Women in Operator Algebras III

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June 25 – 30, 2023

1 Overview

This report details the organization and scientific progress made at the third Women in Operator Algebras workshop held at the Banff International Research Station in Banff, Canada. WOA III was a 5-day workshop held from June 25 to June 30, 2023. Although predominantly an in-person event, 32% (12 of 38 participants) participated virtually.

To bolster the numbers and success of women in higher mathematics, various research fields have formed networks and held workshops for the purpose of initiating women-led and all-women research projects; among the first were "Women in Number Theory" and "Women in Topology". Women are particularly under-represented in the field of operator algebras, with estimates indicating that less than 17% of researchers in that area are women ([7]). Moreover, as it is a small field with research nodes spread around the globe—including in Scandinavia, New Zealand, Australia, Germany, Italy, France, Japan, Poland, Canada, and scattered across the U.S.—geographic isolation can compound the marginalization of women in the field. For these reasons, the "Women in Operator Algebras" (WOA) network was formed several years ago. As the most recent effort in WOA's mission to broaden participation in the operator algebras community, WOA III provided a vital opportunity for women in operator algebras to

1. conduct cutting-edge collaborative research, and
2. build a network of women working in the field to drive future research collaborations.

Notably, these aims were accomplished while facilitating

1. horizontal integration amongst researchers from PhD-granting and primarily undergraduate institutions, and
2. vertical integration across career levels.

In particular, the format from the BIRS workshops Women in Numbers, where the focus is on research in groups on current problems, provided the organizational groundwork for achieving these successes.

Leaders for WOA III group projects were solicited directly via emails to well-known senior and mid-career women researchers, as well as indirectly via bulk emails to lists for women in operator algebras and to major conferences in operator algebras. The workshop was advertised over social media (in particular, the Operator Algebras Facebook group).

Group leaders supplied a 1-2 page description of their proposed project. Some projects were led by leaders from past iterations of WOA, while a few were proposed by researchers who had never been involved with WOA. Notably, three fourths of the projects were proposed by past participants of Women in Operator

Algebras workshops who had not previously led projects: Sara Azzali, Sarah Browne, Kristin Courtney, Marzieh Forough, Anna Duwenig, Sanaz Pooya, Lauren C. Ruth, Camila F. Sehnem, Dillian Yang. This marker of professional growth is another measure of the success and long-term impact of these workshops in increasing representation and visibility of women in operator algebras.

Once group leaders and projects were established, applications for participation in the workshop were sought through broad and repeated emails and social media posts. The application process included the descriptions of the projects and asked for a ranked list of the applicants' projects of interest, a brief statement describing how the applicants expertise fitted with the projects and how they would benefit from participating, and a CV. Participants were selected according to the anticipated benefits to their careers, with consideration given to job security and advancement, existing collaborations, and isolation. Graduate students who had established thesis projects, and had the support of their advisors, were also considered. These selection criteria are similar to those used in previous WOA workshops.

Based on the responses, the organizers matched participants to projects according to their preferences and background. In most cases the participants received their preferred choice of project; however in some cases their second or third choice were used in order to keep group sizes to around 3-6 people, to optimize collaborations. One project was removed from the list of research options because of the lack of interest from potential participants, while another one was removed due to scheduling issues with the leaders. In the end, there were a total of eight research projects.

Group leaders were asked to provide background reading material and references for group members and encouraged to organize several virtual meetings in advance of the workshop. This gave the research groups opportunity to build a strong foundation for the week of the WOA III workshop.

The workshop had a total of 38 participants from 15 countries (Belgium, Canada, China, Czech Republic, England, France, Germany, India, Iran, Ireland, Italy, Korea, New Zealand, Norway and the USA). The majority of the workday was spent working in research groups.

A brief overview of each group project was presented on Monday morning. There was a formal talk by Arundhathi Krishnan (Markovianity and the Thompson Monoid F^+) on Tuesday afternoon. Thursday morning featured a discussion about questions related to projects as well as updates on projects. Thursday afternoon featured a formal talk by Preeti Luthra (C*-envelopes and nuclearity of operator systems). Final presentations of research projects were given on Friday morning. Networking and informal discussions took place during the downtime throughout the week. In particular, there was a moderated discussion about workplace practices and issues on Wednesday evening after dinner. These opportunities enabled women to

1. have access to advice and feedback on career matters, and
2. forge connections between junior women and established researchers.

2 Overview of operator algebras, recent developments and open problems

The study of operator algebras was initiated in the late 1920's to formalize the algebra of observables in quantum mechanics; this branch of functional analysis now intersects with many fields, including algebra, dynamical systems, mathematical physics, ergodic theory, quantum information theory, and topology. The subject has seen many exciting developments over the last several decades. Among these are the work of Jones and Popa on the classification of finite factors; the solution of Tsirelson's problem in quantum information theory; and the recent advances in the classification program for unital, simple, separable, nuclear C*-algebras, which was initially formulated by Elliott in the early 1990's. Some of the fundamental open problems in the area are the classification of non-simple nuclear C*-algebras, completion of the Toms-Winter conjecture, the classification of subfactors of certain indices, and the isomorphism question for free group factors.

3 Scientific progress made during the workshop: reports from the research groups

A rich dictionary of quantum analogues of classical subjects like directed graphs and harmonic analysis exists nowadays, as well as enriched operator algebraic perspectives of index theorems and elliptic operators on manifolds. The WOAIII projects proposed by the research leaders, who are well-known or rising experts in the field, deal with a wide variety of topics, reflecting the array of perspectives and research that falls under the umbrella of operator algebras. The research field not only gives a rigorous context for phenomena from quantum physics, but also for ideas from quantum information, as shown by extensive work on quantum channels and entanglement in recent years. For example, the project by Emanuela Sasso and Sachi Srivastava looks at decoherence of Gaussian Quantum Markov Semigroups, a phenomenon that may appear when there is interaction between the open quantum system and the environment. The concept is related to the ergodic behaviour of Quantum Markov Semigroups.

Non-commutative geometry is another major area of research for operator algebraists. Fundamental open problems in the area are the Baum-Connes conjecture, which suggests a link between the K-theory of the reduced C*-algebra of a group and the K-homology of the classifying space of proper actions of that group, and the coarse Baum-Connes conjecture. Two of the projects deal with the conjectures: the one by Sanaz Pooya and Hang Wang, and the one by Sara Azzali and Sarah Browne.

A few of the workshop's projects are related to the Elliott classification program and the rich structure theorems for C^*-algebras made possible by that framework. For example, dynamical systems yield an abundant supply of operator algebras, and some of the projects aim to advance our understanding of such structures and study the regularities properties of the associated C^*-algebras. The projects of Courtney-an Huef and Forough-Strung both focus on some aspects of dynamics and classification.

Questions about dynamics and equilibrium states can also be defined on C^*-algebras. Camila Sehnem's project studies such phenomena on noncommutative solenoids, which are C^*-algebras which can be identified with inductive limits of noncommutative tori, while Duwenig-Yang's project looks at non-amenable groups acting on a k-graph self-similarly and Cartan subalgebras of self-similar k-graph C^*-algebras.

Lastly the project led by Ruth sits at the intersection of operator algebras and finance. It is motivated by a result in options pricing called the Breeden-Litzenberger formula, and seeks to find an operator algebraic version of this result, by replacing the measure space with a von Neumann algebra, a common theme in operator algebras.

3.1 Actions of groups on C^*-algebras of twisted groupoids, and implications for group actions on classifiable C^*-algebras

Group members: Anshu (National Institute of Science Education and Research), Kristin Courtney (group co-leader; University of Münster), Magdalena Georgescu (Independent), Astrid an Huef (group co-leader; Victoria University of Wellington), and Maria Grazia Viola (Lakehead University)

Research synopsis and progress

The study of C^*-algebras is often described as the study of noncommutative topological spaces. Hence noncommutative generalizations of topological dynamics are a topic of intensive study in the field. These are studied via the crossed product C^*-algebra $A \rtimes H$ arising from an action of a locally compact Hausdorff group H on a C^*-algebra A (it is the C^*-algebraic version of a semi-direct product of groups). Of particular interest is determining, at the level of the dynamics, whether a given crossed product fits into the framework of the celebrated classification theorem (the culminating result of a major research program– see [37]). A useful tool in studying and modeling C^*-algebras is the theory of locally compact, Hausdorff and étale groupoids. The recent breakthrough in [27] (along with [35] and [13]) shows that all C^*-algebras which fit into the classification program admit twisted groupoid models, or equivalently, they possess a certain distinguished subalgebra called a Cartan subalgebra. From [5], a discrete group acting on a C^*-algebra that admits a twisted groupoid model (including the so-called classifiable C^*-algebras) and leaves the canonical Cartan subalgebra invariant, corresponds to an action on the underlying twisted groupoid. This gives rise to the natural question: can we study group actions on C^*-algebras with twisted groupoid models via the corresponding actions on the groupoids? We made the first major strides towards this goal during WOA III at BIRS by proving the following theorem:

Theorem: Let $\mathcal{G}$ be a locally compact, Hausdorff, second-countable groupoid with a Haar system Let α be an action of a discrete group H on $\mathcal{G}$ satisfying an appropriate invariance condition on the Haar system. Then α induces an action $\tilde{\alpha}$ of H on $\mathrm{C}^*_\mathrm{r}(\mathcal{G})$ such that $\mathrm{C}^*_\mathrm{r}(\mathcal{G}) \rtimes_{\tilde{\alpha},\mathrm{r}} H$ and $\mathrm{C}^*_\mathrm{r}(\mathcal{G} \rtimes H)$ are canonically isomorphic.

We can show analogous results for

1. the full crossed product of the full groupoid C*-algebra $\mathrm{C}^*(\mathcal{G})$ by H, and
2. an action of H on a twist over $\mathcal{G}$.

We have also made promising progress in extending the above results to actions of locally compact groups. In future work, we plan to explore how properties of the action of H on $\mathcal{G}$ and properties of the semi-direct product $\mathcal{G} \rtimes H$ translate to those of the action H on $\mathrm{C}^*(\mathcal{G})$ and the crossed product $\mathrm{C}^*(\mathcal{G}) \rtimes_r H$ (and likewise in the full and twisted settings).

3.2 Self-similar k-graph algebras

Group members: Dawn Archey (University of Detroit Mercy), Anna Duwenig (group co-leader; KU Leuven), Shanshan Hua (University of Oxford), Kathryn McCormick (California State University, Long Beach), Rachael Norton (Macalester College), and Dilian Yang (group co-leader; University of Windsor)

Research synopsis and progress

A *self-similar k-graph* is a pair (G, Λ) consisting of a (discrete) group G and a k-graph Λ such that G acts on Λ in a 'self-similar' way: Loosely speaking, that means that Λ also 'acts' on G, and that the two 'actions' align in an appropriate sense. Self-similar k-graphs were first studied by Li-Yang [26], and they generalize Exel-Pardo's self-similar directed graphs [16].

The self-similar k-graph C*-algebra $\mathcal{O}_{G,\Lambda}$ associated to (G, Λ) is a C*-algebra universal for pairs (U, S) of representations, where U is a unitary representation of G on some C*-algebra $\mathcal{A}$ and S is a representation of the graph C*-algebra $\mathcal{O}_\Lambda$ of Λ on $\mathcal{A}$, such that U and S encode the self-similar action of (G, Λ). This class of C*-algebras encompasses many known important C*-algebras as special cases, such as k-graph C*-algebras introduced by Kumjian-Pask, Exel-Pardo algebras, Katsura algebras, and Nekrashevych algebras.

Li-Yang have studied self-similar k-graph C*-algebras in a series of works, including their properties [26], and under some conditions, their KMS states [24] and ideal structure [25].

During the week at BIRS, our different backgrounds enabled us to formulate and investigate various questions that arise for this new concept of self-similar k-graph algebras, such as the following.

Question: In [24], under some strong conditions, the authors obtain a canonical Cartan subalgebra for a self-similar k-graph. Is it possible to remove some of the restrictions?

Progress: To gain traction, we investigated a self-similar k-graph that is not locally faithful and found a likely candidate for a Cartan subalgebra in that particular case. We would like to generalize this.

Question: Can we find an example of a non-amenable group acting on a k-graph self-similarly?

Progress: Yes. In fact, we have found a whole family of examples.

Question: C*-algebra theory is replete with product-type constructions which turn given input data (such as a group action on a C*-algebra) into a new object (in this case, a crossed product C*-algebra), and then one studies the interaction between the two. What kind of product-type constructions make sense for a self-similar k-graph (G, Λ)?

Benefits of the WOA format: Months before the conference began, each participant was placed in a group in which they would be working on one of the proposed projects. Not only did this allow us to get to know each other via email beforehand, but it also allowed each member of our group to familiarize herself with the necessary mathematical background. This was particularly important since each group member has expertise in a different area. The assigned readings enabled everyone to feel more confident and contribute to the research during the week at BIRS.

Future plans: The group intends to have biweekly online meetings to make progress on the research project in their on-going collaboration. Furthermore, the group will meet at the *Fields Institute* in Toronto ON, Canada, during the *Thematic Program on Operator Algebras and Applications* in late 2023.

3.3 Equilibria on Toeplitz extensions of higher-rank noncommutative solenoids

Group members: Becky Armstrong (University of Münster), Mahya Ghandehari (University of Delaware), Larissa Kroell (University of Waterloo), and Camila F. Sehnem (group leader; University of Waterloo)

Project description Solenoids are inverse limits of tori. Latrèmoliére and Packer considered in [23] *noncommutative solenoids*. Following the equivalent characterisation of a noncommutative torus as a twisted group C^*-algebra of $\mathbb{Z}^2$, they defined a noncommutative solenoid to be a certain twisted group C^*-algebra, where the group involved is an abelian discrete group with a solenoid as its Pontryagin dual. They also described noncommutative solenoids as inductive limits of rotation algebras, that is, *noncommutative tori*.

Brownlowe, Hawkins and Sims [10] introduced a notion of Toeplitz extensions of the noncommutative solenoids of [23] as direct limits of certain building blocks, motivated by the description of a noncommutative solenoid as a direct limit of rotation algebras. Each of these building blocks is a Toeplitz-type extension of a noncommutative torus, obtained by replacing one of the two canonical unitary generators in the presentation of a noncommutative torus by an isometry. They studied KMS states and phase transitions of these Toeplitz noncommutative solenoids under natural dynamics. Following this construction, Afsar, an Huef, Raeburn and Sims [1] defined higher-rank versions of Toeplitz noncommutative solenoids using Toeplitz-type extensions of higher-dimensional noncommutative tori as building blocks (see [31, 33] for an account on higher-dimensional noncommutative tori), and studied their phase transition under a dynamics determined by a subgroup of the gauge action.

More recently, a higher-rank Toeplitz noncommutative torus was introduced in [2] using twisted semigroup C^*-algebras. The building blocks used in [1] are quotients of these Toeplitz noncommutative tori for particular choices of 2-cocycles. The main results on phase transitions of [2] generalize those of [1] on phase transitions for their Toeplitz extensions of higher dimensional noncommutative tori.

The purpose of this project is to define higher-rank Toeplitz noncommutative solenoids using twisted semigroup C^*-algebras, thus following [2] and the original approach proposed by Latrèmoliére and Packer in [23]. Once this is done we aim to investigate phase transitions on these Toeplitz noncommutive solenoids. This approach should allow us to generalize the main results of Afsar, an Huef, Raeburn and Sims [1].

Progress made during the week: We have greatly benefited from the workshop at BIRS as we had a very productive week, especially taking into account that this was the first time we all met in person. The group discussed the main concepts needed in the project and defined the class of semigroups and twisted semigroup C^*-algebras we aim to study. We found a concrete class of examples which fit into our approach and are not covered by previous work.

Future plans: We have allocated group members to certain tasks and we plan to meet as a group on Zoom for a week later this year. We plan to apply for funding to visit a research institution as a group to continue working on our project in person.

3.4 Explicit computations around the Baum–Connes conjecture

Group members: Sara Azzali (group co-leader; University of Bari), Sarah Browne (group co-leader; University of Kansas), Indira Chatterji (University of Cte d'Azur), Maria Paula Gomez Aparicio (Universit Paris-Saclay), Therese Landry (UC Santa Barbara), and Hang Wang (East China Normal University)

The project "Explicit computations around the Baum–Connes conjecture" is a continuation of the project at WOA2. We had not met as an entire group for some time and we had a new group member so we spent some of the week working through definitions and results to be on the same page and be able to connect ideas as a team. Indira Chatterji and Hang Wang were both online, so we had scheduled times during the day to meet with them, given their time zone differences.

Activities during the week at BIRS:

1. Summarized the key aspects of goal intentions and directions for project.
2. Discussed and worked through Lafforgue's construction of the γ-element.

Specifically we worked through the definitions of hypotheses for the construction including:

- uniformly locally finiteness
- weakly δ-bolic property
- weakly δ-geodesic property
- strongly bolic property

and then we looked at examples and non-examples of these concepts. We also made connections with other references (Julg-Valette), some we had already explored as well as more recent work of (Brodski-Guentner-Higson [8] and Brodski-Guentner-Higson-Nishikawa [9])

3. We began working on the initial case for the project in order to progress to the main case we want to consider.
4. Therese gave a talk "Non-commutative analogues of geometric structures for twisted group C^*-algebras using length functions on groups" on her work and made connections.

Specifically Therese started speaking about her work with Carla Farsi, Nadia Larsen and Judith Packer, and spoke about the twisted structures and connections to some of the material we had spoken about during the week.

Our plan since BIRS is to meet on Zoom fortnightly and also distribute tasks among the group members as these arise. We also want to give Therese time to finish talking to us about her work with Farsi, Larsen, and Packer. We plan to apply for funding to be able to visit each other as an entire group or in subgroups as is possible given schedules. We hope to meet in-person within the next year as a group as well additionally with both Indira Chatterji and Hang Wang whom we spoke with online during our time here in Banff.

3.5 Some aspects of Gaussian Quantum Markov Semigroups

Group members: Priyanga Ganesan (University of California at San Diego), Arundhathi Krishnan (Munster Technological University), Yulia Kuznetsova (Université de Franche-Comté, Sarah Plosker (Brandon University), Emanuela Sasso (group co-leader; University of Genova), Sachi Srivastava (group co-leader; University of Delhi)

Our project is based on Gaussian Quantum Markov Semigroups (G-QMS). Recall that a QMS is called Gaussian when the underlying algebra is $\mathcal{B}(\Gamma(\mathbb{C}^d))$, i.e., the algebra of all bounded operators on the Fock space $\Gamma(\mathbb{C}^d)$, and the predual semigroup acting on trace class operators on $\Gamma(\mathbb{C}^d)$ preserves Gaussian states. Our goal is to investigate irreducibility and decoherence, phenomena that could appear when there is interaction between the open quantum system and the environment. In particular we have that the environment induces decoherence if the system can be decomposed in two parts: one that evolves as a closed quantum system and a second part which disappears in long time. Both the concepts of decoherence and irreducibility are related to the ergodic behaviour of a QMS. Our work starts from [12, 14, 18, 11] (decoherence for uniformly continuous QMSs) and [3, 4, 36] (G-QMSs)

During the week of the workshop, we started a lively and productive collaboration. Our plan was to study irreducibility and decoherence for G-QMSs. However, we immediately realized that [17] had already (recently) solved the characterization of irreducibility in $d-$dimension. So we approached the problem of decoherence and were able to achieve the following for a Gaussian QMS $(\mathcal{T}_t)$ with faithful normal invariant state ρ.

- Characterize the decoherence free algebra $N(\mathcal{T})$ completely when ρ is Gaussian and deduce that EID holds.
- Find a description of the set of fixed points $\mathcal{F}(\mathcal{T})$ of $(\mathcal{T}_t)$ in terms of Weyl operators and certain matrices that depend on the GKLS representation of the generator of the QMS.
- Express the fixed points set as
$$\mathcal{F}(\mathcal{T}) = \bigoplus p_i \mathcal{F}(\mathcal{T}) p_i,$$
where p_i are minimal projections on $\mathcal{F}(\mathcal{T})$ and $p_i\mathcal{F}(\mathcal{T})p_i$ is a type-I factor, and then define the set of reduced semigroups: $\mathcal{T}_t^i(p_i x p_i) := p_i \mathcal{T}_t(x) p_i$ on $p_i B(\mathfrak{h}) p_i$. This allows us to deduce important properties of $\mathcal{F}(\mathcal{T})$.

Our work thus far gives rise to several conjectures, involving behaviour of the minimal projections, faithful invariant states of the G-QMS and also $\mathcal{F}(\mathcal{T})$.

Plans for the near future: We plan to address our conjectures and obtain results about the structure of faithful invariant states for a G-QMS. Our long term goal is to investigate EID for G-QMSs. Finally by starting from [32] we want to explore how we can use some valid results for quantum channel for QMSs.

We will meet online to continue working on our project. We applied to the Fields Thematic Program on Operator Algebras and Applications and we have obtained funds for a research week in November in Toronto. We have also invited Veronica Umanità (University of Genova) to join this group project.

3.6 High dimensional expanders and the coarse Baum-Connes conjecture

Group members: Sherry Gong (Texas A&M University), Sanaz Pooya (group co-leader; Stockholm University), Hang Wang (group co-leader; East China Normal University), and Kun Wang (Texas A&M University)

The aim of our project is to construct new counterexamples of the coarse Baum-Connes conjecture for a metric space X with bounded geometry, via higher dimensional expanders. The coarse Baum-Connes conjecture states that the coarse assembly map

$$\mu : \lim_{d\to\infty} K_*(P_d(X)) \to K_*(\mathrm{C}^*(X))$$

is an isomorphism. Let X be a countable family of higher dimensional expanders as in [28], with a uniform spectral gap at 0. Motivated by Higson's counterexamples to the coarse Baum-Connes conjecture [20], we would like to construct the associated higher Kazhdan projection p in the Roe algebra $\mathrm{C}^*(X)$. Then the hope is to establish functionals

$$\phi_1, \phi_2 : K_0(\mathrm{C}^*(X)) \to \frac{\prod \mathbb{R}}{\bigoplus \mathbb{R}}$$

so that $\phi_1(p) \neq \phi_2(p)$ while $\phi_1 = \phi_2$ over the image of the coarse assembly map μ.

During the focus week, we looked into relevant papers by Lubotzky [28], by Higson [20], by Higson-Lafforgue-Skandalis [21], by Willett-Yu [38], by Drutu-Nowak [15] and by Sawicki [34], as well as the survey paper by Gomez-Aparicio, Julg and Valette [19]. First, we observed that all existing counterexamples employ the idea of Higson [20]. Also, we noticed that most papers except for the one by Willett-Yu [38] construct the obstructing projections based on groups, rather than from expanders. Based on these findings, for the higher dimensional case, we need to find a suitable replacement to the notion of "large girth", so that the functionals τ_i for $i = 0, 1$ can be constructed. We also need to verify that the associated projection p is a ghost so that applying τ_0, one has $\tau_0(p) = 0$, or we need to find a characterization of p so that $\tau_0(p) \neq \tau_1(p)$.

At the end of the workshop, we made a plan to have regular online group meetings till the end of the year and a one-week gathering in 2024. Part of the group members will gather at some conferences during the summer of 2023. We expect to have a first draft of the article in the near future.

3.7 Nuclear dimension of crossed products by Hilbert bimodules over commutative C^*-algebras

Group members: Marzieh Forough (group co-leader; Czech Technical University in Prague), Zahra Hasanpour Yakhdani (University of Tehran), Ja A Jeong (Seoul National University), Preeti Luthra (University of Delhi), Melody Molander (University of California San at Diego), Karen Strung (group co-leader; Institute of Mathematics, Czech Academy of Sciences)

This project is set out to study the nuclear dimension of crossed products by (full) Hilbert bimodules over commutative C*-algebras. This class of crossed products coincides with the class of C*-algebras of vector bundles twisted by homeomorphisms. The nuclear dimension is a noncomumtative version of covering dimension of topological spaces introduced by W. Winter and J. Zacharias (2010). This notion has played a key role in the classification program. Computing the nuclear dimension for C*-algebras, in particular, the ones which come from some sort of dynamical systems, has attracted a lot of attention. I. Hirshberg and J. Wu (2017) obtained an upper bound for the nuclear dimension of a crossed product of $C_0(X)$ by any automorphism where X is a locally compact Hausdorff space of finite covering dimension. Any such crossed product can be viewed as a C*-algebra of a trivial vector bundle twisted by a homeomorphism. The goal of this project is to extend the result of Hirshhberg and Wu to the case that the underlying vector bundle is not trivial. In particular, we aim to study the nuclear dimension of C*-algebras of a vector bundle twisted by a homeomorphism when the base space of the underlying vector bundle has finite covering dimension.

During the workshop at BIRS, we computed an upper bound for the nuclear dimension of the C*-algebra of a line bundle twisted by a periodic homeomorphism. Continuing working on this project after the workshop, we came up with an approach to deal with the general case by using our result for periodic homeomorphisms.

We are planning to continue our discussion via zoom and a shared overleaf. After some more progress, those members who are interested enough and collaborate actively will continue this project.

3.8 Towards an Operator-Algebraic Breeden–Litzenberger Formula

Group members: Sarah Reznikoff (Virginia Tech), Lauren C. Ruth (group leader; Mercy College), Lydia de Wolf (Siena College)

Background on Breeden–Litzenberger Formula

This project is motivated by a result in options pricing called the **Breeden–Litzenberger formula**. Let K be a random variable representing the strike price of an option. Let X be a random variable representing the stock value after one year. The payoff will be $\max\{X-K,0\}$. Suppose you know the expected value of your payoff for every value k of the strike price. Surprisingly, it is then possible to determine the density function of the stock value, $f_X(x)$. Define

$$C(k) = E\left[\max\{X-k,0\}\right].$$

Breeden and Litzenberger showed in [6] that

$$\frac{d^2}{dk^2}C(k) = f_X(k).$$

Their proof uses the calculus of finite differences, since they are working with discrete, real-world data. In the continuous setting, the proof begins as follows.

$$C(k) = E\left[\max\{X-k,0\}\right] = \int_0^\infty \max\{x-k,0\} f_X(x)dx = \int_k^\infty (x-k) f_X(x)dx$$

To differentiate this with respect to k requires the formula for **differentiation under an integral with variable limits**:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} g(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial k} g(x,t)dx + g(b(t),t)\frac{db}{dt} - g(a(t),t)\frac{da}{dt} \tag{1}$$

Assuming finite expectation, we have

$$\frac{d}{dk}\lim_{b\to\infty}\int_k^b (x-k)f_X(x)dx = \lim_{b\to\infty}\left(\frac{d}{dk}\int_k^b (x-k)f_X(x)dx\right)$$
$$= \lim_{b\to\infty}\left(\int_k^b \frac{\partial}{\partial k}(x-k)f_X(x)dx + ((b)-k)\frac{d(b)}{dk} - ((k)-k)\frac{d(k)}{dk}\right)$$
$$= -\int_k^\infty f_X(x)dx + 0 - 0 = -\left(1-\int_{-\infty}^k f_X(x)dx\right) = -(1-F_X(k)) = F_X(k)-1$$

where F_X is the cumulative distribution function corresponding to the density function f_X. Differentiating once more, we obtain

$$\frac{d}{dk}\left(F_X(k)-1\right) = f_X(k)$$

thus the density function of X has been recovered as the second derivative of the expected value of max$\{X - k, 0\}$ with respect to k.

Project Goals and Preparation

We seek to find the operator algebraic version of the Breeden–Litzenberger formula by replacing the measure space with a von Neumann algebra, a common theme in our field. In this non-commutative setting, the "expectation" from probability becomes the "trace" of an operator. For example, Ishan, Peterson and Ruth generalized the notion of measure equivalence to "von Neumann equivalence" in [22]. The experience of working on that paper prepared Ruth to undertake the proposed project with Reznikoff and de Wolf.

Our WOA III team is uniquely positioned to attack this problem, which sits at the intersection of operator algebras and finance: Ruth and Reznikoff have both taught courses in actuarial science (Ruth for Exam P, on probability, and Reznikoff for Exam FM, on financial mathematics); and Ruth and de Wolf have both passed actuarial exams.

Project Progress

The difficulty in the Breeden–Litzenberger generalization appears to lie in finding the right notion of "differentiation under the integral," formula (1) above. We thank Ben Hayes for the suggestion that what we are looking for might be closable derivations ([29], [30]). During WOA III, we reviewed pre-requisites for understanding the predual of a von Neumann algebra, and we concluded by reviewing pre-requisites for working with closable derivations. We look forward to applying what we have learned during WOA III towards our project in the coming months.

Impact on Participants' Careers

This fall, Reznikoff will begin serving as department chair at Virginia Tech, de Wolf will start a new position at Siena College, and Ruth will enter her fourth year on the tenure track at Mercy College. WOA III has provided us with a crucial opportunity to advance our research while managing heavy service responsibilities and teaching loads.

4 Outcomes of the Meeting

Underrepresentation of women is a particular concern in operator algebras, where it is particularly acute. Certain women will particularly benefit from the opportunity offered by the workshop: those with positions at small teaching-oriented colleges, those who have just finished their PhD and need multiple projects to move forward with a research career, and those who have postdoctoral or tenure-track positions in departments with no suitable collaborators.

Strengthening the research careers of women in the field will have broader impacts as well. As the participants return to their institutions, they will be better able to mentor and inspire women undergraduates and graduate students who may pursue higher degrees or careers in mathematics. In addition, having more accomplished and research-active women will improve the existing culture in operator algebras for the next generation.

All of the groups indicated that they will be continuing their collaboration virtually, with several groups meeting in the near future at the Fields Institute during the Thematic Program on Operator Algebras and Applications that is occurring in the Fall of 2023. Other groups will attempt to meet in-person at various upcoming conferences or plan their own research visits. These are good indications that group members see the benefits of the collaborations, and that the research may result in peer-reviewed journal articles in the long-run. Research groups have also been encouraged to seek further funding from other programs such as

1. MSRI's *Summer Research in Mathematics*,
2. Oberwolfach, and
3. the American Institute of Mathematics' *SQuaREs* program.

A follow-up survey will be administered to project leaders in six or eight months to determine projects' progresses, developments and career impacts.

The beneficial outcomes, both tangible and intangible, from the WOA workshops demonstrate the value of such an event in providing support and networking opportunities, particularly for women and others from underrepresented gender groups.

The organizers led a discussion about a potential fourth iteration of WOA, and the response was overwhelmingly positive. Numerous participants put their names forward or nominated women who were not in attendance to help with the organizational aspects of a WOA IV, tentatively set for 2026.

5 Acknowledgement

We are grateful to BIRS for hosting our workshop. We thank the Clay Foundation, the Foundation Compositio Mathematica and the Fields Institute for Research in Mathematical Sciences for travel funds for selected participants. All the graduate students and early career researchers who requested it received travel support to attend the workshop.

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