Topological Orbit Equivalence for Cantor Minimal Systems

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The project is an ongoing one; the first paper appeared in 1995. The program is to classify, up to orbit equivalence, group actions and, more generally, étale equivalence relations on Cantor speaces. During the RIT period at BIRS, we worked on the case of minimal, free $\mathbb{Z}^2$ actions on Cantor sets.

The following set of notes was written by Ian Putnam as a survey of the current state of the program. The RIT result is covered in the last section.

1 Introduction

We will be considering dynamical systems, usually minimal, on a Cantor set. By a Cantor set, we mean a compact, totally disconnected metric space with no isolated points. For dynamical systems, we include free actions of a countable group by homeomorphisms. However, our definition, which follows below, will include more general systems. The main problem is to understand the orbit structure of such systems. Specifically, if we are given two such systems, is there a homeomorphisms between the underlying spaces which carries the orbits of one system to the orbits of the other?

This is the natural extension to the topological case of the program in ergodic theory initiated by Henry Dye [Dy], who considered invertible measure preserving transformations of a Lebesgue space. This was continued by many others, most notably Krieger[Kr1] and Connes, Feldman and Weiss [CFW]. In another direction is the Borel case.

2 Group actions

Suppose that X is a compact metric space and that G is a countable, abelian group. (In fact, abelian is not really required here.) Suppose that φ is an action of G on X by homeomorphisms. That is, for every a in G, there is a homeomorphism $\varphi^a : X \to X$ which satisfy $\varphi^{a+b} = \varphi^a \circ \varphi^b$ and $\varphi^0(x) = x$, for all a, b in G and x in X.

We assume that the action is *free*; that is, if $\varphi^a(x) = x$ for some x in X and a in G, then $a = 0$. We say that the action is *minimal* if, for every point x in X, its orbit under the action, $\{\varphi^a(x) \mid a \in G\}$, is dense in X.

3 Étale equivalence relations

The group actions which we described above are our main objects of interest. However, we will expand the class of dynamical systems which we are considering. This extended class will be equivalence relations endowed with some extra structure. It includes the case of a group action by considering the relation in which the equivalence classes are the orbits. This extension is not merely done for the sake of maximal generality; we will use some of the others (AF-relations) in an essential way.

We begin with some notation and basic ideas. For the first part, we will allow more general topological spaces than just Cantor sets. We let X be a compact metric space and we consider an equivalence relation R on X. Shortly, we will restrict to the case that R has countable equivalence classes.

We let r and s (for range and source) denote the two canonical projections from R to X; $s(x, y) = x, r(x, y) = y$. We say that R is *minimal* if every equivalence class is dense in X.

Definition 1 *Let X be a compact metrizable space, R be an equivalence relation on X and $\mathcal{T}$ be a topology on R. We say that $R, \mathcal{T}$ is étale if*

1. *$\mathcal{T}$ is Hausdorff, second countable and σ-compact,*
2. *the diagonal $\{(x, x) \mid x \in X\}$ is open in R,*
3. *the maps $r, s : R \to X$ are local homeomorphisms; that is, for every (x, y) in R, we may find an open set U in $\mathcal{T}$ such that $r(U)$ and $s(U)$ are open in X and $r : U \to r(U)$ and $s : U \to s(U)$ are homeomorphisms,*
4. *if U and V are open sets as above, then the set*

   $$UV = \{(x, z) \mid (x, y) \in U, (y, z) \in V, \text{ for some } y\}$$

   is also open and
5. *if U as above is open, then so is $U^{-1} = \{(x, y) \mid (y, x) \in U\}$.*

When $\mathcal{T}$ is understood, we simply say that R is étale.

(We make some remarks on the terminology, which comes from the theory of groupoids. An equivalence relation is also a principal groupoid. The term 'étale' is relatively recent; in the past these have also been known as 'r-discrete groupoids with counting measure as Haar system'. See [Ren, Pat, GPS2].)

This may seem an unusual definition, so we spend some time elaborating on it. The idea is that $\mathcal{T}$ provides R with the structure of a dynamical system. The key point is item 3. Consider (x, y) in R and let U be as in part 3 of the definition. Consider the restriction of s to U, $s|U$. The map $\gamma = r \circ (s|U)^{-1}$ is a homeomorphism from $s(U)$ to $r(U)$, both open subsets of X. The graph of γ is simply U, which is contained in R. So we may think of R as being made up of the graphs of local homeomorphisms of X. This collection is closed under composition (part 4) and under taking inverses (part 5). Part 2 of the definition is the analogue of freeness of an action.

Let R be an étale equivalence relation on X. If U is an open subset of R as in part 3 above, then we refer to U as a *graph*.

It is probably worthwhile to give a very simple example of a relation which does not admit such a topology. Let R be the relation on the unit interval $[0, 1]$ whose classes are all singletons, except for $\{0, 1\}$. It is clear that there is no local homeomorphism from a neighbourhood of 0 to a neighbourhood of 1, taking 0 to 1 and whose graph is contained in the relation.

Here are some basic facts to keep in mind. We state them without proof.

1. There are relations, even minimal ones, which admit no such topology.

2. If R is étale, then its equivalence classes are countable. (One shows that, for any x in X, $r^{-1}\{x\} \subset R$ is discrete and closed and then uses σ-compactness.)

3. If R is étale and compact in the topology, then the topology must be the relative topology from $R \subset X \times X$. Moreover, in this case, the equivalence classes are finite. In fact, there is a uniform upper bound on the number of elements in an equivalence class.

4. If R is étale and has an infinite equivalence class, then $\mathcal{T}$ is *not* the relative topology of $X \times X$.

5. If R has an étale topology, it may not be unique. (This may seem a little surprising. Just to make things clear, the topology on X is fixed.)

We note one easy consequence of the definitions.

Theorem 2 *Suppose that R is an étale equivalence relation on X and that R' is an open subequivalence relation of R. Then R' is also étale, in the relative topology from R.*

Finally, we note that there is a natural notion of invariant measure for étale relations. A measure μ is R-invariant [Ren] if

$$\mu(r(U)) = \mu(s(U)),$$

for all graphs $U \subset R$. We let $M(X, R)$ denote the set of all R-invariant probability measures on X. There is a notion of amenability for relations. Describing this would take us too far afield, but we note that amenable relations always possess such measures [Ren]. We will have nothing to say about the case that there are no finite R-invariant measures.

3.1 Group actions (revisited)

We recall the situation of the earlier section, where X is a compact metric space, G is a countable abelian group and φ is a free action of G on X. Our relation of interest in this case is

$$R_\varphi = \{(x, \varphi^a(x)) \mid x \in X, a \in G\}.$$

To topologize R_φ, we use the fact that the map sending (x, a) in $X \times G$ to $(x, \varphi^a(x))$ is bijective, since the action is free. We give G the discrete topology and $X \times G$ the product topology and use the map to transfer this to R_φ. In other words, a sequence, $(x_n, \varphi^{a_n}(x_n))$ converges to $(x, \varphi^a(x))$ in this topology if and only if x_n converges to x in X and a_n converge to a in the discrete topology. (See [Ren].)

This means that the graph of each homeomorphism φ^a is a compact open set in R_φ. Letting U be this graph, the associated map γ from our earlier discussion is just φ^a. One can think of this as the special case where our maps γ are actually given by global rather than just local homeomorphisms of X.

Theorem 2 takes on a new significance in this context: if one considers $R = R_\varphi$ arising from a group action, it is possible that open subequivalence relations need not be themselves group actions. More than just possible, this will be a critical step for us later.

3.2 AF-relations

In this section, we introduce one of the most important classes of étale relations called AF-relations [Ren, GPS2]. The terminology (which actually comes from C^*-algebra theory) represents 'approximately finite'. In these examples, the underlying space is totally disconnected.

Definition 3 *An étale relation R on X is an* AF-relation *if X is compact, metrizable and totally disconnected and if there are*

$$R_1 \subset R_2 \subset \cdots$$

such that $\cup_n R_n = R$ and $R_n \subset R$ is a compact open subequivalence relation, for each n.

We will have much more to say about these examples in a later section. For the moment, we want to point out that by including these relations, we are expanding significantly from group actions by noting the following.

Theorem 4 ([GPS2]) *Let φ be a free action of a countable group G on a compact, totally disconnected metric space, X. The relation R_φ is an AF-relation if and only if the group G is locally finite; that is, there is an increasing sequence of finite subgroups of G, $G_1 \subset G_2 \subset \cdots$ whose union is G.*

We give a sketch of the proof. We first suppose that G is locally finite and choose a sequence of subgroups as in the theorem. For each $n \geq 1$, we let

$$R_n = \{(x, \varphi^a(x)) \mid x \in X, a \in G_n\}.$$

It is easy to see that each R_n is a compact open subrelation of R_φ and their union is R_φ.

To prove the converse statement, we suppose that R_φ may be written as an increasing union of compact open subrelations $R_n, n \geq 1$. Select a finite subset F of G. We will argue that the subgroup of G generated by F, denoted $< F >$ will be finite. It is fairly easy to see that this then implies that G is locally finite. Consider U, the union of the graphs of the elements of F, which is compact in R_φ. Since the R_n form an increasing open cover, U is contained in R_n, for some n. Since R_n is a subrelation, it is fairly easy to check that the graph of any element $< F >$ is also in R_n. This means that the orbits of any point in X under $< F >$ is finite. By the freeness of the action, this implies that $< F >$ is finite.

4 C^*-algebras (briefly)

The notion of an étale equivalence relation comes from C^*-algebra theory. We will not use any C^*-algebra theory in the remainder of these notes or even in the complete proofs. However, we take a few moments to give some idea of the connections. More information may be found in [Ren, Pat].

A C^*-algebra, A, (briefly) is a *-algebra over the complex numbers equipped with a norm $\| \cdot \|$. That is, we have addition, scalar multiplication and a ring multiplication, which is, in general, not commutative. There is also a conjugate linear involution $a \to a^*$ satisfying $(ab)^* = b^* a^*$. The ring multiplication need not, in general, have a unit, but the algebras we construct here will be unital. Of course, the algebraic operations should be continuous in the topology coming from the norm. Moreover, regarded as a metric space with $d(a, b) = \| a - b \|$, A should be complete. Finally, the norm should satisfy the C^*-condition, $\| a^* a \| = \| a \|^2$, for all a in A. This condition may seem obscure to the non-expert, but is really quite powerful.

The first example is the complex numbers, with $*$ being complex conjugation and the usual norm. The second example is $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices. The $*$ operation is conjugate transpose and the norm is

$$\| a \| = \sup\{\| a\xi \|_2 | \ \xi \in \mathbb{C}^n, \| \xi \|_2 = 1\},$$

for all a in $M_n(\mathbb{C})$, where $\| \cdot \|_2$ denotes the l^2-norm on $\mathbb{C}^n$. This example can be easily generalized to the algebra of bounded linear transformation on a complex Hilbert space, by replacing $\mathbb{C}^n$ by the Hilbert space.

If R is an étale equivalence relation on a space X (not necessarily Cantor), we may construct a C^*-algebra as follows. Let $C_c(R)$ denote the set of continuous, compactly supported complex-valued functions on R. It is a linear space in an obvious way. The product and involution are defined by the formulas

$$\begin{aligned} (f \cdot g)(x, y) &= \sum_{(x,z) \in R} f(x, z) g(z, y) \\ f^*(x, y) &= \overline{f(y, x)}, \end{aligned}$$

for all f, g in $C_c(R)$ and (x, y) in R. It is a subtle point here that the product $f \cdot g$ is again in $C_c(R)$. The proof uses the étale property of R.

The formula above for the product should remind one of matrix multiplication. Indeed, if $X = \{1, 2, \ldots, n\}$ and $R = X \times X$, then this algebra is just $M_n(\mathbb{C})$.

The issue of a norm is more subtle. For each point x_0 in X, one can consider the Hilbert space of l^2 sequences on its equivalence class, $[x_0]$. Each f in $C_c(R)$ defines a linear transformation on this Hilbert space by

$$f\xi(x) = \sum_{(x,y)\in R} f(x,y)\xi(y),$$

for x in $[x_0]$. This transformation is bounded and we define

$$\| f \| = \sup\{\| f \|_{x_0} \mid x_0 \in X\},$$

where $\| \cdot \|_{x_0}$ denotes the operator norm for the Hilbert space $l^2[x_0]$. Again in the finite case above, this gives the same norm on $M_n(\mathbb{C})$. Of course, it is necessary to prove that this supremum is finite. Finally, the algebra $C_c(R)$ is usually not complete in this norm. We complete it to obtain a C^*-algebra which is denoted by $C_r^*(R)$ called the reduced C^*-algebra of R. The reason for the subscript r and the term 'reduced' is that there are other choices for the norm. The one above is arguably the most interesting. For amenable equivalence relations, all (reasonable) norms are the same.

5 Isomorphism and Orbit equivalence for étale relations

There are two natural notions of equivalence between two étale relations which we describe now. Classifying systems up to orbit equivalence is our main objective.

Definition 5 ([GPS2]) *Let (X, R) and (X', R') be two étale relations.*

1. *We say that (X, R) and (X', R') are* orbit equivalent *and write $(X, R) \sim (X', R')$ if there is a homeomorphism $h : X \to X'$ such that*

$$h \times h(R) = R'.$$

That is, the map h carries R-equivalence classes exactly to R'-equivalence classes.

2. *We say that (X, R) and (X', R') are* isomorphic *and write $(X, R) \cong (X', R')$ if there is a homeomorphism $h : X \to X'$ such that*

$$h \times h(R) = R'$$

and such that $h \times h : R \to R'$ is a homeomorphism.

The first notion is probably the most natural one for dynamics. However, much of what we are doing really uses the topology on R and this makes the second important. It is worth pointing out here that what is really going on is that the topologies which are given to our relations are not usually unique. Given an orbit equivalence h from R to another relation, we may use $(h \times h)^{-1}$ to transfer the other topology back to R, but it may not agree with the original from R.

Let us also take a moment here to explain why we concentrate on totally disconnected spaces. Just to be specific, suppose that φ is a free action of the group G on the compact, connected space X and ψ is a free action of the group H on the compact, connected space Y. Also suppose that $h : X \to Y$ is an orbit equivalence. This means that, for x in X and a in G, we may find b in H such that $h(\varphi^a(x)) = \psi^b(h(x))$. Fix a for the moment and for each b in H, let B_b be the set of x where the equation above holds. The sets $B_b, b \in H$ form a countable partition of the space X. It is fairly easy to check that each of these sets is closed. By a result of Sierpinski, as X is connected, one of these sets must be all of X and the rest are empty. We will not pursue this, but it allows a rather precise (and very restrictive) description of the map h.

We have hinted at the importance of AF-relations. It leads us to the following definition.

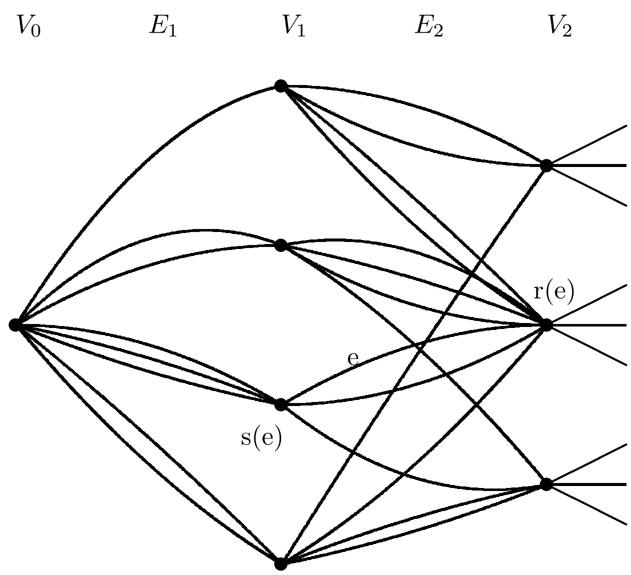
Definition 6 ([GPS2]) *Let (X, R) be an étale relation with X totally disconnected. We say that (X, R) is* affable *if (X, R) is orbit equivalent to an AF-relation (X', R').*

The reason for the terminology follows from the last comment of the previous paragraph. If (X, R) is orbit equivalent to an AF-relation, we may use the orbit map to transfer the topology to R. That is, R may be given a new topology in which it is AF. So that R is 'AF-able' or affable.

6 The construction of AF-relations

One thing which was missing from our definition of AF-relations earlier was a general method for producing such systems. We present this now.

We begin with a Bratteli diagram: a locally finite, infinite directed graph as shown below. (See [HPS, Ef].)



It consists of a vertex set V which is partitioned into a sequence of non-empty finite sets, $V_n, n \geq 0$, and an edge set, E, which is also partitioned into a sequence of non-empty finite sets, $E_n, n \geq 1$. Each edge e in E_n has a source, $s(e)$ in V_{n-1}, and a range, $r(e)$ in V_n. (This is a different use of the terms range and source than earlier, but it should not cause any confusion.) For simplicity, we assume that V_0 consists of a single vertex and for every other vertex v, $r^{-1}\{v\}$ and $s^{-1}\{v\}$ are non-empty. (There are no sources, other than in V_0, or sinks.)

The space X is the set of infinite paths in the diagram. That is,

$$X = \{(e_1, e_2, \ldots) \mid e_n \in E_n, r(e_n) = s(e_{n+1}), n \geq 1\}.$$

It is given the relative topology from the product space $\Pi_n E_n$ in which it is compact, metrizable and totally disconnected. For each $N \geq 0$, we define

$$R_N = \{(e, f) \mid e, f \in X, e_n = f_n, \text{ for all } n > N\}.$$

This set is given the relative topology of the product $X \times X$. It is easy to check that R_N is a compact étale equivalence relation (and hence each equivalence class is finite), and that R_N is an open subset of R_{N+1}, for all N. We define

$$R = \cup_{N=0}^{\infty} R_N$$

and it is given the inductive limit topology.

It is not difficult to check that R is an étale equivalence relation on X. It is worth considering for a moment the local homeomorphisms we described earlier. Suppose that $(e_1, e_2, \ldots, e_N)$ and $(f_1, f_2, \ldots, f_N)$ are two finite paths in the diagram, ending at the same vertex $v = r(e_N) = r(f_N)$. We let U be the clopen set in R_N

$$U = \{(e', f') \mid e'_n = e_n, f'_n = f_n, n \leq N, e'_n = f'_n, n > N\}.$$

The map γ is

$$\gamma(e_1, e_2, \ldots, e_N, e'_{N+1}, e'_{N+2}, \ldots) = (f_1, f_2, \ldots, f_N, e'_{N+1}, e'_{N+2}, \ldots)$$

for all sequences of the form $(e_1, e_2, \ldots, e_N, e'_{N+1}, e'_{N+2}, \ldots)$ in X.

We denote X by $X(V, E)$ and $R = R(V, E)$.

Theorem 7 ([GPS2]) *Let R be an AF-relation on a totally disconnected compact metrizable space X. Then (X, R) is isomorphic to $(X(V, E), R(V, E))$, for some Bratteli diagram V, E.*

7 Invariants for Cantor étale relations

We will introduce two invariants for Cantor minimal systems. These will be ordered abelian groups [Go, Ef]. By an order on an abelian group, G, we mean a subset G^+ such that $G^+ \cap (-G^+) = \{0\}$, $G^+ + G^+ \subset G^+$ and $G^+ - G^+ = G$. The set G^+ is usually referred to as a positive cone. An order in the usual sense is obtained by setting $a \geq b$ if and only if $a - b$ is in G^+. Also, our ordered groups will have a distinguished positive element. Such an element, u, is called an order unit if for every a in G^+, we have $nu - a$ is in G^+, for some $n \geq 1$.

We let $C(X, \mathbb{Z})$ denote the continuous $\mathbb{Z}$-valued functions on X. It is an abelian group with the operation of pointwise addition. If E is a clopen subset of X, we let χ_E denote its characteristic function, which is in $C(X, \mathbb{Z})$.

We define $B(X, \varphi)$ to be the subgroup of $C(X, \mathbb{Z})$ generated by the functions $\chi_{r(U)} - \chi_{s(U)}$, where U is a compact graph in R.

We define $B_m(X, R)$ to be the subgroup of $C(X, \mathbb{Z})$ generated by the functions f such that $\int_X f d\mu(x) = 0$, for all μ in $M(X, R)$. (In the case that there are no invariant probability measures, we have $B_m(X, R) = C(X, \mathbb{Z})$.) It is clear that $B(X, R) \subset B_m(X, R)$.

Definition 8 *Let R be an étale relation on the Cantor set X. We define*

$$D(X, R) = C(X, \mathbb{Z})/B(X, R),$$

with positive cone

$$D(X, R)^+ = \{[f] \mid f \in C(X, \mathbb{Z}), f \geq 0\},$$

and order unit $u = [1]$.

We also define

$$D_m(X, R) = C(X, \mathbb{Z})/B_m(X, R),$$

with positive cone

$$D_m(X, R)^+ = \{[f] \mid f \in C(X, \mathbb{Z}), f \geq 0\},$$

and order unit $u = [1]$.

Although the notation and context are slightly different, a version may be found in [HPS]. Notice that $D_m(X, R)$ is a quotient of $D(X, R)$.

Earlier, we had two notions, isomorphism and orbit equivalence, between Cantor étale relations. We now spell out the precise sense in which our 'invariants' are invariant.

Theorem 9 *Let (X, R) and (X', R') be two étale relations on Cantor sets. If $h : X \to X'$ is a homeomorphism which implements an isomorphism between the relations, then the map $h^*[f] = [f \circ h]$, $f \in C(X', R')$, is an isomorphism from $D(X', R')$ to $D(X, R)$ mapping $D(X', R')^+$ onto $D(X, R)^+$ and preserving the order units. If $h : X \to X'$ is a homeomorphism which implements an orbit equivalence between the relations, then the map $h^*[f] = [f \circ h]$, $f \in C(X', R')$, is an isomorphism from $D_m(X', R')$ to $D_m(X, R)$ mapping $D_m(X', R')^+$ onto $D_m(X, R)^+$ and preserving the order units.*

The first statement is quite easy. For the second, it is fairly easy to check that an orbit equivalence will induce a bijection between the sets of invariant measures of the two systems.

8 The invariants of an AF-relation

The structure of the invariants introduced in the last section are quite well-understood for AF-relations.

First, we want to describe briefly how, if one is given a Bratteli diagram V, E, the invariant $D(X(V, E), R(X, R))$ can be computed. For each $n \geq 0$, let $\mathbb{Z}^{V_n} = \{f : V_n \to \mathbb{Z}\}$. This group is given the simplicial or standard order, $f \in \mathbb{Z}^{V_n +}$ if and only if $f(v) \geq 0$ for all v in V_n. The edge set E_n gives a group homomorphism $\alpha_n : \mathbb{Z}^{V_{n-1}} \to \mathbb{Z}^{V_n}$ as follows. Either one can consider E_n as providing a rectangular adjacency matrix and the homomorphism is simply multiplication by this matrix, or equivalently, we have, for f in $\mathbb{Z}^{V_{n-1}}$,

$$\alpha_n(f)(v) = \sum_{r(e)=v} f(s(e)), v \in V_n.$$

This provides an inductive system of ordered abelian groups.

Theorem 10 (See [HPS]) *For an AF-relation, $(X(V, E), R(V, E))$, the group $D(X(V, E), R(V, E))$ is the inductive limit of the system $(\mathbb{Z}^{V_n}, \alpha_n)$, in the category of ordered abelian groups.*

Such a group is called a *dimension group*. A fundamental result in the subject is the following result.

Theorem 11 (Effros-Handelman-Shen [Ef, Go]) *A countable, ordered abelian group G, G^+ is a dimension group if and only if*

1. *it is unperforated: if g is in G and ng is in G^+ for some positive integer n, then g is in G^+, and*
2. *it satisfies the Riesz interpolation property: if g_1, g_2, h_1, h_2 are in G such that $g_i \leq h_j$ for all $1 \leq i, j \leq 2$, then there exists g in G such that $g_i \leq g \leq h_j$ for all $1 \leq i, j \leq 2$.*

The point is that it is relatively easy to find groups which satisfy the two conditions of the theorem. To any such group, we may find an AF-relation, (X, R), having $D(X, R)$ isomorphic to that group.

An *order ideal* in an ordered abelian group G, G^+ is a subgroup H such that $H \cap G^+$ generates H as a group and whenever g in G^+ and h in $H \cap G^+$ satisfy $g \leq h$, then g is in H. A dimension group is *simple* if the only order ideals are 0 and G. (See [Go].)

Theorem 12 (See [HPS]) *An AF-relation (X, R) is minimal if and only if the associated dimension group $D(X, R)$ is simple.*

9 The strategy for orbit equivalence results

We now take a few lines to set out our strategy, without being too precise about all the terms, for proving orbit equivalence results. There are three steps:

1. Classify minimal AF-relations.

2. Let R be a minimal AF-relation on a Cantor set X. Suppose that $Y_1, Y_2 \subset X$ are 'small' closed subsets and $\alpha : Y_1 \to Y_2$ is a homeomorphism. Show that the relation

$$R \vee Graph(\alpha)$$

   is orbit equivalent to R. Here, $\vee$ denotes the equivalence relation generated by the two sets.

3. For a free, minimal action φ of the group G on the Cantor set X, find a sequence $R_1 \subset R_2 \subset \cdots$ of compact open subequivalence relations of R_φ, whose union, denoted by R, is minimal, and Y_1, Y_2, α as above such that

$$R_\varphi = R \vee Graph(\alpha).$$

With these steps, the problem of orbit equivalence for actions of the group G is reduced to that of AF-relations. Of course, the third step above depends on the group G. We will see that we have a complete answer for the case $G = \mathbb{Z}$ and a partial one for the group $G = \mathbb{Z}^2$.

This is the same strategy used by Dye in the ergodic measure preserving case. For the first step, there is only AF-relation in this case, up to orbit equivalence. For the second step, the meaning of 'small' in the definition of the sets Y_1 and Y_2 is measure zero. Then the result needed for that step is trivial. The third step is done by using Rohlin partitions for the appropriate group. It is the classic Rohlin lemma for $G = \mathbb{Z}$. This can be extended to include amenable groups.

10 Classification of AF-relations

One of the most important features of AF-relations is that they may be classified up to isomorphism and also (at least in the minimal case) up to orbit equivalence by the invariants we have discussed.

Theorem 13 (Elliott-Krieger [Kr2]) *For AF-relations (X, R), the triple $(D(X,R), D(X,R)^+, [1])$ is a complete invariant for isomorphism.*

Building on this, one may also obtain the following result, but the hypothesis of minimality is also needed.

Theorem 14 (Giordano-Putnam-Skau [GPS1]) *For minimal AF-relations (X, R), the triple $(D_m(X,R), D_m(X,R)^+, [1])$ is a complete invariant for orbit equivalence.*

We will not discuss the proofs of these. The second result appears in [GPS1] as a consequence of the classification for $\mathbb{Z}$-actions. In hindsight, this seems to be putting the proverbial cart before the horse. A direct proof can be given, and it now seems much more logical to proceed with this result first.

11 The absorption theorem

We now turn our attention to the second step of our strategy. That is, showing that a minimal AF-relation may be enlarged slightly and remain orbit equivalent to the result. Here, the topological case is much more subtle than, say, the measurable. The precise result, which we refer to as the absorption theorem, follows below.

Theorem 15 ([GPS2]) *Let X, R be a minimal AF-relation. Suppose that Y_1 and Y_2 are closed subsets of X and $\alpha : Y_1 \to Y_2$ is a homeomorphism such that the following hold.*

1.
$$R \cap (Y_1 \times Y_2) = \emptyset,$$
2. *$\mu(Y_1) = \mu(Y_2) = 0$, for all μ in $M(X, R)$,*
3. *$R \cap (Y_i \times Y_i)$ is an étale relation on Y_i, for $i = 1, 2$,*
4. *α is an isomorphism from $(Y_1, R \cap (Y_1 \times Y_1))$ to $(Y_2, R \cap (Y_2 \times Y_2))$.*

Then the relation
$$R \vee Graph(\alpha)$$
is orbit equivalent to R.

12 Minimal $\mathbb{Z}$-actions

Our main objective is the classification up to orbit equivalence. However, this seems a good time to note the following result regarding isomorphism for the relations coming from $\mathbb{Z}$-actions.

Theorem 16 (Boyle, see [GPS1]) *Let φ and ψ be two minimal $\mathbb{Z}$-actions on Cantor sets. The relations R_φ and R_ψ are isomorphic if and only if φ is conjugate to ψ or to ψ^{-1}.*

Now we return to the problem of orbit equivalence, concentrating on the group $G = \mathbb{Z}$.

Theorem 17 (Giordano-Putnam-Skau [GPS1]) *Let φ be a minimal action of $\mathbb{Z}$ on the Cantor set X. Then the relation R_φ is orbit equivalent to an AF-relation. (That is, R_φ is affable.)*

The following is an immediate consequence of this result and Theorem 14.

Corollary 18 *The triple $(D_m(X, R), D_m(X, R)^+, [1])$ is a complete invariant for orbit equivalence for the class of Cantor systems consisting of minimal AF-relations and minimal $\mathbb{Z}$-actions.*

We will sketch a proof of Theorem 15, showing how the absorption theorem of the last section is used.

Begin by selecting a sequence of clopen sets $U_1 \supset U_2 \supset \cdots$, whose intersection is a single point y. For each $n \geq 1$, let R_n denote the relation generated by $\{(x, \varphi^1(x)) \mid x \in X - U_n\}$. Notice that if any U_n were empty, R_n would be R_φ. As it is, since U_n is open and φ is minimal, any point in X will enter U_n after a finite number of iterations of φ or φ^{-1}. From this, it follows that the R_n-equivalence class of the point in X is finite. A slightly more careful analysis involving the continuity of the return times of φ on U_n shows that R_n is compact and open. It is clear that $R_n \subset R_{n+1}$, for all $n \geq 1$. We let $R = \cup_n R_n$, which is an AF-relation. It is easy to check that R is minimal. In fact, every R-class is also a φ-orbit, except for the orbit of the point y. We let $Y_1 = \{y\}$, $Y_2 = \{\varphi^1(y)\}$ and $\alpha = \varphi^1$. We are then in a position to apply the absorption Theorem 15. (It is surprisingly easy here to check the hypotheses.) Moreover, we have

$$\begin{aligned} R_\varphi &= (\cup_n R_n) \vee \{(y, \varphi^1(y))\} \\ &= R \vee Graph(\alpha) \\ &\sim R \end{aligned}$$

and we are done.

13 Minimal $\mathbb{Z}^2$-actions

The results of this section are quite recent and are in preparation [GPS3]. We consider a minimal, free action, φ, of the group $\mathbb{Z}^2$ on the Cantor set X. Before we can state our main result, we need some basic notions about cocycles. The first definition is a standard one, although our interest is only in integer-valued cocycles. The next two are new, as far as we know.

Definition 19 *Let R be an étale equivalence relation on X. A* cocycle *or more accurately a* 1-cocycle *for R is a continuous function*

$$\theta : R \to \mathbb{Z}$$

such that

$$\theta(x, z) = \theta(x, y) + \theta(y, z),$$

for all $(x, y), (x, z)$ in R.

Definition 20 *Let (X, φ) be a minimal free $\mathbb{Z}^2$ Cantor system and let C be a subset of $\mathbb{Z}^2$. A cocycle θ for R_φ is* positive *with respect to C if $\theta(x, \varphi^n(x)) \geq 0$ for all $x \in X$ and $n \in C$.*

We say that θ is strictly positive if it is positive and θ is a proper as a map from $\{(x, \varphi^n(x)) \mid x \in X, n \in C\}$ to $\mathbb{Z}$.

Definition 21 *Let (X, φ) be a minimal free $\mathbb{Z}^2$ Cantor system and let θ be a cocycle for R_φ. For any positive integer M, we write $\theta \leq M^{-1}$ if $|\theta(x, \varphi^n(x))| \leq 1$, for all x in X and n in $\mathbb{Z}^2$ with $|n| \leq M$, where $|n| = |(n_1, n_2)| = \max\{|n_1|, |n_2|\}$ denotes the L^∞ norm on $\mathbb{Z}^2$.*

For any a, b in $\mathbb{Z}^2$ which generate it as a group, we define

$$C(a, b) = \{ia + jb \in \mathbb{Z}^2 \mid i, j \geq 0\}.$$

Theorem 22 *Let (X, φ) be a free, minimal $\mathbb{Z}^2$ Cantor system. Suppose that for every pair of generators, a, b, of $\mathbb{Z}^2$ and every positive integer M, there is a cocycle θ such that θ is strictly positive on $C(a, b)$ and $\theta \leq M^{-1}$. Then R_φ is orbit equivalent to an AF-relation. (That is, R_φ is affable.)*

Corollary 23 *The triple $(D_m(X, R), D_m(X, R)^+, [1])$ is a complete invariant for orbit equivalence for the class of Cantor systems consisting of minimal AF-relations, minimal $\mathbb{Z}$-actions and free, minimal $\mathbb{Z}^2$-actions satisfying the hypotheses of Theorem 22.*

The actual theorem has a slightly weaker version of the hypothesis. In any case, the condition is a little strange and we have very little insight at this point whether or not it is reasonable. We know of no free, minimal $\mathbb{Z}^2$action which does not satisfy the condition. We know of two classes of examples which do satisfy the hypothesis which we describe now.

Let p be prime (although the result is surely true for any integer greater than 1). Let X be the p-adic integers. That is, $X = \Pi_{n \geq 0}\{0, 1, \ldots, p-1\}$. It is a group with addition done coordinate-wise modulo p and with carry over to the right. Suppose that α and β are two elements such that $i\alpha + j\beta = 0$ only if $i = j = 0$ and such that the subgroup they generate is dense. (It is not difficult to find such pairs.) We define a $\mathbb{Z}^2$-action by rotation by α and β; that is, $\varphi^{(i,j)}(x) = x + i\alpha + j\beta$, for all x in X and (i, j) in $\mathbb{Z}^2$. This system satisfies the hypotheses of the theorem.

For the second example, we let S^1 be the circle, which we write as $\mathbb{R}/\mathbb{Z}$. Suppose that α, β are real numbers such that $1, \alpha, \beta$ are linearly independent over the rationals. We begin with the $\mathbb{Z}^2$-action on S^1 obtained by rotating by α and β - see the formula in the last example. It is possible to 'cut' the circle along an orbit of this action (or even countably many orbits). Take a single point and replace it by two points separated by a gap. Repeat this process for each point in its orbit under the $\mathbb{Z}^2$-action, using smaller and smaller gaps. The result is a Cantor set which we denote by X. The action extends to X in an obvious way and this is a free minimal $\mathbb{Z}^2$-system. It also satisfies the hypotheses of the theorem.

We conclude with a few general remarks about the hypothesis of the theorem. Let (X, R) be a minimal Cantor étale equivalence relation. If f is in $C(X, \mathbb{Z})$, then $bf(x, y) = f(y) - f(x)$ is called a *coboundary.* The set of all cocycles form an abelian group under addition. The coboundaries form a subgroup and we let $H^1(X, R)$ be the quotient group. If we now restrict to the case of a free, minimal $\mathbb{Z}^2$-action, φ, we may find a canonical copy of the group $\mathbb{Z}^2$ in $H^1(X, R_\varphi)$. Specifically, for a in $\mathbb{Z}^2$, let $\theta_a(x, \varphi^b(x)) =< a, b >$, where $<, >$ denotes the usual inner product. If every cocycle is equal to one of these (up to coboundaries), then it is not hard to see that the hypothesis of the theorem fails. (It is impossible to make small cocycles from the θ_a.) We note that it does not imply that the conclusion fails.

It is interesting to note that in our first example, we have a short exact sequence

$$0 \to \mathbb{Z} \to H^1(X, R_\varphi) \to \mathbb{Z}[1/p] \to 0.$$

In particular, the group H^1 is rank two, but slightly larger than $\mathbb{Z}^2$. In the second example, $H^1(X, R_\varphi) \cong \mathbb{Z}^3$. So in both cases, the cohomology is slightly larger than just $\mathbb{Z}^2$, but it is sufficiently large to provide enough cocycles for application of the theorem.

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