Mathematical aspects of the physics with non-self-adjoint operators

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This workshop aimed at facilitating interdisciplinary collaborations between the mathematical analysis and mathematical physics communities, with a central role played by the theory of non-self-adjoint operators. From the point of view of applications, developing modern physical subjects, such as quantum mechanics with non-Hermitian operators, superconductivity, hydrodynamics, metamaterials and optics were covered. High impact mid-term aspirations included the development of rigorous tools allowing understanding of complex physical phenomena such as revivals, cloaking and turbulence. On the mathematical side, invited participants comprised a mix of experts in mathematical physics, numerical, semiclassical, functional and harmonic analysis, as well as applied mathematics and PDEs.

The workshop was a follow-up of a series initiated 12 years ago:

2010 ESF Exploratory Workshop, Prague (CZ), 27 participants;

2013 ICMS Workshop, Edinburgh (UK), 37 participants;

2015 AIM Workshop, San Jose (US), 23 participants;

2017 CIRM Conference, Marseille (FR), 71 participants;

2021 CIRM Conference, Marseille (FR), 83 participants.

See the dedicated webpage: `http://nsa.fjfi.cvut.cz`. The goal of these meetings including the present BIRS Workshop has been to:

- bring together scientists from different communities who would not ordinarily interact in the same scientific environment, although they tackle non-self-adjoint or other non-standard spectral problems;
- maintain a stable, regularly updated list of open problems and macro-research directions in the field, see `http://nsa.fjfi.cvut.cz/problems.html` for details;
- encourage interactions between participants so that the challenging problems can be discussed and addressed from various perspectives reflecting a diverse expertise and background;
- create a forum to report on the progress in solving these problems and to discuss arising obstacles;
- stimulate collaborations in between research communities with members at various stages of their career.

We strongly believe that these goals were achieved during the BIRS Workshop in July 2022.

1 Overview of the Field

Many physical systems can be described by partial differential equations which in turn generate operators between Banach spaces. A well-known illustration of such interplay is quantum mechanics together with the spectral theory of self-adjoint operators in a Hilbert space. However, in several branches of physics like hydrodynamics, damped systems, quantum resonances, superconductivity or balanced loss/gain materials, the occurring partial differential equations contain non-symmetric terms and thus lead to non-self-adjoint spectral problems.

Due to the lack of tools such as the Spectral Theorem or the Variational Principles, which cannot be formulated in general, non-self-adjoint phenomena are very different from their self-adjoint counterpart and include high spectral instabilities, divergence of expansions in eigenvectors and unreliabile numerical/asymptotic approximations. It has been well documented that the analysis of canonical non-self-adjoint models relies on sets of mathematical tools that are normally fragmented. These include notions of generalized coercivity, semi-classical and microlocal analysis, pseudodifferential operators, method of multipliers and generalized Lieb-Thirring or Hardy-type inequalities.

Specific research direction of current interest in the field, the majority of which were strongly represented this workshop, are the following.

- **Spectral analysis**

  Although the spectrum alone does not encode exhausting information about the structure of a non-self-adjoint operator, the localization of its discrete eigenvalues and essential spectrum is of fundamental importance. In recent years we have seen substantial progress in eigenvalues bounds of Lieb-Thirring type in particular for Schrödinger and Dirac operators with complex potential.

- **Resolvent estimates and pseudospectra**

  The pseudospectra of a linear operator measure its non-self-adjointness. The latter results in typical phenomena such as spectral instability, wild eigenvector basis properties or transient growth of solutions of the associated time evolution problems. By now, well-established results on the resolvent norm in the traditional semiclassical regime employing microlocal analysis and semiclassical WKB have been extended also to some non-semiclassical cases and for operators in relativistic quantum mechanics and damped wave systems. Besides the analytic results, an important aspect is a reliable numerical approximation of pseudospectra of differential operators, so that they can be used to examine properties of concrete physical systems.

- **Evolution problems, non-linearity and dispersive quantisation**

  Stability analysis of solutions of non-linear time-evolution PDEs, e.g. in the context of hydrodynamics or superconductivity, often leads to non-self-adjoint linearizations which normally exhibit non-standard properties. One such a property is the transient growth of solutions followed by an eventual exponential decay. Another is the non-solvability for a dense set of initial data, despite existence of a dense family of eigenfunctions. A quantitative control of such effects is still beyond the reach of currently available methods. In parallel to this, recent results about dispersive quantisation (quantum revival) have come to light in the non-self-adjoint setting. The latter only involve exactly solvable models. Therefore it is natural to ask about the extent to which they can be treated via a perturbative approach.

- **Classical PDE methods and non-self-adjoint linear problems**

  Unlike variational and traditional functional analytical tools, multiplier methods as well as Carleman and Strichartz estimates, are often independent of self-adjointness. Their successful implementation was recently turned into new spectral results leading to showing absence of complex eigenvalues of Schrödinger operator with L^p complex potentials. Despite of the applicability of these powerful techniques, many questions remain about the qualitative aspect of the spectrum such as its nature and stability properties. Also, the question of the extent to which the corresponding operators are close (or not) to their self-adjoint counterpart.

2 Presentation Highlights

2.1 Spectral analysis

Jean-Claude Cuenin reported on the recent progress in estimates on eigenvalues of Schrödinger operators with complex potentials. He presented the recent counterexample to the Laptev-Safronov conjecture found jointly with S. Bögli. He further discussed structural assumptions on the potential leading to improvements of the eigenvalue estimates. In particular, based on the joint work with K. Merz, he showed how the decay assumptions can be weakened under randomization or sparsification of the potential.

Lucrezia Cossetti presented how the method of multipliers, originally developed in the dispersive PDEs context, can be employed in spectral theory. In particular, multipliers were identified as a new tool to establish the absence of point spectrum for both self-adjoint and non-self-adjoint operators. Moreover, the technique allows to detect physically natural repulsive and smallness conditions on the potentials to guarantee total absence of eigenvalues. Based on joint works with L. Fanelli and D. Krejčiřík, the developments of the method were reviewed and recent results concerning self-adjoint and non-self-adjoint Schrödinger as well as relativistic operators were presented. Specifically, matrix-valued Schrödinger operators, Pauli and Dirac types and Schrödinger operators on domains with boundary were discussed.

Martin Vogel (online) reported on results on spectra and in particular eigenvectors localization of noisy non-selfadjoint Toeplitz matrices. First, he reviewed the results on probabilistic Weyl laws for the eigenvalue asymptotics for small random perturbations of non-selfadjoint semiclassical pseudo-differential operators, Berezin-Toeplitz quantizations of compact Kähler manifolds and Toeplitz matrices. Based on the joint work with A. Basak and O. Zeitouni, he showed results on eigenvector localization and delocalization of large non-selfadjoint Toeplitz matrices with small random perturbations. The presented numerical studies and experiments were very helpful to illustrate the occurring phenomena as well as the obtained analytic results.

Zhiqin Lu (online) discussed the essential spectrum of the Laplacian on differential forms. He presented the result, obtained jointly with N. Charalambous, stating that for a complete non-compact Riemannian manifold whose curvature goes to zero at infinity, its essential spectrum of the Laplacian on differential forms is a connected set. In particular, the case of form spectrum when the manifold is collapsing at infinity was studied.

2.2 Resolvent estimates and pseudospectra

Tanya Christiansen presented a new approach, developed jointly with K. Datchev, to studying the low energy behavior of the resolvent of a self-adjoint Schrödinger operator, in particular in the complicated case in dimension two. It was explained that this new strategy is suitable for a large class of perturbations, including some non-self-adjoint ones, of $-\Delta$ on $\mathbb{R}^2$.

Yaniv Almog reported on the joint work with B. Helffer on the stability of symmetric flows in a two-dimensional tunnel. In 2015 Grenier, Guo, and Nguyen have established instability of these flows in a particular region of the parameter space, affirming formal asymptotics results from the 1960s. As a new result, it was presented that these flows are stable outside this region in parameter space. More precisely, the Orr-Sommerfeld operator

$$\mathcal{B} = \Big(-\frac{d^2}{dx^2} + i\beta(U + i\lambda)\Big)\Big(\frac{d^2}{dx^2} - \alpha^2\Big) - i\beta U'',$$

defined on

$$D(\mathcal{B}) = \{u \in H^4(0,1)\,,\ u'(0) = u^{(3)}(0) = 0 \text{ and } u(1) = u'(1) = 0\},$$

has a bounded inverse on the half-plane $\Re\lambda \geq 0$ for $\alpha \gg \beta^{-1/10}$ or $\alpha \ll \beta^{-1/6}$.

Antonio Arnal (online) presented new resolvent estimates for the generator G of the one-dimensional damped wave equation with unbounded damping coefficient $a(x)$. It was shown that the norm of the resolvent operator $\|(G-\lambda)^{-1}\|$ is approximately constant as $|\lambda| \to +\infty$ on vertical strips of bounded width contained in the closure of the left complex half-plane. This complements lower bounds on the resolvent norm based on a pseudomode construction by Arifoski and Siegl. The main step of the proof relies on a precise asymptotic analysis of the norm of the inverse of $T(\lambda) = -\Delta + q(x) + 2\lambda a(x) + \lambda^2$, the quadratic operator associated with G. The results were illustrated by an example with quadratic damping and potential where more explicit information on the spectrum can be obtained.

Tho Nguyen (online) introduced a model of a non-self-adjoint biharmonic operator with $i\mathrm{sgn}$ potential

$$L = \frac{d^4}{dx^4} + i\mathrm{sgn}(x)$$

which is a generalization of the Schrödinger case studied earlier by Henry and Krejčiřík. He presented results on the spectrum of this model and in particular both-sided resolvent estimates based on the analysis of the explicit resolvent kernel. This was followed by new resolvent norm estimates for biharmonic operator with a general complex potential. The strategy of the proof relies on a generalization of a non-semi-classical pseudomode construction for Schrödinger operators by Krejčiřík and Siegl.

2.3 Semigroups, time evolution, stability

Bernard Helffer revisited the proof of the classical Gearhart-Prüss-Huang-Greiner theorem yielding an explicit estimate on the norm of the semigroup $S(t)$ in terms of bounds on the resolvent of the generator. Based on a joint work with J. Sjöstrand and more recently also with J. Viola, he presented new improvements, partially motivated by a paper of D. Wei, and discussed the optimality of the results.

Perry Kleinhenz (online) first gave an overview of classical results for the damped wave equation and then focused on the torus with damping that does not satisfy the geometric control condition. This setup is interesting because two dampings with the same support can result in different decay rates. The behavior of the damping near the boundary of its support determines these decay rates, but it is not clear what properties are of utmost importance. Leading candidates are Hölder regularity and a "derivative bound condition" in which the size of the damping controls the size of the gradient of the damping. Results in two model cases, polynomial and oscillatory damping, were discussed. In this framework a resolvent estimate or quasimode construction for the associated non-self-adjoint stationary operator determine which of these two properties determine the sharp decay rate.

Mariana Haragus (online) reported on new results on nonlinear stability of spectrally stable Lugiato-Lefever periodic waves. The Lugiato-Lefever equation is a nonlinear Schrödinger-type equation with damping, detuning and driving, derived in nonlinear optics by Lugiato and Lefever (1987). In a previous contribution, the linear asymptotic stability of spectrally stable periodic waves to perturbations which are localized, i.e., integrable on the real line, were studied. In the presentation, it was shown how the results found at the linear level can be used to analyse the asymptotic nonlinear stability of these periodic waves. In particular, decay rates for localized perturbations were obtained and these are precisely the same as the ones found for the linear problem.

Marjeta Kramar Fijavž presented results on the transport equation on metric graphs. First abstract results on the generation of C_0-semigroups by first order differential operators on $\mathrm{L}^p(\mathbb{R}_+\mathbb{C}^\ell)\times\mathrm{L}^p([0,1],\mathbb{C}^m)$ with general boundary conditions were given. It was found that in many cases the generation property can be characterized in terms of the invertibility of a matrix associated to the boundary conditions. The abstract results were further used to study well-posedness of transport equations on non-compact metric graphs.

Milena Stanislavova reported on the stability analysis of the periodic waves for the Benney and Zakharov systems. The former is a model for interaction of short and long waves in resonant water waves, the latter describes Langmuir turbulence in plasma. The particular interest is in the periodic traveling waves, which are constructed and studied in detail. For the Zakharov system, it was shown that periodic dnoidal waves are spectrally stable for all natural values of the parameters. For the Benney system, it was proved that the periodic dnoidal waves are spectrally stable with respect to perturbations of the same period. For a different parameter set, snoidal waves of the Benney system were constructed, which exhibit instabilities in the same setup. The presented results are the first instability results in this context. The used approach, which allows for a definite answer for the entire domain of parameters, relies on the instability index theory developed by Kapitula, Kevrekidis, and Sandstede 2005, Lin and Zeng 2022 and Pelinovsky 2005. Even though the linearized operators are explicit, the spectral analysis requires subtle investigation of matrix Schrödinger operators in the periodic context, revealing some interesting features.

Atanas Stefanov (open problem) presented an open problem on the stability of solitons in the 1D Dirac model. In this case, the spectral stability (of all waves) appears to be wide open and the main technical difficulty is an absence of semi-boundedness of the considered operators.

Borbala Gerhat presented a new approach to find closed realizations with non-empty resolvent set of unbounded operator matrices without a relative boundedness within the matrix entries. The cornerstones of the method are on one hand a Schur complement which is dominant in a suitable sense with respect to the entries and a distributional approach on the other hand. This allows for passing from a well-behaved representation of the Schur complement (for instance by its sesquilinear form) to a well-behaved representation of the operator matrix and to establish an equivalence between the (point and essential) spectra of matrix and Schur complement. The abstract results were illustrated by a semigroup generation result for a wave equation with accretive (differential) damping in a weighted space.

2.4 Hardy and Rellich inequalities

Yehuda Pinchover presented new results on optimal Hardy-weights for elliptic operators with mixed boundary conditions. Families of optimal Hardy-weights for a subcritical linear second-order elliptic operator (P, B) with degenerate mixed boundary conditions were constructed. By an optimal Hardy-weight for a subcritical operator, it is meant a nonzero nonnegative weight function W such that $(P - W, B)$ is critical, and null-critical with respect to W. The presented results rely on a recently developed criticality theory for positive solutions of the corresponding mixed boundary value problem.

František Štampach (open problem) posed on open problem concerning a recent investigation of discrete Rellich inequalities on the half-line done jointly with B. Gerhat and D. Krejčiřík.

2.5 Dirichlet-to-Neumann operators

Denis Grebenkov (online) gave a fresh insights onto diffusion-controlled reactions via Dirichlet-to-Neumann operators. The talk summarized recent developments of an encounter-based approach to diffusion-controlled reactions, which is based on the concept of the boundary local time. While conventional theories of restricted diffusion towards a reactive target in a confining Euclidean domain rely on the spectral properties of the Laplace operator with mixed boundary conditions, the new approach involves a version of the Dirichlet-to-Neumann operator. Some probabilistic interpretations and applications of this approach will be provided. Even though such pseudo-differential operators have been studied in the past, relations between the spectrum of the Dirichlet-to-Neumann operator and the geometric structure of the domain remain poorly understood, especially in non-self-adjoint settings.

Tom ter Elst reported on recent results involving the diamagnetic inequality for the Dirichlet-to-Neumann operators. Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary Γ. The presentation examined the Dirichlet-to-Neumann operator $\mathcal{N}$ on $L_2(\Gamma)$ associated with a second-order elliptic operator

$$A = -\sum_{k,j=1}^{d} \partial_k (c_{kl}\, \partial_l) + \sum_{k=1}^{d} (b_k\, \partial_k - \partial_k (c_k \cdot)) + a_0.$$

A criterion for invariance of a closed convex set under the action of the semigroup of $\mathcal{N}$ was discussed with interesting applications. Roughly speaking, the criterion says that if the semigroup generated by $-A$, endowed with Neumann boundary conditions leaves invariant a closed convex set of $L_2(\Omega)$, then the 'trace' of this convex set is invariant for the semigroup of $\mathcal{N}$. The application involved using this invariance to prove a criterion for the domination of semigroups of two Dirichlet-to-Neumann operators. Then apply it to prove the diamagnetic inequality for such operators on $L_2(\Gamma)$.

2.6 Eigenfunctions, diagonalization and numerics

Jeff Ovall spoke about an open problem on properties of the coefficients of an (elliptic) differential operator, together with domain geometry and boundary conditions was discussed in this talk. These turn out to cause some eigenvectors of the operator to be strongly spatially localized in relatively small regions of the domain. A better understanding of where such eigenvectors are likely to localize, and for which eigenvalues this localization occurs, was discussed to be of practical interest in the design of certain structures having desired acoustic or electromagnetic properties. Over the past decade, advances have been made in the mathematical understanding of localization for certain classes of operators, and a few techniques have been put forward that reliably predict regions of localization for eigenvectors whose eigenvalues are low in the spectrum, and even provide reasonable estimates of the smallest eigenvalue having an eigenvector localized in a given region. During the talk it was shown that, there is significant room for development of computational techniques that may be needed in practice for specific design problems, and may also lead to more refined conjectures on the theoretical side. An approach, possibly better suited for a more thorough *numerical* investigation of eigenvector localization, was described in the talk. It allows for exploration deeper into the spectrum and provides clear quantitative control over how strongly localized an eigenvector must be within a region before it is accepted as such. Theoretical justification of the approach, as well as numerical results of a partial realization of the associated algorithm that serves as a proof-of-concept was provided.

Francis White reported on recent L^p-bounds for eigenfunctions of analytic non-self-adjoint operators with double characteristics. New approaches involving dynamical systems for establishing bounds for L^p-norms of low-lying eigenfunctions of non self-adjoint semiclassical pseudodifferential operators with double characteristics were discussed. Most notably, the main theorem improved the already known results in the case of pseudodifferential operators with analytic symbols. As an open question, it was discussed whether such bounds continue to hold under less restrictive regularity assumptions. The main ingredients in the proof were Fourier-Bros-Iagolnitzer (FBI) transform techniques and complex Hamilton-Jacobi theory.

Catherine Drysdale showed that, using the quasi-basis structure to understand the mechanisms of fluid mechanics phenomena, it is possible to approximate solutions accurately. Concretely, the model involved real non-self-adjoint Ginzburg Landau (RnsaGL) equation where the linear operator gives rise to a quasi-basis structure. The quasi-basis structure of the problem allowed to look at the transfer of energy between eigenmodes in a system that is both non-normal and nonlinear. The importance of non-normality versus nonlinearity has been a topic of debate concerning where difficulty modelling Fluid Mechanics systems lies. Although only a toy model, the quasi-basis structure allowed to illustrate the ramifications of non-normality and nonlinearity contemporaneously in contrast to tools such as the plotting of pseudospectra that only illuminates the linear phenomena.

Beatrice Pelloni discussed various novelties in the theory of odd-order linear differential operators. A review of known results was presented for these operators on bounded domains. The door to these results have been unlocked by the understanding of the behaviour of third-order boundary values problems. These problems have been studied over the last 20 years by means of the Unified Transform approach originally due to Fokas. In some non-self-adjoint cases, this approach yields a spectral diagonalisation of the operator. More generally, it was highlighted that, the dependence of these problems on the specific boundary conditions is fundamentally different from the even-order case. Novel and surprising examples arise for "Dirichlet-type" boundary conditions, as well as for quasi-periodic and time-periodic ones were presented in detail.

Dave Smith described a new form of diagonalization for linear two point constant coefficient differential operators with arbitrary linear boundary conditions. Although the diagonalization is in a weaker sense than that usually employed to solve initial boundary value problems (IBVP), it was shown that it is sufficient to solve IBVP whose spatial parts are described by such operators. It was argued that the method described may be viewed as a reimplementation of the Fokas transform method for linear evolution equations on the finite interval. The results could be extended to multipoint and interface operators, including operators defined on networks of finite intervals, in which the coefficients of the differential operator may vary between subintervals, and arbitrary interface and boundary conditions may be imposed; differential operators with piecewise constant coefficients were thus included.

2.7 Revival

George Farmakis presented details about the solution to a large class of linear dispersive PDEs under periodic boundary conditions, including for example the free linear Schrödinger equation and the Airy PDE, which exhibit a dichotomy at rational and irrational times. At rational times, the solution is decomposed in a finite number of translated copies of the initial condition. Equivalently, when the initial function has a jump discontinuity, then the solution also exhibits finitely many jump discontinuities. On the other hand, at irrational times the solution is known to become a continuous, but nowhere differentiable function. These two effects form the so-called revival and fractalisation phenomenon at rational and irrational times respectively.

The talk gave an overview emphasising on the phenomenon of revivals and discuss recent extensions outside the classical theory.

Peter Olver (online) continued describing the revival phenomenon by showing new directions in the fractalization, quantization and revival of dispersive systems. Recent developments and open directions in the study of the phenomenon were presented. Revival in linear and nonlinear dispersive systems, including the effects of boundary conditions, coupling dispersive systems to Lamb oscillators, new manifestations in integro-differential equations modeling interface dynamics, and a study of the Fermi-Pasta-Ulam-Tsingou problem for coupled nonlinear oscillators, were all surveyed in this talk.

2.8 Solvability Complexity Index

Jonathan Ben-Artzi delivered a talk about quantifying the complexity of computing resonances. The Solvability Complexity Index (SCI) Hierarchy is a classification of the complexity of problems that take infinitely long to compute. The simplest example is computing the spectrum of an infinite matrix: one can ask whether there exists an algorithm which reads ever increasing finite sections and approximates its spectrum correctly (being spectrally exact and with no pollution). It turns out that this is impossible: an algorithm that can compute the spectrum of any such infinite matrix requires exactly three limits, hence we say that this problem has SCI=3. The talk included recent results that apply this theory to the computation of scattering resonances: both quantum and classical. In both cases it is shown that a single limiting procedure suffices, i.e. SCI=1, by constructing explicit implementable algorithms. In the case of quantum scattering by a compactly supported potential this is done by obtaining quantitative estimates of the bordered resolvent. For scattering by a compact obstacle in Rd this is done by expressing the Dirichlet-to-Neumann operator as a compact perturbation of the identity and approximating this compact perturbation.

2.9 Metamaterials

Graeme Milton gave an insight on several non-self-adjoint problems in the theory of composites. This included various important problems, including those with energy loss such as those described in the quasistatic limit at constant frequency by effective complex moduli including dielectric and viscoelastic ones, and others such as conduction in the presence of a magnetic field and convection enhanced diffusion. Powerful tools in analyzing these problems include utilizing the analytic properties of the effective moduli as a function of those of the component phases, and generalizations of a technique of Gibiansky and Cherkaev for converting these non-self-adjoint problems into self-adjoint ones. More recently the presenter investigated wave propagation in certain space-time microstuctures exhibiting a type of PT symmetry and found stable wave propagation for a range of parameter values, but exponential blow-up in time outside this range. The talk reviewed these and other exciting results in the field.

2.10 Various topics

Juan Manuel Pérez Pardo discussed the controllability on infinite dimensional quantum systems. The development of quantum information and computation technologies is deeply related with the ability to control quantum systems. The techniques of finite dimensional control have been successful in this development but have intrinsic limitations. In this talk he presented new perspectives that are enabled by control in infinite dimensional quantum systems. The talk introduced results on the existence of solutions of the time-dependent Schrödinger equation and their stability and use them to prove controllability of some quantum systems.

Piero D'Ancona showed techniques for scattering for the non-linear Schrödinger equation (NLS) with variable coefficients on the line. In recent years an efficient framework was established to prove scattering for nonlinear dispersive equations, based on the combination of concentration-compactness principles and induction on energy arguments. Originally developed by Kenig and Merle, the framework has been adapted to several equations with constant coefficients. The presence of potential perturbations or variable coefficients introduces new difficulties due to unisotropy. D'Ancona reported on new results, obtained in collaboration with A. Zanni, concerning scattering for a defocusing, subcritical NLS in one space dimension, with fully variable coefficients.

Ivana Alexandrova (online) analyzed the microlocal structure of the semi-classical scattering amplitude for Schrödinger operators with a strong magnetic and a strong electric fields at nontrapping energies. For this purpose we develop a framework and establish some of the properties of semi-classical-Fourier-integral-operator-valued pseudodifferential operators and prove that the scattering amplitude is given by such an operator.

Rakesh Kumar (online) reported on advancements on the mathematical analysis of non-self-adjoint eigenvalue problem for the bent waveguides and its challenges.

3 Open problems and informal mathematical discussion

During the workshop we held two open problem sessions, which also involved bespoken informal mathematical discussions. We arranged these sessions at the end of the day on Tuesday and on Thursday. We asked individuals for problems in advance and chose those which we felt represented the interest of the majority of attendants. Most participants contributed to the discussions with original ideas. The delegates who presented open problem were Štampach, Stefanov, Milton, D'Ancona, Pérez Pardo, Ben-Artzi, Helffer and Ovall. Some of these contributed problems will be added to the list mentioned above.

4 Widening participation contribution

At the beginning of the workshop we ran a session entitled "Athena Swan and the Mathematical Sciences". This widening participation charter was launched recently in Canada, but has a long history of success in the UK. B. Pelloni who has been involved in various aspects of the award at Heriot-Watt University (UK) described its structure, levels of distinction, challenges and added values for institutions. A lively discussion followed between participants. Among other ideas, it was commented that in other countries, mechanisms to ensure support for staff exist at a different level, but the systematic collection and classification of EDI data is fragmented.

The Programme Coordinator, present at the event, commented that the awards scheme could benefit BIRS by channelling an application through the University of British Columbia. Most participants agreed that the scheme could be significant for collecting data about the existing institutional challenges, supporting staff on equality matters and that more should be done to promote such types of charters accross the mathematical community in America and Europe.

There is now the chance of discussions between BIRS and the ICMS in the UK for an exchange of proactive ideas on equality and diversity in Research Stations.

5 List of participants

1. Ivana Alexandrova (Albany, NY, US) online
2. Yaniv Almog (Karmiel, IL)
3. Antonio Arnal (Belfast, UK) online
4. Jonathan Ben-Artzi (Cardiff, UK)
5. Sabine Boegli (Durham, UK) online
6. Lyonell Boulton (Heriot-Watt, UK)
7. Cristina Câmara (Lisbon, PT) online
8. Tanya Christiansen (Columbia, MO, US)
9. Lucrezia Cossetti (Karlsruhe, DE)
10. Jean-Claude Cuenin (Loughborough, UK)

11. Piero D'Ancona (Rome, IT)
12. Karys Dickson (Belfast, UK) online
13. Catherine Drysdale (Paris, FR)
14. Luca Fanelli (Bilbao, SP) online
15. George Farmakis (Heriot-Watt, UK)
16. Marjeta Fijafž Kramar (Ljubljana, SI)
17. Borbala Gerhat (Prague, CZ)
18. Denis Grebenkov (Paris, FR) online
19. Mariana Haragus (Besancon, FR) online
20. Bernard Helffer (Nantes, FR)
21. Michael Hitrik (Los Angeles, CA, US) online
22. Perry Kleinhenz (Michigan State, MI, US) online
23. Katya Krupchyk (Irvine, CA, US) online
24. Rakesh Kumar (Jodhpur, IN) online
25. Romana Kvasničková (Prague, CZ)
26. Richard Laugesen (Urbana-Champaign, IL, US) online
27. Zhiqin Lu (Irvine, CA, US) online
28. Graeme Milton (Salt Lake City, UT, US)
29. Boris Mityagin (Columbus, OH, US) online
30. Tho Duc Nguyen (Prague, CZ) online
31. Peter Olver (Minneapolis, Minnesota, US) online
32. Jeff Ovall (Portland, OR, US)
33. Beatrice Pelloni (Heriot-Watt, UK)
34. Juan Manuel Pérez Pardo (Madrid, SP)
35. Domenic Petzinna (Heriot-Watt, UK)
36. Yehuda Pinchover (Haifa, IL)
37. Gustavo Ponce (Santa Barbara, CA, US) online
38. Petr Siegl (Graz, AT)
39. Dave Smith (Yale-NUS Singapore, SG)
40. František Štampach (Prague, CZ)
41. Milena Stanislavova (Birmingham, AL, US)
42. Atanas Stefanov (Birmingham, AL, US)
43. Tom ter Elst (Auckland, NZ)

44. Martin Vogel (Strasbourg, FR) online
45. Tuyen Vu (Prague, CZ) online
46. Ruoyu Wang (Evanston, IL, US) online
47. Francis White (Los Angeles, CA, US)

5.1 Some statistics

- female participants: 15/47
- early career participants (up to 10 years after PhD): 16/47
- online participants: 21/47
- female and early career speakers: 16/30
- online speakers: 10/30

6 Comments on the hybrid mode

The hybrid mode of the workshop was well supported technically. In our opinion the Research Station's setting is robust and works very well. The IT support was excellent and diligent. Since the majority of the participants and speakers were on-site in this workshop, it was possible to localize most of the online talks to one day (Thursday). We strongly encourage the use of this model as it simplifies the transitions between presentations in different delivery modes.

The online part of the workshop worked well, probably the best it could from the technical and formal point of view. The online talks had a high standard and occasional questions were asked. However, we strongly believe that on-site participation is crucial for the full success of this type of scientific events. In our opinion, the online participation cannot replace a full workshop experience.

A hybrid mode might have its advantages in particular in case of severe travel restrictions, rapidly changing situation or sudden circumstances preventing a participant to attend in person. Last minute changes in the timetable present serious challenges to the organizers, irrespective of the circumstances surrounding these changes. It is substantially more difficult to re-arrange a schedule in the middle of a workshop, than switch the mode of a given presentation.

Unfortunately, several participants who were strongly committed to travel to Banff, were unable to do so because of visa-related issues. We could sort these issues by switching some of the talks to online mode without seriously altering the timetable. However we felt that the full research experience of these online participants was of a lesser quality compared to those attending the event in person.

7 Outcome of the Meeting

From some of the private comments made by participants, we strongly believe that the meeting was a success. It achieved its original goals of consolidating cross-generational participation of the research community interested in non-self-adjoint spectral theory and its applications. We received this positive feedback from many of the participants, who praised the program and style of the event, as well as the environment facilitated by the Banff Centre and the BIRS. We are very grateful to the staff present at the Research Station during the workshop for their support before, during and after the event. For us it was a true pleasure to organise this workshop.