Algebraically integrable domains

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The research efforts of our team were dedicated to algebraically integrable domains. The first study of such domains goes back to Newton and is connected to Kepler's laws of planetary motion. Let K be an infinitely smooth bounded domain in $\mathbb{R}^n$. For a non-zero vector ξ in $\mathbb{R}^n$ and a real number t, consider the cut-off functions of K':

$$V_K^+(\xi, t) = \mathrm{vol}_n(K \cap \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq t\}),$$
$$V_K^-(\xi, t) = \mathrm{vol}_n(K \cap \{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq t\}).$$

The domain K is called algebraically integrable if the two-valued function $V_K^{\pm}$ is an algebraic function, which means that there is a polynomial P of $n+2$ variables such that

$$P(\xi_1, \xi_2, \cdots, \xi_n, t, V_K^{\pm}(\xi, t)) = 0$$

for all ξ and t such that $\langle x, \xi \rangle = t$ intersects K.

In Lemma XXVIII of his Principia [7], Newton proved that there are no algebraically integrable ovals in $\mathbb{R}^2$. Three centuries later, Arnold asked for extensions of Newton's result to other dimensions and non-convex domains; see problems 1987-14, 1988-13, and 1990-27 in his book [5]. Arnold's problem in even dimensions was solved by Vassiliev [8], who showed that there are no algebraically integrable bounded domains with infinitely smooth boundaries in $\mathbb{R}^{2n}$. It is still an open problem whether in odd dimensions the only algebraically integrable smooth domains are ellipsoids.

In order to attack this problem Agranovsky [1] introduced a related concept of polynomially integrable domains. Let K be a bounded domain in $\mathbb{R}^n$. The parallel section function of K in the direction $\xi \in S^{n-1}$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{\langle x, \xi \rangle = t\}), \qquad t \in \mathbb{R}.$$

We say that K is polynomially integrable if there is an integer N such that

$$A_{K,\xi}(t) = \sum_{m=0}^{N} a_m(\xi) t^m \tag{1}$$

for all ξ and t such that $\langle x, \xi \rangle = t$ intersects K.

It is not difficult to see that ellipsoids are polynomially integrable in odd dimensions (but not in even). Agranovsky asked whether these are the only polynomially integrable domains in Euclidean spaces. He also gave a partial answer to this question: there are no polynomially integrable bounded domains with smooth boundaries in even dimensions. In odd dimensions, Agranovsky proved that such domains must be convex and obtained some results towards the affirmative answer. Koldobsky, Merkurjev, and Yaskin [6] showed that the only infinitely smooth polynomially integrable convex bodies in odd dimensions are ellipsoids, thus completing the solution of the problem. In [2] Agranovsky partially solved Arnold's problem (for the so-called domains free of real singularities) by reducing it to the polynomially integrable case.

As we mentioned above, the parallel section function of ellipsoids is not polynomial in even dimensions. However, the square of this function is polynomial for ellipsoids in all dimensions. Thus it is natural to ask if ellipsoids are the only bounded domains in $\mathbb{R}^n$ such that

$$(A_{K,\xi}(t))^2 = \sum_{m=0}^{N} a_m(\xi) t^m$$

for all ξ and t such that $\langle x, \xi\rangle = t$ intersects K.

In $\mathbb{R}^2$ this question has been answered affirmatively by Agranovsky [4] for domains with algebraic boundaries. In higher dimensions the problem is open. During our time at BIRS we worked on this problem by identifying possible approaches via the Fourier transform and connections to floating bodies.

In addition to the question described above we also worked on closely related problems. For example, Agranovsky in his paper [3] suggested to study unbounded surfaces that are polynomially integrable near their boundaries. He conjectured that only quadrics in odd dimensions can have this property.

Another question was raised by Yaskin in [9]. Let K be a convex body in $\mathbb{R}^n$. Consider the following generalization of the parallel section function.

$$A_{K,m,\xi}(t) = V_m(K \cap \{\langle x, \xi\rangle = t\}), \qquad t \in \mathbb{R},$$

where V_m is the m-th intrinsic volume. If $m = n-1$, this definition coincides with the one given above. If $m = n-2$, this function gives the surface area of the relative boundary of $K \cap \{\langle x, \xi\rangle = t\}$.

It can be checked that for ellipsoids $A_{K,m,\xi}(t)$ is a polynomial in t precisely when m is even. However, it is an open problem whether ellipsoids are the only bodies with this property when m is even.

As we mentioned above, there are no polynomially integrable domains in even dimensions. It is natural to ask what conditions (similar to (1)) we should imposed on the parallel section function to obtain ellipsoids in even dimensions. We were able to identify such a condition, and based on the results obtained in Banff, we are preparing a manuscript tentatively titled "Polynomially integrable domains in even-dimensional spaces".

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