# Singularities Formations in Nonlinear PDEs (Online)

Robert Jerrard (University of Toronto),
Monica Musso (University of Bath),
Juncheng Wei (University of British Columbia)

September 26–October 1, 2021

## 1 Introduction

This five-day workshop gathered world-wide experts in formation of singularities in several different mathematical models that involve partial differential equations (PDEs). These equations arise in situations as diverse as the long-term evolution of temperature and winds on the earth's surface due to climate change, the spreading of a tumor, changes in the behavior of light near a black hole, the evolution of stock markets or the shape an ice mountain takes as it melts. Solutions to PDE can be interpreted as attainable situations. They may exhibit ”singularities”, namely places or instants where they ”blow-up” or exhibit irregular behavior. Examples abound in nature: one may think of the breaking of a wave, the formation of black holes, meteorological phenomena such as tornados; similar phenomena are found in physical systems such as liquid crystals and superconductors. It is of enormous interest to predict ”how” and ”when” singularities occur, since they indicate situations where the original model may collapse. Similar PDEs can model natural phenomena that appear to be completely different, which makes them an intriguing and complex object of study.

The main purpose of this workshop is to share high quality information related to the contemporary research in the fields of partial differential equations, dispersive equations and geometric flows, with the objective of working on common research problems and favoring the formation of students and young researchers.

The meeting brought together two groups of mathematicians: one working on geometric partial differential equations (PDEs), especially in construction and classification of concentrations in semilinear elliptic equations including Allen-Cahn, Ginzburg-Landau equations, and a second group, in fluid mechanics, including specialists in dispersive equations, 3D Euler and Navier-Stokes equations, and fluid mechanics. The main goal of the workshop was to explore deeper the connections between these two fields and inspire participating researchers into new developments, exposing them to methods and phenomena in singularity formations discovered separately in the geometric PDE and fluid equations sides.

## 2 Overview and Recent Developments in the Field

Several partial differential equations (PDEs) and many geometric flows are characterized by the formation of singularities in their solutions. This occurs when solutions either become concentrated on low-dimensional sets, implying a loss of regularity, or some expressions that depend on the solution become arbitrarily large (blow-up in time). Detecting or ruling out formation of singularities is one of the most fascinating and difficult topics in modern advanced research in Mathematics. One of the seven Millennium Problems of the Clay Institute concerns the global in time regularity of solutions to the classical Navier-Stokes equations. The key obstacle for regularity is the potential formation of singularities. The same question arises for the Euler equations of fluid mechanics and numerous problems with physical or mathematical background. Singularity formation is a central aspect also in the study of geometric flows. Understanding singularities in the Ricci flow was a key element in Perelman's proof of the Poincaré conjecture. Singularity formation also arises in static problems, in solutions that break down when some parameter of the problem approaches a limiting value. This is the typical case in phase transition problems. Detecting singularities is a difficult task. It requires a detailed understanding of approximate solutions exhibiting the formation of singularities, for example in the form of asymptotic expansions, and a mathematical framework to prove the existence of full solutions rather than approximate ones, for instance by means of perturbative arguments.

The four selected topics covered in the workshop are the following.

**Blow-ups in energy critical parabolic and dispersive equations**. Many nonlinear parabolic and dispersive equations have solutions that develop singularities as time tends to either a finite value $T$ or infinity. It is of general interest to study whether and how singularities appear, such as rate, profile and stability. Relevant questions include: large time asymptotic for global solutions, decomposition of generic solutions into sums of decoupled solitons in non-integrable settings, description of critical phenomenon for blow up in the Hamiltonian situation, stable or generic behavior for blow up on critical dynamics, global existence for defocusing supercritical problems and blow up dynamics in the focusing cases, global existence for large data energy critical wave maps and profile decomposition, semi-linear wave equations with Ginzburg Landau or Allen-Cahn type nonlinearities with extreme energy concentration along time-like minimal surface in Minkowski space.

**Singularities in Euler equations.** A fascinating field within the theory of nonlinear PDEs is the analysis of the motion of fluids. The classical system of PDEs that models the dynamics of an inviscid, incompressible fluid is given by the Euler equations, first formulated in 1755, relating the vector-valued velocity field of the fluid and its pressure. This problem can be understood as a nonlocal transport equation and its analysis is in many ways extraordinarily subtle. In entire space or in a domain under no-flux boundary conditions existence of a unique smooth solution global in time for smooth initial data holds in 2 dimensions. On the other hand, delicate oscillatory phenomena inherent to the problem give rise to non-uniqueness of weak solutions. In 3 dimensions, whether finite time singularities may arise is one of the main unsolved problems in PDE theory. Relevant questions include: desingularization of vortex dynamics, dynamics of vortex filaments, vortex rings and fronts in Euler flows, finite-time blow-up for Boussinesq equations, vortex filament for Gross-Pitaevskii equation of superfluidity when dimension is 3.

**Singularities in Geometric Flow.** Geometric flows have shown to be powerful tools to solve fundamental problems in Riemannian Geometry. The Yamabe problem consists in finding metrics on a given Riemannian manifold with constant scalar curvature. This is a variational problem and it has been solved using refined techniques from Nonlinear Analysis. An alternative approach that has been exploited is to follow the associated flow: this method of proof is to begin with any Riemannian manifold and flow it using the heat flow for the Yamabe energy, known as the Yamabe flow, to obtain the constant scalar curvature metric for which one is searching, provided the flow is defined for all time and no singularity or blow-up are formed. Singularity formation for geometric flows thus becomes a central issue, apart from being a challenging and interesting problem in its own. Relevant questions include: singularity formation in the harmonic map flow and possible type II blow-ups, and the analysis of their asymptotic stability, type II blow-up for the Ricci flow, and the possibility of continuation after blow-up, new constructions of type II blow-up for the mean curvature flow.

**Sharp interfaces and singularities in evolutionary equations.** In 1978 the famous Italian mathematician Ennio De Giorgi made the following conjecture: if the solution to the Allen-Cahn equation is monotone in one direction, then the solution is essentially one-dimensional, at least when dimension is less than 9. The counterexample by del Pino, Kowalczyck and Wei shows indeed that the critical dimension is at most 9: for this dimension, and higher, they found a monotone solution to the Allen-Cahn equation which is not one-dimensional. This solution exhibits a highly concentrated interface, passing from being identically 1 to being identically -1 (up to exponentially small errors) in a very tiny region, which can be though as a thin neighborhood of the Bombieri De Giorgi Giusti minimal surface in dimension 8. Quite interesting results have been obtained in recent years on solutions with sharp interfaces for parabolic and hyperbolic equations, depending on a small parameter, with Allen-Cahn or Ginzubur-Landau type of non-linearity. Relevant questions include: finite time and infinite time evolution for the interfaces, multiple interfaces, asymptotic estimates, stability. Recently there are exciting developments on the use of singularly perturbed Allen-Chan in finding minimal surfaces on manifolds, thereby proving Yau's conjecture. See the work of Chodosh and C. Mantoulidis and their groups.

# 3 Presentation Highlights

In accord with the main goals of the meeting, the program included lectures by specialist in various aspects of singularity formations in nonlinear PDEs. Specifically, several lectures were devoted to the potential singularities of the classical Euler and Navier-Stokes systems (M. del Pino, T. Hou, Kieselev). Related lectures on turburlence and blow-up mechanism on Euler include (C. Collot, N. Masmoudi, J. Hao, J. Wu). There are also singularity formations studies in critical elliptic, heat and dispersive equations from different groups (elliptic: Pistoia, parabolic: Souplet, Wang, Fila, King, Davila, dispersive: J. Krieger, W. Schlag, Pusateri, ...)

Sharp interfaces and singularity formations also appear in geometric flows and evolutionary equations (Allen-Cahn, Ginzburg-Landau). Classification of singularity formation on mean curvature flows are discussed (Panagiota Daskalopoulos, Mantoulidis). Higher-dimension concentration sets are difficult subjects to study and some important progresses are made (F.H. Lin, Contreras, C. Wang, Y. Sire).

Some highlights of the talks include

- Del Pino's proof of Leapfrogging dynamics for 3D Euler
- Hou's numerical evidence of potential singularity of 3D incompressible Euler equations and the nearly singular behavior of 3D Navier-Stokes equations
- Daskalopoulos' classification of Velazquez Type II cone singularity in Mean curvature flow
- Chodosh's computation of the p-widths on surfaces, using in particular Liu—Wei's analysis of entire solutions to the sine-Gordon equation on the plane. In particular, he proved that the p-widths on a surface correspond to immersed geodesics (instead of geodesic nets) and computed the entire p-width spectrum of $S^2$ yielding the constant in the Liokumovich—Marques—Neves Weyl law in this dimension.
- Contreras' proof of the existence of local minimizers of Ginzburg-Landau theory with the number of vortices to a power of $\frac{1}{\epsilon}$.

# 4 Scientific Progress Made

The speakers brought many very interesting results to the conference. These were presented in their lectures and discussed right after or during times reserved for more informal discussions. Some of the theorems presented were brand new results, not yet available in the literature. The dissemination of the such results was one of the main goals of the conference. We also consider it a major achievement of the meeting that somewhat disjoint groups of scientists working on related problems exchanged their points of view. Combining strengths of various approaches will inevitably lead to progress in these research projects.

In the following we summarize the main points of individual talks. Also, if available, we provide references to publications containing detailed proofs of some results presented in the lecture; or we quote other papers of the lecturer where similar techniques have been used.

- **Blow-ups in energy critical parabolic and dispersive equations**
  - **Joachim Krieger**: *Recent developments in singularity formation of nonlinear waves*
    Joachim discussed some recent results and formulated some conjectures on singularity formation in the context of geometric wave equations.
  - **Charles Collot**: *On the derivation of the Kinetic Wave Equation in the inhomogeneous setting*
    Charles considered the kinetic wave equation arising in weak wave turbulence theory. He discussed its derivation as an effective equation from dispersive waves with quadratic nonlinearity for the microscopic description of a system. He approached the problem of the validity of this kinetic wave equation through the convergence and stability of the corresponding Dyson series. He identified certain nonlinearities, dispersion relations, and regimes, and for which the convergence indeed holds almost up to the kinetic time (arbitrarily small polynomial loss).
  - **Fabio Pusateri**: *Internal modes and radiation damping for quadratic KG in 3d*
    Fabio considered quadratic Klein-Gordon equations with an external potential $V$ in $3+1$ space dimensions. He addressed the question of whether such solutions persist under the full nonlinear flow. He showed that all small nonlinear solutions slowly decay as the energy is transferred from the internal mode to the continuous spectrum, provided a natural Fermi golden rule holds.
  - **Jiahong Wu**: *Stabilization and prevention of potential singularity formation*
    Jiahong presented two examples of the smoothing and stabilizing phenomenon for coupled PDE systems that prevents potential finite-time singularity formation.
  - **Angela Pistoia:** *Critical Lane-Emden systems*
    Angela presented some recent results concerning non-degeneracy, existence and multiplicity of solutions to a Lane-Emden critical system.
  - **Philippe Souplet:** *Some recent Liouville type results and their applications*
    Philippe discussed some recent Liouville type theorems and their applications. The problems under consideration include the Lane-Emden equation and the diffusive Hamilton-Jacobi equation.
  - **John R. King**: *Some blow-up and post-blow-up results for quasilinear reaction diffusion*
    John presented some formal asymptotic results on blow-ups for quasilinear reaction diffusions.
  - **Marek Fila**: *Solutions with snaking singularities for the fast diffusion equation*
    Marek presented construction of solutions of the fast diffusion equation, which exist for all time and and are singular on some curve $\Gamma(t)$. Some precise description of the behavior of the solutions near $\Gamma(t)$ are given. .
  - **Christian Seis**: *Leading order asymptotics for fast diffusion on bounded domains*
    Christian presented results on quantification of the rate of convergence to rescaled solutions to subcritical steady-states for the fast-diffusion equations uniformly in relative error, showing the rate is either exponentially fast (with a rate constant predicted by the spectral gap) or algebraically slow (which is only possible in the presence of zero modes).
  - **Liqun Zhang**: *The blow up solutions to Boussinesq equations on R3 with dispersive temperature*
    Liqun presented construction of $C^{1,\alpha}$ finite time blow-up solution for Boussinesq equations where temperature has diffusion and finite energy.
  - **Juan Davila**:*Blow-up for the Keller-Segel system in the critical mass case*
    Juan considered the Keller-Segel system in the plane with an initial condition with suitable decay and critical mass $8\pi$. He found a (non-radial) function $u_0$ with mass $8\pi$ such that for any initial condition sufficiently close to $u_0$ and mass $8\pi$, the solution is globally defined and blows up in infinite time. The profile and rate of blow-up are also found.
  - **Jean Dolbeault**: *Two non-conventional inequalities*
    Jean gave a survey on two inequalities: (1) Reverse Hardy-Littlewood-Sobolev inequalities, (2) Two-dimensional logarithmic inequalities.

  - **Kelei Wang**: *Nonexistence of Type II blowups for an energy critical nonlinear heat equation* Kelei considered the energy critical heat equation in dimensions 7 or higher. He showed that any positive blow-up must be of Type I. Some ideas from geometric measure theory and a reverse inner-outer gluing mechanism are used.

- **Singularities in Euler equations** .

  - **Manuel del Pino**: *Dynamics of concentrated vorticities in 2d and 3d Euler flows*

    Manuel's talk considered a classical problem that traces back to Helmholtz and Kirchhoff on the understanding of the dynamics of solutions to the Euler equations of an inviscid incompressible fluid, when the vorticity of the solution is initially concentrated near isolated points in 2d or vortex lines in 3d. He discussed some recent results on the existence and asymptotic behaviour of these solutions, and described, with precise asymptotics, interacting vortices, and travelling helices. Finally he presented a recent result on rigorously establishing the law of motion of "leapfrogging vortex rings", originally conjectured by Helmholtz in 1858.

  - **Thomas Hou**: *Potential singularity of 3D incompressible Euler equations and the nearly singular behavior of 3D Navier-Stokes equations*

    Thomas gave a survey on potential singularity for 3D incompressible Euler and Navier-Stokes equations. He first discussed the finite time blowup of the 2D Boussinesq and 3D Euler equations with $C^{1,\alpha}$ initial velocity and boundary, extending Elgindi's recent work on the Euler singularity. Then he presented some new numerical evidence that the 3D incompressible Euler equations with smooth initial data develop a potential finite time singularity at the origin.

  - **Alexander Kiselev**: *Boundary layer models of the Hou-Luo scenario*

    Alexander discussed a 2D extension of the 1D Hou-Lou equation which models singularity formation for the 3D Euler equation. By exploring this 2D model some insights into the mechanics of boundary layer where extreme growth of vorticity is observed are obtained. Furthermore he isolated a regularization mechanism and built a simplified model around it which is globally regular.

  - **Hao Jia:** *Some recent progress on asymptotic stability for shear flows and vortices*

    Jia gave a survey on some recent work on nonlinear asymptotic stability of the two dimensional incompressible Euler equations, with a focus on shear flows and vortices. Some open problems are also discussed.

  - **Nader Masmoudi**: *Recent advances in the Nonlinear inviscid damping*

    Nader considered some recent developments in the inviscid damping theory of 2D Euler. He presented some new results including the optimality of the Gevrey regularity required to get the nonlinear damping as well as the generalization to the asymptotic stability around more general monotone flows.

  - **Slim Ibrahim:** *Revisit singularity formation for the inviscid primitive equation*

    Slim discussed recent progress on qualitative properties of singularity formation to the primitive equation.

- **Singularities in Geometric Flow**.

  - **Yong Yu:** *Patterns in spherical droplets*

    Yong introduced the spherical droplet problem in the Landau-de Gennes theory. With a novel bifurcation diagram, he found solutions with ring and split-core disclinations, which theoretically confirms the numerical results of Gartland and Mkaddem in 2000.

  - **Carlos Román:** *Vortex lines in the 3D Ginzburg-Landau model of superconductivity.*

    Carlos presented a sharp estimate on the first critical field of the applied magnetic field above which the vortex lines occur. He also reported some studies on the onset of vortex lines and derivation of an interaction energy for them.

- **Changyou Wang:** *Partial regularity of a nematic liquid crystal flow with kinematic transport effects*

  Changyou considered an non-corotational Beris-Edwards Q-tensor Ericksen vectorial model that Includes kinematic transport parameters for molecules of various shapes and show that there exists a global weak solution in dimension three, which is smooth away from a closed set with Hausdorff dimension at most 15/7.

- **Christos Mantoulidis:** *Mean curvature flow with generic initial data*

  Christos discussed why the mean curvature flow of generic closed surfaces in $\mathbb{R}^3$ avoids asymptotically conical and non-spherical compact singularities. He also discussed why the mean curvature flow of generic closed low-entropy hypersurfaces in $\mathbb{R}^4$ is smooth until it disappears in a round point.

- **Yannick Sire**: *A new Ginzburg-Landau approximation for the heat flow of harmonic maps with free boundary and partial regularity of weak solutions*

  Yannick reported on recent results on a new approximation of harmonic maps with free boundary which allows to better capture the boundary behavior and construct weak solutions of the associated heat flow. He presented a small energy criterion which allows to prove partial regularity of the solutions.

- **Panagiota Daskalopoulos**: *Type II smoothing in Mean curvature flow* Panagiota studied the Type II smoothing in mean curvature flow in the setting of Velazquiz's cone singularity. She established the short time existence of Velázquez' formal continuation, and verified that the mean curvature is also uniformly bounded on the continuation.

- **Sharp interfaces and singularities in evolutionary equations.**

  - **Wilhelm Schlag:** *Asymptotic stability for the Sine-Gordon kink under odd perturbations*

    Wilhelm described the recent asymptotic analysis on the Sine-Gordon evolution of odd data near the kink. The proofs do not rely on the complete integrability of the problem in a direct way, in particular do not use the inverse scattering transform.

  - **José A. Carrillo:** *Nonlocal Aggregation-Diffusion Equations: entropies, gradient flows, phase transitions and applications*

    Jose gave an overview of recent results understanding the bifurcation analysis of nonlinear Fokker-Planck equations arising in a myriad of applications such as consensus formation, optimization, granular media, swarming behavior, opinion dynamics and financial mathematics. He presented several results related to localized Cucker-Smale orientation dynamics, McKean-Vlasov equations, and nonlinear diffusion Keller-Segel type models in several settings.

  - **Ping Zhang:** *On global hydrostatic approximation of hyperbolic Navier-Stokes system with small Gevrey class 2 data*

    Ping considered a hyperbolic version of the Navier-Stokes equations obtained by using Cattaneo heat transfer law instead of Fourier law, evolving in a thin strip. The formal limit of these equations is a hyperbolic Prandtl type equation. He proved the existence and uniqueness of a global solution to these equations under a uniform smallness assumption on the data in Gevrey 2 class.

  - **Otis Chodosh:** *The p-widths of a surface*

    Otis discussed recent work concerning the p-widths on surfaces, using in particular Liu—Wei's analysis of entire solutions to the sine-Gordon equation on the plane. In particular, he proved that the p-widths on a surface correspond to immersed geodesics (instead of geodesic nets) and computed the entire p-width spectrum of $S^2$ yielding the constant in the Liokumovich—Marques—Neves Weyl law in this dimension.

  - **Yoshihiro Tonegawa:** *Existence of canonical multi-phase mean curvature flows*

    Yoshihiro presented a recent existence result for multi-phase Brakke flow starting from arbitrary partition with locally finite co-dimension 1 Hausdorff measure which improves on his own work with Lami Kim in 2017. The new aspect is that the flow has a character of BV solution, a notion introduced by Luckhaus-Sturzenhecker in 1995, in addition to being a Brakke flow.

– **Andres Contreras**: *Stable vortex configurations with unbounded vorticity in Ginzburg-Landau theory*

  Andres presented results on the existence of local minimizers of Ginzburg-Landau theory with prescribed vorticity for a wide range of external fields and treating for the first time a number of vortices comparable to a power of $\frac{1}{\epsilon}$.

– **Tai-Peng Tsai:** *Finite energy Navier-Stokes flows with unbounded gradients induced by localized flux in the half-space*

  Tai-Peng gave explicit global pointwise estimates to the Stokes system in the half space. Then he used the above solution as a profile to construct solutions of the Navier-Stokes equations which also have finite global energy and unbounded normal derivatives due to the flux.

– **Fang-Hua Lin**: *Relaxed Energies, Defect measures and Minimal Currents*

  After a brief discussion for harmonic map problems from a three-ball into the two-sphere, Fanghua gave a review on an open problem posed by R.Schoen, the notions of relaxed energy, minimal connection and some results in the late 1980s by several groups. Then he presented results on a higher dimensional version of these studies, and also a solution to an open problem proposed by Brezis-Mironescu recently.

# 5 Outcome of the Meeting

The workshop brought together four groups of mathematicians. The first group work on traditional elliptic/geometric PDEs, especially in construction and classification of concentration solutions of semilinear elliptic equations/systems. The second group, works with parabolic PDEs, specialists in critical heat equations and mean curvature and geometric flows. The third group works on dispersive PDEs, like wave equations and kinetic PDEs. The fourth group work on physical fluids, emphasizing numerical and physical motivations. This workshop provided a good chance for these four groups of researchers to meet and interact with each other. It gave many opportunities to continue existing collaborations and to stimulate new mathematical ideas. Many participants saw new ideas arising from interaction with others who brought their own perspectives. A number of new collaborations appear to have arisen from the meeting, and the work of other collaborations was advanced. The large number of open problems that were presented in the talks will certainly encourage research in the area by participants and their Ph.D. students or Postdoctoral researchers.