23w5003: Interactions Between Topological Combinatorics and Combinatorial Commutative Algebra

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1 Overview of the Field

Starting with the pioneering work of Stanley and Hochster in the early seventies, commutative algebra methods have become an essential part of geometric and algebraic combinatorics, and more specifically face enumeration theory. The connecting language between commutative algebra and combinatorics is that of monomial ideals in polynomial rings. For instance, the Stanley–Reisner ideal of a simplicial complex is a monomial ideal generated by monomials representing the non-faces of the complex. The algebraic properties of this ideal are strongly related to the combinatorial and topological invariants of the complex in question. The Stanley–Reisner connection has been used heavily by researchers in combinatorics. In fact, some of the most beautiful results in the theory of face numbers of simplicial (and more general) complexes on one hand and graded Betti numbers of monomial ideals on the other were proved by using this language along with a subtle combination of algebraic and geometric arguments (e.g., Lefschetz elements, generic and not-so-generic initial ideals, local cohomology, rigidity, etc.).

The purpose of this workshop was to bring together two groups of researchers who work on similar problems in this research area from different points of view and often in parallel: combinatorialists who use methods of commutative algebra, and commutative algebraists who use combinatorics. One of the bridges between the two fields is homological algebra — the modern language of commutative algebra; indeed, many homological invariants become computable once tools from topological combinatorics are used. Both fields have developed their own sets of techniques, open problems, and fundamental theorems. What has become apparent over the last few decades is that a result in any of these two fields usually has a major impact on the other, transforming the long term objectives in both fields. Consequently, it is important to have to encourage ongoing communication between the two groups.

This workshop was part of the larger program to bring to allow each side to learn the language and terminology of the other. Learning the language is crucial since an obscure concept in one field often becomes much more transparent when described in the language of the other. One spectacular example is the Cohen–Macaulay property: although this property is very well-behaved in algebra, its algebraic description is sometimes quite technical compared to the description offered by topological combinatorics. In particular, Reisner's theorem [39] that provides a combinatorial characterization of Cohen–Macaulay Stanley–Reisner rings has been a cornerstone of much research in both fields, and as such, greatly illustrates the need for researchers working in these fields to become conversant in both languages.

As we describe in more detail below, our mini-workshop allowed groups of researchers from different areas to work together. Preliminary work during the week at the Juniper Hotel resulted in promising new results about unobstructed simplicial complexes, minimal Cohen-Macaulay simplicial complexes, h-vectors of doubly Cohen-Macaulay simplicial complexes, affine stresses of prime polytopes, and powers of a simplexes. A nice feature of each problem is that they can be attacked using techniques of combinatorial commutative algebra and topological combinatorics.

2 Recent Developments and Open Problems

The recent years has seen progress and development in a number of problems in topological combinatorics and combinatorial commutative algebra. We describe five broad areas where there has been significant progress. The topics described below are not meant to be comprehensive; however, these topics are closely related to some of the problems and talks at the workshop.

Around the g-conjecture. Applications of Stanley–Reisner rings to the theory of face enumeration is a classical topic in Stanley–Reisner theory. The recent biggest development on this topic is a solution of the algebraic g-conjecture [28, 41, 42], which states that the Stanley–Reisner ring of a triangulation of a sphere has the strong Lefschetz property with respect to a generic linear system of parameters. The solution was first announced by Adiprasito [2], and, two years later, a much shorter proof was found by Papadakis and Petrotou [38]. Moreover, Adiprasito, Papadakis, and Petrotou [3] later announced extensions of these results to a much larger class of simplicial complexes, including triangulated manifolds and doubly Cohen–Macaulay complexes. Their works are closely related to the theory of stress spaces, which is a higher-dimensional analogue of both: the framework rigidity, studied in combinatorics, and Macaulay's inverse systems, studied in commutative algebra.

There is still a number of interesting open problems related to the g-conjecture. One direction is to have more understandings on stress spaces. Papadakis and Petrotou proved that in characteristic 2, the Stanley–Reisner ring of a triangulation of a sphere has a property, which they call anisotropicity. Combinatorial or geometric meaning of anisotropicity is not fully understood, and more applications of this property is expected. Also, a recent study of Novik and Zheng on affine spaces [35, 36, 37] suggests a new connection between commutative algebra and affine stresses. Also, for some beautiful subclasses of simplicial complexes, such as *balanced simplicial complexes* and *centrally symmetric simplicial complexes*, several variations of the g-conjecture positing the strong Lefschetz property w.r.t. certain special linear systems of parameters remain open. For balanced complexes, partial affirmative results were obtained by Cook, Juhnke-Kubitzke, Nevo, and Murai [14] during a BIRS workshop. For centrally symmetric complexes, several new developments are due to Klee, Nevo, Novik and Zheng [29] as well as Novik and Zheng [35]. We expect the solution of the g-conjecture will continue to inspire further interesting research problems. Indeed, while at the workshop, one group discussed h-vectors of doubly Cohen–Macaulay complexes which is closely related to the work of Adiprasito, Papadakis, and Petrotou, and the other group delved deeper into the theory of affine stress spaces. A summary of their new works can find in a later section.

Stanley–Reisner rings and topological invariants. According to a result of Bruns and Gubeladze [11], an isomorphism of two Stanley–Reisner rings as $\mathbf{k}$-algebras implies an isomorphism of the corresponding simplicial complexes, and hence there should be a way to extract various topological invariants of a simplicial complex from its Stanley–Reisner ring. Yet, at present our understanding of the impact of classical topological invariants other than Betti numbers (over a field) on the combinatorics of triangulations is practically nil. In the last few years, Bagchi and Datta [8] proposed new invariants — σ- and μ-numbers that appear to encode more subtle topological/geometric information about the complex in question than the Betti numbers do. For instance, these μ-numbers were recently used by Murai and Novik [31] to prove an inequality on the face numbers of a triangulated manifold in terms of its fundamental group. This seems to indicate that it would be interesting to trace various topological invariants such as the fundamental group, intersection homology, characteristic classes, etc. in the Stanley–Reisner ring of the complex in question. Any progress on finding such connections would be of great significance; in fact, understanding the properties of Stanley-Reisner rings was a central theme for a number of problems discussed at the workshop (see the next sections for more details).

Buchsbaum rings and beyond. A classical way of applying the Stanley–Reisner theory to combinatorics of simplicial complexes is via the use of the Cohen–Macaulay property. Recently, several breakthroughs in topological combinatorics were made by considering the more general Buchsbaum property. For instance, using this approach, Adiprasito and Sanyal [4] solved a long-standing problem on the combinatorial complexity of Minkowski sums of polytopes. Another example is work of Novik and Swartz [33], followed by work of Murai [30], who proved a 30-year old conjecture of Kalai [27] on face numbers of triangulated manifolds. While several deep applications of this approach are known, the theory is still developing and some of its basic properties are not well understood. One interesting question in this area is "what can be said about free resolutions of Buchsbaum complexes and, in particular, homology manifolds?" For homology spheres, the resolution is symmetric by the Gorenstein property, but an analogue of this symmetry for homology manifolds is missing, and an algebraic study of such a problem would be very interesting. Another very tempting direction is to try to generalize known results on face numbers of manifolds to pseudomanifolds. It follows from results of [32, 34] that a certain quotient of an Artinian reduction of the Stanley–Reisner ring of a homology manifold is Gorenstein. An analogous Gorenstein algebra can be defined for pseudomanifolds by using Macaulay's inverse system and Adiprasito-Papadakis-Petrotou [3] recently showed the Lefschetz property for this algebra. However, how to relate this result to combinatorics and topology of pseudomanifolds is still a very big question.

Edge ideals and flag complexes. The study of flag complexes is closely related to both topological combinatorics and commutative algebra. Indeed, on the combinatorial side, it would be of great interest to understand face numbers of flag complexes — a problem that is equivalent to understanding Hilbert functions of edge ideals, which is a classical topic in commutative algebra.

There are many deep open problems on flag complexes. One of them is an unpublished conjecture of Kalai on face numbers of Cohen–Macaulay flag complexes. A recent breakthrough on this combinatorial conjecture is due to Caviglia, Constantinescu, and Varbaro [12]. They used a result of Abedelfatah [1] who gave a partial affirmative answer to the famous Eisenbud–Green–Harris conjecture in commutative algebra. Current developments in the field were also described by in the talk of de Holleben (see more). A number of proposed problems presented at the workshop were also about edge ideals. Moreover, when studying simplicial complexes, the class of edge ideals and flag complexes is a nice subclass to test conjectures and provide intuition since the corresponding monomial ideals are quite simple (i.e., generated by quadratic monomials).

A facet ideal dictionary. Over the past twenty years a substantial amount of work has been done in understanding relations between simplicial complexes and their corresponding facet or edge ideals. This area has been developing parallel to the Stanley–Reisner theory with many overlaps along the way. Resolutions of facet ideals, their Cohen–Macaulay properties, their Rees algebras and much more are constantly under investigation. While Stanley-Reisner theory relies on topological combinatorics, facet ideal theory borrows techniques from hypergraph and matroid theories. What is lacking, is a direct dictionary between the two languages. For example, Reisner's criterion of Cohen–Macaulayness does not have a clear counterpart in facet ideal theory. Researchers have been getting closer to such a characterization via describing the dual notion of linear resolutions [9, 10, 13]. Identifying gaps in the dictionary and working on translations is a key problem going forward. At the workshop, one group worked on improving our understanding of facet ideals and simplicial complexes that support the resolutions of their powers. Additional information about this group is provided below.

3 Structure of the Workshop

Unlike a more traditional BIRS workshop, the emphasis of our workshop was to encourage new collaborations and to give participants the opportunity to work on new problems. During the morning of the first day, numerous participants described potential research problems (the talks ranged from 5-25 minutes).

Before lunch on the first day , all participants were required to rank which project they would like to work on (some of the problems had been circulated before the start of the program). The organizers then placed the participants into groups of five or six researchers, with the aim of having a balancing of researchers from combinatorial commutative algebra and topological combinatorics. For the remainder of the week, the groups worked together for some of the mornings and most of the afternoons. The Juniper Hotel provided numerous rooms for groups to work together.

At the end of each day, each group would give a 5-10 minute progress report. On the last day, each group gave a slightly longer progress report which included their plans for moving their projects forward.

In addition to the ample time to work on projects, research talks with further questions were presented on Tuesday and Wednesday morning. Like the talks on Monday, these talks were made available to remote participants (they are also available on the BIRS website).

4 Presentation Highlights

As mentioned above, the focus of our workshop was to encourage new collaborations and work on new problems. Consequently, we encouraged speakers to talk about potential problems and recent developments in one of the two fields to encourage discussion about new projects. Because we wanted to leave a significant amount of time for groups to work together, we had fewer talks than a regular BIRS workshop. Regardless, the talks were well received by all participants (both local and remote) for summarizing recent developments and for promoting possible research questions. Below we highlight the topics of the six talks.

Cotangent cohomology for matroids (Alexandru Constantinescu) In this talk, Constantinescu described a problem related to the cotangent cohomolomogy module. The first cotangent cohomology module T^1 describes the first order deformations of a commutative ring. For Stanley–Reisner rings, this module has a purely combinatorial description: its multigraded components are given as the relative cohomology of some topological spaces associated to the defining simplicial complex. When the Stanley–Reisner ring is associated to a matroid, Constantinescu gave an explicit formula for the dimensions of these components. Furthermore, he showed that T^1 provides a new complete characterization for matroids. This talk was based on a joint work with William Bitsch [7]. Constantinescu lead a working group on problems related to this talk while at the workshop; for progress on this problem, see the next section.

Homological invariants of ternary graphs (Thiago de Holleben) In 2022, Jinha Kim [26] proved a conjecture by Engström [23] that states the independence complex of graphs with no induced cycle of length divisible by three is either contractible or homotopy equivalent to a sphere. These graphs are called ternary. A direct corollary is that the minimal free resolution of the edge ideal of these graphs is characteristic-free. In this talk, Thiago de Holleben showed how to apply this result to give a combinatorial description of projective dimension and depth of the edge ideals of ternary graphs. As a consequence, he was able to give a complete description of the multigraded Betti numbers of edge ideals of ternary graphs in terms of its combinatorial structure and classify ternary graphs whose independence complex is contractible.

Some results and questions in Stanley–Reisner theory motivated by commutative algebra (Hai Long Dao) In this talk, Hai Long Dao discussed several new algebraically-motivated directions in the study of simplicial complexes. They are: minimal Cohen-Macaulay complexes (which include many interesting old and new constructions in combinatorial topology), higher nerve complexes (which capture numerous algebraic invariants) and acyclicity results suggested by Kodaira vanishing. Some of these problems were inspired by Dao's papers [20, 21]. Dao lead a research group at the workshop that looked at some of these problems; the progress of this group is recorded below.

Alexander duals of symmetric simplicial complexes and Stanley–Reisner ideals (Uwe R. Nagel) It is known that any ascending chain $(I_n)_{n\in\mathbb{N}}$ of related squarefree monomial ideals, where I_n is invariant under the action of the symmetric group $Sym(n)$ on n letters, enjoys strong stabilization properties. For example, there are finitely many polynomials whose $Sym(n)$-orbits generate I_n if n is sufficiently large. In this talk Uwe Nagel discussed properties of the corresponding chain of Alexander duals $(I_n^\vee)_{n\in\mathbb{N}}$. It does not have the same stabilization properties. However, it turns out the minimal generating set of $I_n^\vee$ can be described explicitly and that the number of orbit generators is given by a polynomial in n for sufficiently large n. As an application, one obtains that, for each $i \geq 0$, the number of i-dimensional faces of the associated Stanley–Reisner complexes of I_n is also given by a polynomial in n for large n. The needed arguments include a novel combinatorial tool, which is called *avoidance up to symmetry*, and methods from discrete geometry for counting lattice points in polyhedra. This talk was based upon Uwe Nagel's joint work with Ayah Almousa, Kaitlin Bruegge, Martina Juhnke-Kubitzke and Alexandra Pevzner [5].

Garland method, its extensions and potential new applications (Volkmar Welker) In this talk Volkmar Welker introduced participants to the Garland method that comes from geometric group theory and how it could be applied to simplicial complexes Δ. For example, suppose you wish to show that $H_i(\Delta, \mathbb{Q}) = 0$ (this method only works for homology with coefficients in a field of characteristic 0). One approach to this problem is to consider the 1-skeleta (graphs) of the links of all $(i-1)$-simplices in Δ. If the smallest non-zero eigenvalue of the normalized graph Laplacian of all those 1-skeleta is $> i/(i+1)$, then $H_i(\Delta, \mathbb{Q}) = 0$. Welker then discussed his recent and ongoing work with Eric Babson which extends this method to chain complexes satisfying certain rather weak conditions. Welker met with interested participants after the talks to further discuss the Garland method.

Perfect matchings and Alexander duals (Russ Woodroofe) In this talk Russ Woodroofe first reviewed the topological view of the classical Alexander dual of a simplicial complex. He then gave an alternative construction that may be well suited for the independence complex of a graph with a perfect matching. The classical Alexander dual complex can be regarded as the Alexander duality for simplices. Woodroofe suggested to consider the Alexander dual over a cross polytope. The corresponding dual complex is not necessary a simplicial complex, but is a cubical complex since it is a subcomplex of a cube, and has an advantage that it is much smaller than the classical Alexander dual simplicial complex.

Problem Proposals (Various) In addition to the formal talks described above, a number of short presentation were given. These very short presentations described possible research projects. Presentations were given by:

- Christos A. Athanasiadis on h-vectors (this problem is described in more detail below).
- Susan Morey on powers of a simplex (this problem is described in more detail below).
- Sara Faridi on the subadditivity problem.
- Isabella Novik on affine stresses (this problem is described in more detail below).
- Vic Reiner on a question in invariant theory.
- Adam Van Tuyl on Betti splittings.
- Russ Woodroof on Alexander duality.

Although a number of potential problems were discussed, the workshop only focused on five of these problems. For the other problems that were proposed, we have elected not to give further details here since the originators of the problems way wish to pursue them as part of their own research program or give these problems to their own students.

5 Scientific Progress Made

On the first day of the workshop, five small research groups (5-6 participants in each group) were made to tackle some of the proposed problems. During the work at the Juniper Hotel, each group made progress on their problem. We expect that some, if not all, of these projects will result in a future publications. Below is a summary of each problem and its progress during the five days of the workshop.

Problem: Cotangent Cohomology for simplicial complexes and matroids

Group Participants:

Ayah Almousa (University of Minnesota - Twin Cities),

Alexandru Constantinescu (Freie Universitt Berlin),

Patricia Klein (Texas A&M University),

Thi Thnh Nguyłn (McMaster University),

Anurag Singh (Indian Institute of Technology Bhilai),

Lorenzo Venturello (Universit di Pisa).

This group investigated the question of when a simplicial complex is unobstructed. A simplicial complex Δ on $[n]$ is *unobstructed* if $T^2(\Delta)$, the second cotangent cohomolgy module of its Stanley-Reisner ring, vanishes. The $\mathbb{Z}^n$-grading of the Stanley-Reisner ring is inherited by T^2, and, due to work of Altmann and Christophersen, understanding $T^2(\Delta)$ boils down to understanding $T^2_{-b}(\mathrm{link}_\Delta A)$ for all $b \in \{0,1\}^n$ and $A \in \Delta$. These vector spaces are described as the relative cohomology of two combinatorially defined topological spaces.

The group started by considering the lowest dimensional cases. For zero-dimensional complexes, being unobstructed is equivalent to having at most 3 vertices. This gives the first of three conditions which characterized unobstructedness in dimension one:

Theorem. *A one-dimensional simplicial complex Δ on $[n]$ is unobstructed if and only if the following three conditions hold:*

(i) Every vertex is contained in at most three edges.

(ii) Every cycle is a dominating set.

(iii) If $\{v, w\} \notin \Delta$, removing v, w and their common neighbors from Δ one gets a connected space.

These are strong restrictions and the group hopes to be able to list all the complexes that fulfill them. However, there are infinitely many unobstructed one-dimensional simplicial complexes, so the general setting in higher dimension was not pursed. In the second part of the workshop, the group focused on matroid complexes. They were able to compute the dimensions of all the multigraded components of T^2 for the uniform matroids $U_n^r = \{F \subseteq [n] \ : \ \#F \le r\}$ with $r < n$:

$$\dim_{\mathbb{C}} T^2_{-b}(U_n^r) = \begin{cases} 0 & \text{if } \#b \neq 2, \\ r \cdot \binom{n-2}{r} - \binom{n-1}{r} + 1 & \text{if } \#b = 2. \end{cases}$$

This implies that the uniform matroid U_n^r is unobstructed if and only if $r \geq n-2$. They conjecture that if a matroid is unobstructed, then its simplification is a matroid of corank at most 2. They were able to prove this for rank two matroids, and they have some partial results for higher ranks as well.

Problem: Minimal Cohen–Macaulay Simplicial Complexes

Group Participants:

Hai Long Dao (University of Kansas),

Anton Dochtermann (Texas State University),

Jay Schweig (Oklahoma State University),

Adam Van Tuyl (McMaster University),

Russ Woodroofe (University of Primorska).

This group focussed on minimal Cohen–Macaulay complexes. Fix a field k and let Δ be a simplicial complex. We say that Δ is *minimal Cohen–Macaulay* (over k) if it is Cohen–Macaulay and removing any facet from the facet list of Δ results in a complex which is not Cohen–Macaulay. Any Cohen–Macaulay complex can be obtained from a minimal one by shelling moves, thus a systematic study of these objects seem worthwhile. This group obtained some results: Cohen–Macaulay complexes of codimension at most two are not minimal Cohen-Macaulay (unless they are simplices). This complements what we know: the smallest example of a (non-trivial) minimal Cohen–Macaulay complex over $\mathbb{R}$ is the 6 vertex triangulation of the projective plane. The group also gave broad new constructions of minimal Cohen–Macaulay complexes, and used them to show that these complexes can have free faces, might not come from triangulations of manifolds, and might not be closed under barycentric division. The gorup also investigated the general question of what 'nice' properties (e.g. Cohen–Macaulayness, shellability, vertex-decomposability, etc.) of a simplicial complex are destroyed when we remove facets, for instance from the i-skeleton of a simplex. Some interesting patterns emerged, which the group hopes to continue studying after the workshop.

Problem: h-vectors of simplicial complexes

Group Participants:

Christos Athanasiadis (National and Kapodistrian University of Athens),

Susan Cooper (University of Manitoba),

Martina Juhnke-Kubitzke (Universität Osnabrück),

Kazunori Matsuda (Kitami Institute of Technology),

Victor Reiner (University of Minnesota),

Volkmar Welker (Philipps-Universitaet Marburg).

This group worked on the problem to decide whether the inequalities

$$\frac{h_0(\Delta)}{h_d(\Delta)} \leq \frac{h_1(\Delta)}{h_{d-1}(\Delta)} \leq \cdots \leq \frac{h_{d-1}(\Delta)}{h_1(\Delta)} \leq \frac{h_d(\Delta)}{h_0(\Delta)} \tag{1}$$

hold for the h-vector $(h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$ of any $(d-1)$-dimensional doubly Cohen–Macaulay (over some field) simplicial complex Δ (see [40] for basic definitions). This problem was posed in [6], where it was motivated by questions on the real-rootedness of face polynomials of triangulations of simplicial complexes (such as barycentric subdivision). There was little evidence in favor of an affirmative answer prior to the workshop.

During the workshop, the team thoroughly discussed new interesting special cases and was able to either confirm the inequalities, or investigate the problem further and connect it to other topics within enumerative and algebraic combinatorics, or suggest possible generalizations, or provide evidence in favor of an affirmative answer. For instance, it was shown that the inequalities hold if the entries of the h-vector of Δ are replaced by those of the f-vector $(f_{-1}(\Gamma), f_0(\Gamma), \ldots, f_{d-1}(\Gamma))$ of a $(d-1)$-dimensional Cohen–Macaulay simplicial complex Γ, or with the coefficients of a polynomial of degree d having all its complex roots in the interval $[-1, 0]$, and that they are preserved when Δ is replaced by its simplicial join with a zero-dimensional complex, or with the boundary complex of a simplex. Special attention was paid to order complexes (for instance, of Boolean and subspace lattices), balanced complexes and their rank-selected subcomplexes, matroid complexes and complexes with a convex ear decomposition. These special cases connect the problem to topics such as permutation enumeration and the combinatorics of pure O-sequences and allow for natural q-analogues and equivariant versions of the inequalities to be formulated. Some first attempts to prove the inequalities for pure O-sequences (instead of h-vectors of matroid complexes), and thus to generalize the main results of [25], were made. The team intends to continue to investigate some of these aspects of the problem in the future.

Problem: Affine Stresses

Group Participants:

Sankhaneel Bisui (University of Manitoba),

Selvi Kara (University of Utah),

Satoshi Murai (Waseda University),

Uwe Nagel (University of Kentucky),

Isabella Novik (University of Washington),

Jose Samper (Pontificia Universidad Católica de Chile).

This group looked at the following problem:

Problem. *Let $P \subset \mathbb{R}^{2k}$ be a simplicial $2k$-polytope. Is it true that, if the Stanley–Reisner ideal of P has no generators of degree $\geq k+2$, then*

$$\operatorname{Socle}\big(\mathbb{R}[P]/\big((\Theta_P, \ell)\mathbb{R}[P]\big)\big)_{k-1} = 0?$$

Here $\mathbb{R}[P]$ is the Stanley–Reisner ring of P, Θ_P is the sequence of linear forms determined by the coordinates of the vertices of P, ℓ is the sum of variables, and $\operatorname{Socle}(-)$ *denotes the socle.*

This problem is motivated by the problem of Gil Kalai asking if the space of affine 2-stresses of a prime d-polytope determines the affine type of P. During the workshop, the group worked on some special cases of the problem. Specifically, the group answered the problem in the affirmative in the following three cases:

- Suspensions of simplicial $(2k-1)$-polytopes;
- k-stacked $2k$-polytopes;
- One-point suspensions of simplicial 3-polytopes.

The group thinks that these results provide nice evidence suggesting that the answer to the question is likely to be "yes". They have a number of interesting questions and possible directions to continue their work on the problem, such as: Can we extend the third result to polytopes obtained by taking simultaneous one-point suspensions? What happens if we take edge subdivisions? Can we extend the second result to i-stacked $2k$-polytopes with $i > k$? Can we solve the general case in dimension 4? In the short term, the group is also planning to write a code to compute the dimension of the socle in the problem and check a number of examples in dimensions 4 and 6 using this code. Some members of the group plan to continue working on the problem.

Problem: Powers of a Simplex

Group Participants:

Trung Chau (University of Utah),

Thiago de Holleben (Dalhousie University),

Art Duval (University of Texas at El Paso),

Sara Faridi (Dalhousie University),

Susan Morey (Texas State University),

Liana Şega (University of Missouri-Kansas City).

The group worked on the following problem.

Problem. *Define a topological power for a q-simplex Δ with the following properties:*

$$\Delta^2 = \mathbb{L}_q^2 \quad \Delta^r \subsetneq \mathbb{L}_q^r \quad \textit{for} \quad r \geq 3,$$

where $\mathbb{L}_q^2$ and $\mathbb{L}_q^r$ are simplicial complexes defined in [19, 16].

The end goal is to have a topologically meaningful definition of Δ^r which supports a minimal free resolution of I^r for some monomial ideal I with q generators (I would be the q-extremal ideal defined in [16]) and for any square-free monomial ideal J with q generators

$$\beta_i(J^r) \leq f_i(\Delta^r)$$

where f_i is the number of i-dimensional faces of a complex.

The group happened to include three people who had previous experience with this specific problem, and could therefore break into subgroups pursuing different angles of approach using the expertise of the newly added members.

One direction was a “bottom to top” approach, which consists of describing products of edges in a way that it is consistent with products of triangles, and then the two would be consistent with products of tetrahedra and so on. The group focused on variations of the well-known Minkowski sum of polyhedra, Cartesion products of simplices and similar products, and moved forward adjusting these definitions to build powers with the required properties.

Another approach was a “top to bottom” approach which starts from a very large simplex (representing the Taylor simplex of a power of an extremal ideal) and deletes faces from it via Morse matchings. This was along the lines of the approach taken in [22], and the team in Banff tried to push it further, with limited success.

The team also found a concrete description of the powers of a simplex as an intersection of hyperplanes in coordinates labeled with the faces of the simplex. This new description will be explored further in subsequent work of the group.

Finally, some team members dedicated extra time to programming in Macaulay2 [24] in order to find a concrete description of the expected faces of the powers of the simplex.

This team has scheduled weekly virtual meetings to continue their work.

6 Outcome of the Meeting

Overall, it was felt by both the organizers and participants that the workshop was very successful. First, in terms of new research, it is expected that many of the projects that started at our workshop will eventually result in future publications. Indeed, as apparent from the last section, each group made a significant start on their proposed research problem. At the end of the workshop, each group made a plan on how to build upon the momentum of the workshop.

But secondly, and probably more importantly, it was felt that the workshop was very successful in building new bridges and collaborations between the two research groups. Each group had at least one researcher who had never worked with any of the other members on previous projects. We expect these new collaborations will bear fruit over the years to come.

It is also interesting to highlight that for more than half of the participants, this was their first visit to a BIRS workshop in Banff. All participants were very impressed with the facilities and the professionalism of the BIRS staff. As organizers, we heard from many of the participants that this was one of the best workshops that they had attended.

We would like to conclude by thanking the staff of BIRS for their help organizing this conference (a special recognition to Connor and Jake who helped with local details). We would also like to thank the staff at the Juniper Hotel for a wonderful and productive stay.

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