifolds. Yasha Saveliev described a generalisation of Weinstein conjecture using a locally conformal symplectic formulation, he hinted at how holomorphic techniques can lead to proof of this conjecture in some cases. Giovanni Bazzoni discussed the notion of lcs manifolds of the first kind giving a topological classification of such manifolds. He also discussed lcs structures on Lie groups. Yong-Geun Oh discussed adaptation of

holomorphic curves techniques in lcs manifolds (with a special focus on those of the first kind). Anna Fino discussed the notion of stron Kähler manifold with torsion, she explained how any complex manifolds admit such a structure. Georges Dloussky explained how any complex surface admits an lcs structure (that is not necessarily lck). He related then existence of plurisub-harmonic function to proof of the GSS conjecture. Jean-François Barraud gave a definition of the Novikov fundamental groups (generalising the notion of Novikov homology). He explain how Floer theory on symplectic isotopies with small flux allows to compute such Novikov fundamental groups. Nicolina Istrati reported on work on Kato Manifolds that are manifold admitting a GSS. She described constructions of such manifolds. She then characterised those which admit LCK metric. Gaël Meignez explained a proof of a general existence result of lcs structure on manifold endowed with a non exact closed 1-form and non degenerate 2-form. Mihaela Pilca talked about conformal vector field on lcK manifolds and explain a result proving that every conformal vector field is Killing for a particular metric in the conformal class of the lcK metric. Viviana del Barco described a necessary condition for a nil-manifold to admit a symplectic structure. This implies a classification symplectic nil-manifolds in some restricted cases.

5 Outcome of the Meeting

Despite the online format there was a serious involvement of the speakers and the participants and all talks were lively. The long session of discussions each days of the conference really helped to break the ice and stimulated mathematical discussions aroused from those. On the last day it was clear that some important questions could be tackled both from the complex side and the symplectic side of story. On the complex side one can points that many questions were raised about the topological implication of the existence a potential on LCK manifolds. For the symplectic side general questions about finding a notion of tightness on lcs manifolds that would rigidify the theory. On both side questions about construction of lck/lck manifolds were raised. On all those aspects smaller group of person discussed about possibles results and there are no doubt that in a near future some article by participants will address some of those questions .

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