BIRS Research in Teams Report

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Title: Asymptotic Dynamics of Dispersive Equations with Solitons
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Our collaboration during our stay in the BIRS has resulted in very fruitful outcomes. We have obtained results sufficient for about one and a half papers. In the following we describe the background, what we have obtained, and what we plan to do next.

1 Background

Solutions of dispersive partial differential equations (with repulsive nonlinearities) tend to spread out in space, although they often have conserved L^2 mass. There has been extensive study in this subject, usually referred to as scattering theory. These equations include Schrödinger equations, wave equations and KdV equations. When the nonlinearity is attractive, however, these equations possess solitary wave solutions (solitons) which have localized spatial profiles that are constant in time. To understand the asymptotic dynamics of general solutions, it is essential to study the interaction between the solitary waves and dispersive waves. The matter becomes more involved when the linearized operator around the solitary wave possesses multiple eigenvalues which correspond to excited states. The interaction with eigenvectors is very delicate and very few results are known.

For nonlinear Schrödinger equations with solitons, there are two types of results:

1. Control of the solutions in a finite time interval and construction of all-time solutions with specified asymptotic behaviors (scattering solutions). The first kind of results does not allow sufficient time for the excited states interaction to make a difference. In contrast, for scattering solutions the excited state interaction is effectively eliminated and scattering solutions may indeed be very rare.

2. Asymptotic stability of solitons, assuming the spectrum of the linearized operator enjoys certain properties (for example has only one eigenvalue or has multiple "well-placed"). The data are often assumed to be localized so that the dispersive wave has fast local decay.

Currently, only perturbation problems can be treated for large solitons, while more general results can be obtained for small solitons.

2 What we have done

During our stay in BIRS, we first studied small solutions of the equation

$$i\partial_t \psi = (-\Delta + V)\psi + \lambda |\psi|^2 \psi, \qquad \psi(0,\cdot) = \psi_0 \in H^1(\mathbf{R}^3). \tag{1}$$

We assume ψ_0 is small in H^1, but we do not assume ψ_0 is in $L^1(\mathbf{R}^3)$, as is usually assumed. The equation possesses small solitary wave solutions which do not move in space, and is hence a good first-step model problem.

The importance of H^1 results (i.e., with non-localized data) is in that it is intimately related to the Hamiltonian or conservative structure, and more shortly, persistence global in time, in contrast against weighted L^2, whose smallness persists only for short time due to dispersion, and L^1, which may be instantaneously lost and therefore does not seem to have physical relevance. A related motivation is, as more eigenvalues are present, the dispersive component tends to decay very slowly. It is thus essential to be able to remove the localization assumption on the data.

Assume that $-\Delta + V$ supports only one eigenvalue $e_0 < 0$. There is a family of small nonlinear bound states Q_E satisfying

$$(-\Delta + V)Q + \lambda Q^3 = EQ, \qquad E \sim e_0.$$

They give exact solutions $Q_E(x)e^{-iEt}$ to (1). Let $\mathcal{L}_E$ denote the corresponding linearized operator. For any ϕ sufficiently small in H^1, it can be decomposed as

$$\phi = (Q_E + \xi)e^{-i\omega}$$

for a unique set of $E, \omega \in \mathbf{R}$ and $\xi \in H_c(\mathcal{L}_E)$. Since $\psi(t)$ is uniformly small in H^1, there is a well-defined set of functions $E(t), \omega(t) \in \mathbf{R}$ and $\xi(t) \in H_c(\mathcal{L}_{E(t)})$ such that

$$\psi(t,x) = (Q_{E(t)}(x) + \xi(t,x))e^{-i\omega(t)}. \tag{2}$$

$Q_{E(t)}$ and $\xi(t,x)$ are the solitary and dispersive wave components, respectively. We want to study the asymptotic stability of the solitary wave component and the asymptotic completeness of the dispersive wave component. We have obtained the following results, to be collected in the paper "Asymptotic Stability and Completeness in Energy Space for Nonlinear Schrödinger Equations with Small Solitons."

1. Asymptotic stability and completeness. When ψ_0 is sufficiently small in $H^1(\mathbf{R}^3)$, $\psi(t)$ can be uniquely decomposed as in (2), with differentiable $E(t)$, $\omega(t)$ and $\xi(t) \in H_c(\mathcal{L}_{E(t)})$. We have $E(t) \in (E_{\min}, E_{\max})$ and

$$\|\xi\|_{L^2_t W^{1,6}_x \cap L^\infty_t H^1_x} \leq C \|\psi_0\|_{H^1} .$$

Moreover, there exist $E_\infty \sim e_0$ and $\xi_+ \in H_c(\mathcal{L}_{E_\infty}) \cap H^1$ such that $E(t) \to E_\infty$ and

$$\left\| \psi(t) - Q_{E_\infty} e^{-i\omega(t)} - e^{-iE_\infty t} e^{t\mathcal{L}_\infty} \xi_+ \right\|_{H^1} \to 0, \qquad \text{as } t \to \infty. \tag{3}$$

2. Wave operator. For any set of $E_\infty \in (E_{\min}, E_{\max})$ and $\xi_+ \in H_c(\mathcal{L}_{E_\infty}) \cap H^1$ with $\|Q_{E_\infty}\| + \|\xi_+\|_{H^1}$ sufficiently small, there is a solution $\psi(t)$ of (1) such that (3) holds for some $\omega(t) \in \mathbf{R}$.

3. Examples of slow decay. For any non-increasing function $f(t)$ which goes to zero as $t \to \infty$, there exists a solution $\psi(t)$ of (1), decomposed as in (2), and a sequence t_j, $j = 1, 2, 3, \ldots$ with $t_j \to \infty$ as $j \to \infty$, such that

$$\|\xi(t_j)\|_{L^2_{\text{loc}}} \geq f(t_j). \tag{4}$$

Besides the above results for Eq. (1), we have also estimated explicitly all small eigenvalues of the linearized operator for

$$i\partial_t \psi = -\Delta\psi - |\psi|^{p-1}\psi, \qquad x \in \mathbf{R}^d, \qquad \psi(0, \cdot) = \psi_0, \tag{5}$$

when p is closed to the critical exponent p_c for stability and blow-up, $p_c = 1 + 4/d$. This confirms a picture conjectured by M.I. Weinstein (he made some unpublished computations for the 1D case), with greater details.

3 Next project

We plan to extend the known results for (5) to the Hartree equations

$$i\partial_t \psi = -\Delta\psi - \left(\frac{1}{|x|^\alpha} * |\psi|^2\right)\psi, \qquad \psi(0, \cdot) = \psi_0, \qquad 0 < \alpha < d. \tag{6}$$

The equation is similar to (5) but is subtler due to the nonlocalness of the nonlinearity. The case $\alpha = 2$ corresponds to the critical case $p = p_c$ for (5). The cases $\alpha < 2$ are subcritical. The case $\alpha = 1$ corresponds to the case $p = 1 + 2/d$ for (5) and is the borderline for long-range potential. It is equivalent to the Schrödinger-Poisson system and is of fundamental importance. The stability of vacuum and asymptotic completeness for small solutions in $H^1(\mathbf{R}^3)$ are well known for both repulsive and attractive nonlinearities. We expect similar results about solitons for (5) to hold for (6). We plan to divide the investigation into the following steps.

Linear analysis:

1. No hidden symmetry: this corresponds to the nonexistence of nontrivial solutions for a certain linear equation associated to the linearized operator around the soliton.

2. Wave operator estimates for $\alpha > 1$: Extend the result of K. Yajima – S. Cuccagna on the $L^p - L^p$ estimates for the wave operator for linearized operator to Hartree equations.

The essential difficulty lies on the nonlocalness of the convolution. We restrict $\alpha > 1$ to ensure that the potential terms are of short range as in Yajima-Cuccagna.

3. Modified wave operator estimates for $\alpha = 1$: In this case the potential terms are of long range and the wave operator needs to be modified. We hope we may still prove certain decay and Strichartz estimates for the linear evolution.

Nonlinear analysis:

4. As a preliminary step for Step 5, (and independent of Steps 1–3), we wish to study (1) in the case V supports one eigenvector but is of long range, i.e., $V(x) \sim |x|^{-1}$ as $|x| \to \infty$. We use this step to investigate the effect of the long range potential on the (small) soliton.

5. Nonlinear dynamics of Hartree equation (6) with $\alpha = 1$. Since the linearized operator is expected to possess many eigenvalues by physical arguments, one cannot hope to study the initial value problem using the known machinery. One may, however, study the nonlinear wave operator and construct scattering solutions. For this step we will only consider localized data in weighted space, $\int_{\mathbf{R}^3} |\psi_0(x)|^2(1+x)^{1+\varepsilon}dx < \infty$, as in the small solution case.

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