Banff workshop 22w5025: At the Interface of Mathematical Relativity and Astrophysics

Edgar Gasperín (Instituto Superior Técnico, Lisbon),
José Luis Jaramillo (University of Bourgogne-Franche Comte),
Badri Krishnan (Radboud University),
Juan A. Valiente Kroon (Queen Mary, University of London)

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1 Overview of the Field

Mathematical relativity is concerned with the systematic study of the topological, geometric and analytic properties of generic solutions to the equations of the theory of general relativity —the Einstein field equations. This area of research has been of fundamental importance to explore the physical content of the relativistic theory of gravity formulated by Einstein. On the other hand, the detection of gravitational waves from the merger of black holes and neutron stars, has opened the remarkable possibility of verifying (or disproving) the predictions of mathematical relativity using observational data and numerical simulations.

General relativity is one of the pillars of mathematical physics. As such, it makes use of ideas and methods from diverse areas of mathematics —in particular differential geometry, topology and the theory of partial differential equations. Since its inception, general relativity has been a continuous source of challenging mathematical problems and conjectures which have sparkled the development of novel mathematical techniques and, even, whole research topics. A particular example of this interaction is the notion of mass in general relativity originally arising from physical considerations, which eventually led to one of the milestones of mathematical relativity —the proof of the positivity of the mass by Schoen & Yau [22] and, independently, Witten [25]. This result brings to the fore the intimate relation between relativity and geometric analysis and kick-started the rigorous study of *geometric inequalities*. Another example of this interaction is the use of ideas from differential topology pioneered by Penrose to study gravitational collapse [18] which eventually led to a number of singularity theorems —see e.g. [19]. The key ingredient of the singularity theorems, namely trapped surfaces, have led to a new formulation of all facets of black hole physics with important applications in gravitational wave astronomy, numerical relativity and quantum gravity [20]. A final example of current relevance of the interaction between general relativity and mathematics concerns Lorentzian geometry and the theory of hyperbolic partial differential equations in which the need to understand the physically crucial issues of the stability and the late-time behaviour of dynamic black holes has driven the development of novel mathematical ideas and techniques —see e.g. [7] and references within.

The detection of gravitational radiation from coalescing black holes and neutron stars [8, 9, 10, 12, 11] has led to the remarkable possibility of testing or disproving many of the predictions of mathematical relativity; see e.g. [13, 14]. Research in the mathematical aspects of general relativity is driven to a large extent by the role of this theory in astrophysics —see e.g. [17]. Most of the central problems in the subject are deeply rooted on physical considerations. In the last couple of decades a natural bridge between mathematical relativity and astrophysics has been numerical relativity —i.e. the study of solutions to Einstein equations by means of computer simulations; see e.g. [1, 2] for introductory accounts to this fast-evolving subject. The need to obtain numerically stable and long running simulations, and extracting physical gauge invariant information, fostered further interactions between physicists, astrophysicists and mathematicians. The aftermath of the numerical resolution of the binary problem of black holes by Pretorious saw the conclusion of a longstanding challenge [21]. However, at the same time, it saw the split of the communities of mathematical and astrophysical relativity.

A departure point between the mathematically and astrophysically oriented community in general relativity has been the analysis of the Einstein field equations from a linear or a non-linear perspective. While some phenomena are well captured by the linearised equations, there are effects which have roots in the non-linearities of the Einstein field equations. A promiment example of the latter is the memory effect which can be split into linear and non-linear memory effects, these should ultimately result in a mesurable imprint in the wave form of gravitational waves generated by astrophysical events, hence putting in contact theory, numerics and observations.

2 Recent Developments and Open Problems

The last decade has seen spectacular developments in gravitational wave science in which the detection of the coalescence of astrophysical black holes and, more recently, neutron stars is becoming routine and, in an exciting more recent development, researchers have gained the technical ability of obtaining images of black holes and detecting their shadows. As such, the organising team of this workshop felt that it was time now to bring the communities of mathematical and astrophysical relativity together in a renewed and, hopefully, invigorating dialogue. Both communities enter this new era with a renewed technical and conceptual toolkit. On the mathematical side, there is a clear focus on the use of advanced techniques from the theory of partial differential equations: in particular, geometric analysis and asymptotic analysis. On the astrophysics side, research enjoys a spectacular period supported by three main efforts: the advent of gravitational wave astrophysics; the observational access to imaging the strong field regime in the neighbourhood of black horizons and finally, the consolidation of cosmology as a high precision subject. Mathematical relativity is at a crossroad. We believe that it would be healthy for the field to reassess its goals. In particular, this involves identifying fundamental astrophysical questions for which a coherent and rigorous mathematical formulation can be provided so that the mathematical community can start its investigation, and also evaluating research areas which do not have any potential of being tested observationally. It may well be that the mathematical insights thus obtained may lead to new questions for the theoretical and observational astrophysical community.

The stated aims of the workshop were:

i. To bring together researchers of mathematical relativity and relativistic astrophysics/gravitational wave science, aiming at identifying areas of potential fruitful interaction for both communities.

ii. Set an agenda, for the coming years, of research in mathematical relativity.

3 Presentations

The workshop was organised around thematic days with three presentations per day and ample time for discussion between talks. Given the time difference with Europe, where most of the online participants were based, the talks were scheduled in the morning, Banff time. At the begining of each day there was a brief presentation by the chair introducing the topic of the day, its context in the current research and the particular objectives. The schedule allows for 20 minutes after each talk for questions and discussions. At the end of the talks, around 10 minutes were devoted to a wrap up of the day and to provide a teaser for the session in the following day.

Additional time for (online) scientific discussions was not scheduled and let the individual participants to arrange them as necessary.

3.1 Day 1: Non-linear stability of black holes and other mathematical topics

A particular research problem has dominated the focus of the mathematical relativity community since the mid 2000's: the nonlinear stability of black holes —in particular, that of the Kerr spacetime. Besides the natural physical importance of the issue, this question has led to the development of new mathematical techniques and the reassessment of ideas and tools that lied buried in the literature. The collective work of a number of groups world-wide in almost over two decades have brought tantalisingly close the aim of a full proof of the nonlinear stability of the Kerr spacetime which, undoubtly will appear in the coming years.

3.1.1 Lars Andersson (Max Planck Institute for Gravitational Physics): Remarks on the Black Hole Stability problem

In this talk based on recent work with Thomas Bäckdahl, Pieter Blue and Siyan Ma, I will review some aspects of our approach to the black hole stability problem. The use of a radiation gauge plays an important role and I will discuss some features of the resulting reduced system.

3.1.2 Rita Teixeira da Costa (Princeton): Mode stability for Kerr black holes

The Teukolsky master equations are a family of PDEs describing the linear behavior of perturbations of the Kerr black hole family, of which the wave equation is a particular case. As a first essential step towards stability, Whiting showed in 1989 that the Teukolsky equation on subextremal Kerr admits no exponentially growing modes.

3.1.3 Stefanos Aretakis (Toronto): Observational signatures for extremal black holes

We will present results regarding the asymptotics of scalar perturbations on black hole backgrounds. We will then derive observational signatures for extremal black holes that are based on global or localized measurements on null infinity. This is based on joint work with Gajic-Angelopoulos and ongoing work with Khanna-Sabharwal.

3.1.4 Bridge to Day 2

A key particular aspect in the discussions of Day 1 concerned the control of linear perturbations in black hole background spacetimes, as an intermediate stage in the full proof of black hole stability. This point constitutes the link between the discussion of Day 1 and Day 2, the latter devoted to quasi-normal modes, namely one of the key aspects of the linear perturbation theory.

3.2 Day 2: Quasinormal modes

Quasinormal modes constitute a paradigmatic example of 'meeting point' between mathematical and astrophysical relativity since, on the one hand they play a key role in the mathematical understanding and control of certain aspects of the (linear) stability problem and, on the other hand, the quasinormal frequencies are 'a priori' accessible through gravitational wave observations. Day 2 is devoted to cover a set of points in the broad spectrum of quasinormal modes that are of particular relevance in current research. In this sense, Day 2 starts talk addressing the capability to extract quasinormal modes frequencies from actual observational data, with a particular focus on so-called overtones. The second talk addresses a particular issue concerning such overtones, namely their potential instabilities under ultraviolet perturbations in the black hole environment. Finally, the day concludes with an analysis of integrability issues in quasinormal modes, uncovering a hidden (Darboux) symmetry.

3.2.1 Colin Capano (Max Planck Institute for Gravitational Physics): Observational evidence for quasi-normal modes from astrophysical black holes

The LIGO and Virgo interferometers have detected nearly 100 binary black hole mergers to date. The black holes formed by these mergers have provided our first opportunity to directly observe quasi-normal modes (QNMs) emitted by a perturbed Kerr black hole. However, detecting more than the dominant QNM is challenging. It has been claimed that an overtone of the dominant QNM can be detected at the merger of GW150914 (and other events); the detection of a sub-dominant angular mode has been claimed in the merger GW190521. Both of these claims remain controversial, with conflicting evidence presented by different groups. I will review these detection claims and the evidence for each.

3.2.2 Rodrigo Panosso Macedo (Southampton): Pseudospectrum and black hole quasi-normal mode (in)stability

Black hole spectroscopy is as a powerful approach to extract spacetime information from gravitational wave observed signals. However, quasinormal mode (QNM) spectral instability under high wave-number perturbations has been recently shown to be a common classical general relativistic phenomenon. I will discuss these recent results on the stability of QNM in asymptotically flat black hole spacetimes by means of a pseudospectrum analysis.

3.2.3 Carlos Sopuerta (Institute of Space Sciences, Barcelona): Symmetries in the dynamics of perturbed Schwarzschild Black Holes

There are two important physical processes around black holes that can be well described using relativistic perturbation theory: Scattering of electromagnetic and gravitational waves (and other fields) and quasinormal mode oscillations that take place, for instance, after the coalescence of a black hole binary. It is well-known that these physical processes can be described in terms of gauge-invariant master functions. We have analyzed the space of all the possible master functions for the case of non-rotating black holes and we find two branches of solutions. One branch includes the known results: In the odd-parity case, the most general master function is an arbitrary linear combination of the Regge-Wheeler and the Cunningham-Price-Moncrief master functions whereas in the even-parity case it is an arbitrary linear combination of the Zerilli master function and another master function that is new to our knowledge. The other branch is very different since it includes an infinite collection of potentials which in turn lead to an independent collection master of functions which depend on the potential. We also find that of all them are connected via Darboux transformations. These transformations preserve physical quantities like the quasinormal mode frequencies and the infinite hierarchy of Korteweg-de Vries conserved quantities, revealing a new hidden symmetry in the description of the perturbations of Schwarzschild black holes: Darboux covariance.

3.2.4 Bridge to Day 3

Black hole spectroscopy constitutes an emerging and promising major research program aiming at probing the spacetime dynamics and astrophysics of black holes from the analysis of the observed frequencies of quasinormal modes. However, as discussed in this session and without entering into the underlying reasons for such instability, such research program must address the potential instability of quasinormal overtones, that may affect some of the goals in the program. On the other hand, the integrability concepts discussed in this session, in particular from the passage of Darboux transformation to the infinite conserved quantities of Korteweg-de Vries equations, may offer inights into the degrees of freedom responsible for such quasinormal instabilities. In sum, the effort to reconstruct the bulk spacetime features from scattering-like data (such as a quasinormal mode frequencies), as well as the discussion of algebraic structures involving an infinite number of conserved quantities constitute the link between Day 2 and Day 3.

3.3 Day 3: BMS structures and black holes

The scientific thread in Day 3 session is the discussion of the role of asymptotic symmetries as asymptotic algebraic structures providing insights into geometric and astrophysical features of the bulk of the spacetime. In particular, focus is set on BMS symmetries at null infinity, an 'outer' spacetime 'boundary' corresponding to the far radiative wave zone. In the first talk such BMS structure is extended to the 'inner' boundary of black hole spacetimes with a stationary horizon, namely to so-called non-expanding horizons. In this setting physical charges and fluxes are defined (this will be revisited in last talk's of Day 5), and local degrees of freedom ('gravitational radiation') associated to perturbed non-expanding horizons are discussed. Interestingly, the second talk explores/suggests the possibility that the quasinormal mode ultraviolet instability discussed in Day 2 might be understood in terms of such local degrees of freedom associated with BMS symmetries both at null infinity and the black hole horizon. Finally, the last talk extends such BMS algebraic structures to larger asymptotic groups and, in particular, relates the asymptotic dynamics (namely asymptotic Einstein equations) to the representation properties of the resulting asymptotic symmetry.

3.3.1 Jerzi Lewandowski (Warsaw): Gravitational radiation through non-expanding horizons

It is well-known that blackhole and cosmological horizons in equilibrium situations are well-modeled by non-expanding horizons (NEHs). Multipole moments to characterize their geometry will be introduced. A 1-dimensional extension of the BMS group acts on NEH. These symmetries will be used to define charges and fluxes on NEHs, as well as perturbed NEHs (gravitational radiation). They have physically attractive properties. Also, a new quadrupole formula for gravitational radiation through cosmological horison in de Sitter spacetime will be presented.

3.3.2 Edgar Gasperín (CENTRA, Lisbon): Energy scales and black hole pseudospectra: the structural role of the scalar product

A pseudospectrum analysis has recently provided evidence of a potential generic instability of the black hole (BH) quasinormal mode (QNM) spectrum. Such instability analysis depends on the assessment of the size of the perturbations. This is encoded in the scalar product and its choice is not unique. In this talk, we will address the impact of the scalar product choice, founding it on the physical energy scales of the problem. Applications of the scalar product in the QNM problem will be discussed as well as further insights into potentially geometric structures in the QNM problem brought to the forefront by geometric structures in the QNM problem brought to the forefront by the expression for the energy flux at null infinity.

3.3.3 Roberto Olivieri (Paris Observatory): The Weyl-BMS group and the asymptotic gravitational dynamics

Asymptotic symmetries play an important role and have deep implications in our understanding of gravity. After a short review of asymptotic symmetries at null infinity in Einstein gravity, I will introduce the Weyl-BMS group, a recent extension of the original BMS group, and discuss its main properties. I will also show that the asymptotic Einstein's equations can be derived from the requirement that the Noether charges associated to the Weyl-BMS generators form a representation of the Weyl-BMS algebra.

3.3.4 Bridge to Day 4

The focus on BMS symmetries in Day 3 brings about, on the one hand, the relation with gravitational wave memory. Indeed, research in recent years has demonstrated the close relation between gravitational wave memory, BMS symmetries and scattering of infrared gravitons. On the other hand, the associated charges and fluxes make enter quasi-local quantities into the picture. These two points, namely gravitational wave memory and quasi-local quantities in general relativity, constitute the link to Day 4.

3.4 Day 4: Gravitational memory and quasilocal observables

Day 4 is devoted to the discussion of gravitational memory and extended new gravitational effects associated with sources that are not stationary outside at large distances, on the one hand, and to quasi-local quantities characterising geometrically physical quantities such as gravitational energy or the quasi-local evolution of black holes, on the other hand. Both (related) subjects are archetypical examples of the interaction between mathematical relativity and relativistic astrophysics.

3.4.1 Lydia Bieri (University of Michigan): Gravitational Radiation in General Spacetimes

Studies of gravitational waves have been devoted mostly to sources such as binary black hole mergers or neutron star mergers, or generally sources that are stationary outside of a compact set. These systems are described by asymptotically-flat manifolds solving the Einstein equations with sufficiently fast decay of the gravitational field towards Minkowski spacetime far away from the source. Waves from such sources have been recorded by the LIGO/VIRGO collaboration since 2015. In this talk, I will present new results on gravitational radiation for sources that are not stationary outside of a compact set, but whose gravitational fields decay more slowly towards infinity. A panorama of new gravitational effects opens up when delving deeper into these more general spacetimes. In particular, whereas the former sources produce memory effects that are finite and of purely electric parity, the latter in addition generate memory of magnetic type, and both types grow. These new effects emerge naturally from the Einstein equations both in the Einstein vacuum case and for neutrino radiation. The latter results are important for sources with extended neutrino halos.

3.4.2 José M. M. Senovilla (University of the Basque Country): Pure gravitational energy inside an empty ball

Gravity manifests itself as curvature of spacetime, and its strength can be measured by considering the variations of radius, area and volume of small balls with respect to their counterparts in flat spacetime. These variations can actually be put in relation, via the Einstein field equations, with the energy density of matter at the ball's centre. In this talk I will also consider what happens when the matter energy density vanishes. The elementary geometric quantities still feel the effect of pure gravity, leading to variations that should be related to the gravitational strength or, in simple words, to the gravitational energy density. These variations now involve terms quadratic in the curvature that can be appropriately put in connection with the Bel-Robinson tensor. New definitions of quasi-local gravitational energy arise. Some basic examples will be discussed.

3.4.3 Daniel Pook-Kolb (Max Planck for Gravitational Physics): The ultimate fate of apparent horizons in a binary black hole merger

Apparent horizons are routinely used in numerical relativity to infer properties of black holes in simulations of dynamical systems. Advances in numerical methods allowed us to follow these objects into the interior of merging black holes, revealing how the two original horizons connect (non-smoothly) with the remnant horizon. However, this still left the question of their final fate open. In this talk, I will present our most recent results on axisymmetric head-on mergers, showing that the evolution of apparent horizons is much more intricate than previously thought: In the interior of the newly formed common horizon, the original horizons are individually annihilated by unstable horizon-like structures. This completes our picture of how two black holes become one and provides the analog of the famous pair-of-pants diagram of the event horizon now for the apparent horizon.

3.4.4 Bridge to Day 5

Day 5 does not follow uniquely from the discussion in Day 4 but gathers together, among other inputs, elements from the stability analysis of Day 1, the integrability notions of Day 2, the probe of bulk properties from asymptotic data in Day 3 (and Day 2), and quasi-local notions discussed in Day 4. In this sense, Day 4 culminates the week putting together some of the different elements discussed along the workshop.

3.5 Day 5: binary black hole mergers

Day 5 is devoted to the discussion of some of the most challenging aspects underlying the understanding of binary black hole mergers. In spite of the fact that numerical simulations of binary black hole coalescences have become routine calculations for a broad spectrum of groups worldwide, the (qualitative) understanding of the underlying analytic, geometric and physical mechanisms involved in this problem are far from being correctly understood. This session aims at bringing light into some partial aspects, from the tension between non-linear/linear aspects of the ringdown problem (first talk), the elucidation of the universality and simplicity properties of the binary black hole (merger) waveform (second talk) and, finally, the construction of a robust framework for the reconstruction of black spacetimes in a correlation 'Gravitational Wave Tomography' approach (third talk).

3.5.1 Luis Lehner (Perimeter and University of Guelph): Puzzles and/or insights in the RingDown regime of black hole collisions

Understanding the behavior of black hole relaxation to equilibrium is presenting new challenges at theoretical and practical levels. This talk will discuss some recent developments which are raising new (and revisiting old) questions on this topic.

3.5.2 J.L. Jaramillo (Dijon) and B. Krishnan (Radboud): Simplicity in binary black hole merger waveforms

Before the first successful numerical simulations of binary black hole mergers in 2005, it was considered plausible that the gravitational wave signal could have complicated modulations and even be chaotic. After all, general relativity is a non-linear theory and these non-linearities are especially important near the merger. However, the reality is that the signals are so far seen to be rather simple. This does not mean that the signals are trivial, rather that the complications due to e.g. precession, eccentricity etc. are contained in the deviations from a simple underlying model. In this talk, we will propose a reason for this simplicity based on the framework of "singularity theory" developed by Whitney, Arnold and Thom in the 1960s. We shall propose that certain radiative aspects of binary black hole mergers are similar to other common observed physical phenomena such as caustics and rainbows in optics, and this theory provides hints for deeper mathematical structures in binary black hole dynamics.

3.5.3 Abhay Ashtekar (Penn State): Imaging Horizon Dynamics via Gravitational Wave Tomography

When black holes merge (or form by gravitational collapse), we have a common dynamical horizon whose geometry changes dramatically as it settles down to the final Kerr horizon. This evolution can be invariantly characterized by the dynamics of a set of multipoles. Unfortunately, the causal structure of space-time prevents the outside observers from directly witnessing it. However, thanks to Einstein's equations, this dynamics is encoded in the profiles of gravitational waves observed at infinity. Using results presented in previous talks at this workshop, and from joint work with Neev Khera, I will present a transform that reconstructs the late time dynamics of the horizon geometry using gravitational waves at null infinity. Just as one monitors changes in the internal structure of objects from outside using electromagnetic tomography, one can image the horizon dynamics using gravitational waves at infinity.

4 Scientific progress made

In general terms the main scientific output from this workshop is the fact that two communities which have been partially disconnected in General Relativity in recent years, have been able to interact and get to know the progress and current open problems in each area and hence, identify potential points of interaction. We are confident that this first interaction will lead to eventual collaborations (actually this has already started), academic visits or further conferences. This workshop was, arguably, the first event with such specific aim and we expect it to be the first in a regular series of meetings. Moreover, the fact that the workshop had an hybrid format and the talks were recorded and are available on the BIRS website gives a more tangible outcome of the conference where such interaction happened.

As specific points identified in this workshop for boosting the interaction between mathematical and astrophysical relativists we can highlight:

- Disentangling of non-linear and linear mechanisms in the problem of black hole stability and devising of observational strategies to probe and assess them. This requires a close interaction between mathematical relativists (namely analysts) with astrophysicists.

- Assessment of the black hole spectroscopy program, specifically the evaluation of the actual astrophysical content of quasinormal frequencies, with a particular emphasis in the stability properties of overtones. This will involve the simultaneous and combined analysis from the mathematical and the data analysis perspective.

- Enlarging of the standard 'celestial mechanics' approach to the binary black hole problem to a (complementary) 'correlation approach'. This involves the enlarging of mathematical tools to include scattering techniques, in particular from inverse scattering theory. This entails a 'paradigm shift' in which new gravitational wave observables specifically adapted to scattering theory should be devised. This will require the close interaction between analysts/geometers and gravitational wave data analysts.

- Role of algebraic (asymptotic) spacetime structures in the characterization of gravitational local degrees of freedom and the construction of physical observables to be monitored in gravitational wave data.

- Systematic introduction of integrability tools in the analysis of the strong field regime of gravitational dynamics. This is closely related to the study of (asymptotic) spacetime symmetries and we expect a convergence between different communities with distinct approaches and expertise.

- Identification of 'singularity/catastrophe theory' concepts and tools as potentially relevant notions to understand some of the qualitative aspects of gravitational dynamics in the strong field regime.

5 Outcome of the Meeting

Originally, the workshop was envisaged as fully face-to-face with plenty of opportunities for spontaneous interaction between participants. In view of the still prevalent difficulties in international travel we have to settle for a hybrid workshop with a small but enthusiastic team based in Banff. The hybrid format opens the opportunity to open the workshop to a broader audience which is a good thing.

As two of the organisers (EGG & JAVK) are Mexican researchers working in Europe we made use of the opportunity to reach to the Mexican scientific community. For this we scheduled two events (Tuesday, Thursday) outside the scientific programme:

- **Tuesday:** "Equality, diversity and inclusion in Gravity research".

- **Thursday:** "Work perspectives in gravivation in Mexico" (in Spanish).

Besides some of the attendees of the purely scientific part of the conference the two round tables brought the interaction on these topics with PhD students and posdoc researchers from several institutions, among them, students from Instituto Superior Técnico (IST), Institut de Mathématiques de Bourgogne (IMB) and from the Gravitation division of the Mexican Physics Society (SMF).

In summary, as a characterising feature singularising the present meeting from others in the field, we can conclude the identification of a new (and very specific) window for scientific interaction between i) partial differential equation analysts, ii) theoretical physicists specialised in symmetries, representation theory and integrability, iii) numerical relativists and iv) gravitational wave data analysts has been identified in this workshop. New challenges has been proposed and the required tools to address them have been identified. This outlines a work program for the coming years where key topics not discussed in the present (such as spinors and twistor constructions, as a particular but significant instance) should be incorporated. We conclude that a rich and promising interdisciplinary field of research in the boundary between mathematical relativity and astrophysics has put forward in this BIRS meeting.

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