Monge-Ampere Type Equations and Applications

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The workshop on Monge-Ampere type equations at BIRS was held from August 2 to August 7 in the summer of 2003. It is a focused meeting in the rapidly developing areas related to Monge-Ampère equation, and fully nonlinear equations in general. The workshop presents a unique and timely opportunity to stimulate important new mathematical research and exposition on the subjects. There were 20 one-hour talks during the 5-day workshop to present important recent works. The setting of BIRS is ideal for promoting informal interaction that has proved so fruitful in the past in the field. There has been increasing interaction among researchers on the field In recent years. In a more general context, the workshop served as viechle to foster mathematical interaction between researchers in the field in different areas of the world.

The participants consist of mathematicians from Canada, United States, France, Germany, Australia and China. The participants of the workshop includes both some of the most distinguished mathematicians in the world and many of the top young researchers (including 2 graduate students) in the field. That's a good mixture of both senior and junior researchers. There were great interactions between senior and junior mathematicians working in the area. This kind interaction at the research level is crucial to the continuing high level of advancement of the field. The workshop provided a wonderful educational opportunity for many junior researchers in the field. The combination of talks, problem sessions, informal discussions, and the special environment of BIRS provided this opportunity.

Because of the geographic diversity of the researchers in the field, bringing the participants together for the 5-day workshop strongly facilitate the dissemination of the most recent research ideas and results, which otherwise might not be possible. The atmosphere of the workshop and its surroundings may lead to new collaborations during the workshop and especially in the years following the workshop. Neil Trudinger of the Austranlian National University, one of the prominent mathematicians in the field, told the organizers that this workshop is the best meeting he has ever attended. The combination of the breadth and the cohesiveness of the field of Monge-Ampere type equations certainly has made the 5-day workshop at BIRS have significant impact on the field.

In the following, we summarize the main mathematical activities during the workshop around the main theme of Monge-Ampère and fully nonlinear equations. We divide it into to several sections according to the emphasis on different aspects of the field.

1 Monge-Ampère equation and classical differential geometry

Alexandrov type inequalities for Cartan-Hadamard manifolds.

In his study of elliptic equations and geometry, the great Russian geometer A.D. Alexandrov introduced the simple but important idea of the lower (upper) contact set and the associated normal mapping (generalized gradient map). For a function $u \in C^0(\Omega)$ (Ω a bounded domain in $\mathbb{R}^n$), the lower contact set

$$\Gamma^- = \{y \in \Omega | u(x) \geq u(y) + p \cdot (x - y) \text{ ,for all } x \in \Omega \text{ ,for some } p = p(y) \in \mathbb{R}^n\} \ .$$

Note that u is convex if and only if $\Gamma^- = \Omega$ and if $u \in C^1(\Omega)$ $, p = Du(y)$ in the above definition. Further if $u \in C^2(\Omega)$, the hessian $D^2u \geq 0$ on Γ^-. Associated to $y \in \Omega$ is $\chi(y)$, the set of slopes of lower supporting planes to the graph of u at y (so $\chi(y) = \emptyset$ if $y \notin \Gamma^-$). It is not difficult to see that for $u \in C^2(\Omega \cap C^0(\overline{\Omega})$,

$$|\chi(\Omega)| = |\chi(\Gamma^-| \leq \int_{\Gamma^-} \det D^2u(y) \ dy$$

The extension of these ideas to Riemannian manifolds is surprisingly difficult. Joel Spruck discussed this extension for a large class of Cartan-Hadamard manifolds where we replace the linear functions in the definition of the lower contact set by Busemann functions associated to a given point at infinity. The proof uses in an essential way the ergodic theory of the geodesic flow to compute the Jacobian for the generalized gradient map. In a similar way, he derived for a compact hypersurface inequalities for a generalized Gauss map on the outer contact set. These new inequalities are sharp for radial functions and geodesic balls in H^n.

Convex solutions of elliptic PDE in classical differential geometry.

There is a vast literature on the convexity of solutions of quasilinear elliptic equation in $\mathbb{R}^n$, especially the deformation technique of Caffarelli-Friedman and Korevaar-Lewis via the strong minimun principle. In some recent work of P. Guan-X. Ma on classical problems in differential geometry [7], some fundamental questions the convexity problem for fully nonlinear equations have been put into spotlight. It is needed to generalize results for quasilinear equations to fully nonlinear equations. Some of the work has been done for a large general fully nonlinear elliptic equations. Some applications have been given to problems in classical differential geometry and convex bodies, for example Christoffel-Minkowski problem, prescribed Weingarten curvature and prescribed curvature measure in convex body.

The Minkowski Problem for hedgehogs

In differential geometry, the Minkowski problem is that of the existence, uniqueness and regularity of closed convex hypersurfaces of $\mathbb{R}^{n+1}$ whose Gauss curvature is prescribed as a function of the unit normal vector. More generally, the Minkowski problem concerns the existence and uniqueness of convex bodies of $\mathbb{R}^{n+1}$ whose area measure of order n is prescribed on the unit sphere $\mathbb{S}^n$. This classical Minkowski problem is equivalent to the question of solutions of a Monge-Ampère PDE of elliptic type on $\mathbb{S}^n$. Hedgehogs of $\mathbb{R}^{n+1}$ are the geometrical realizations of formal differences of convex bodies of $\mathbb{R}^{n+1}$. In the case of differences of convex bodies of class C^2_+, these geometrical realizations are (possibly singular and self-intersecting) envelopes parametrized by their Gauss map. They constitute a vector space $\mathcal{H}^{n+1}$ in which one can study a given convex body by splitting it judiciously (i.e. according to the problem under consideration) into a sum of hedgehogs.

The Minkowski problem has a natural extension to hedgehogs. For non-convex hedgehogs, this extended Minkowski problem is equivalent to the question of solutions of a Monge-Ampère PDE of mixed type on $\mathbb{S}^n$.

Yves Martinez-Maure expounded the present state of our knowledge on the subject and will consider certain particular cases. In particular, he illustrated the interest of hedgehogs by showing how to construct counter-examples to an old conjectured characterization of the $2-$sphere related to this extension of Minkowski's problem.

Regularity of Monge-Ampere equations and applications in affine geometry

In X.J. Wang's talk he reported some recent advances by N.S. Trudinger and himself on the regularity of Monge-Ampere equations and applications to the affine Plateau geometry. The affine Plateau problem concerns the existence of smooth affine maximal hypersurfaces subject to certain boundary restrictions. The affine maximal surface equation is a fourth order nonlinear PDE which can be written as a system of two Monge-Ampere type equations. The existence of solutions was obtained by the upper semi-continuity of the affine surface area functional and a uniform cone property of locally convex hypersurfaces. The a priori estimates of strictly convex solutions was established by using Caffarelli and Gutierrez's interior estimates for Monge-Ampere equations. To prove that a maximizer can be approximated by smooth solutions, a boundary Schauder estimate for the Monge-Ampere equation and the global regularity to a second boundary value problem of the affine maximal surface equation were also established. The strict convexity of solutions remains open, except in the two dimensional case.

John Urbas proved the existence of smooth solutions of a class of Hessian equations on a compact Riemannian manifold without imposing any curvature restrictions on the manifold.

Local Isometric Embedding Problem.

The celebrated Nash's isometric theorem states that any n-dimensional Riemannian manifold can be isometrically embedded in to $\mathbb{R}^N$. In general, N is a very big number compare to n. For local isometric embedding, the classical Janet-Cartan theorem tell us that any analytic n-dimensional Riemannian manifold can be isometrically embedded into $\mathbb{R}^{\frac{n(n+1)}{2}}$, the number $\frac{n(n+1)}{2}$ is optimal. One would like to remove the analyticity assumption. The fundamental question regarding the local isometric problem for smooth surface is: can it be locally isometrically embedded into $\mathbb{R}^3$? There are important works of C.S. Lin, which dealt with the cases: (i) Gauss curvature nonnegative and (ii) Gaus curvature changes sign cleanly. The problem is closely related to the local solvability of Monge-Ampère equation. In a surprise development, Nikolai Nadirashvili and Yu Yuan constructed smooth metrics on 2-manifold with nonpositive Gauss curvature which cannot be C^3 locally isometrically embedded in $\mathbb{R}^3$. Moreover, the Gauss curvature of the metric can be made negative except for one point.

On the other hand, Qing Han reported his joint work with J. X. Hong and C.S. Lin. They obtained a result on the local isometric embedding of surfaces, which seems to be a compliment of the examples of Nikolai Nadirashvili and Yu Yuan. They proved that there is a such embedding if the Gauss curvature of metric is non-positive, and the null set of the Gauss curvature has nice structure. In particular, if the Gaus curvature is non-positive and of finite type, the local isometric bembedding problem for surface is solvable. In our view, these results present some important behavior of degenerate hyperbolic Monge-Ampère equation. We should also relate this local problem to the similar global problem on $\mathbb{S}^n$ posted by Yves Martinez-Maure.

This is a new direction of Monge-Ampère equation: try to deal with hyperbolic type of this equation. Almost nothing is known at the moment. Any progress in this direction will be important to the theory of Monge-Ampère equation and will certainly yield some geometric applications.

Prescribing Curvature problem.

Bo Guan presented results from his joint work with Joel Spruck on the Plateau type problem of finding hypersurfaces of constant curvature with prescribed boundary. More precisely, the problem can be formulated as follows: given a smooth symmetric function f of n $(n \geq 2)$ variables and a disjoint collection $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$ of closed smooth embedded $(n-1)$ dimensional submanifolds of $\mathbb{R}^{n+1}$, one asks whether there exist (immersed) hypersurfaces M in $\mathbb{R}^{n+1}$ of constant curvature $f(\kappa[M]) = K$ with boundary $\partial M = \Gamma$ for some constant K, where $\kappa[M] = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of M. Important examples include the classical Plateau problem for minimal or constant mean curvature surfaces and the corresponding problem for Gauss curvature. In this and some earlier work, they introduced two different approaches to the problem: the Perron method and the volume minimizing approximation. These methods are based on the solvability of the problem in the non-parametric setting (the Dirichlet problem) and an important uniform local graph representation property of locally convex hypersurfaces. It would be interesting to extend the methods to other classes of hypersurfaces.

The work of Bo Guan and Joel Spruck relies on the existence of a "sunsolution". A natural geometric question is: when there is a such "subsolution"? Using methods of the h-principle of Gromov, specifically the holonomic approximation theorems, Mohammad Ghomi proved that any compact hypersurface with boundary immersed in Euclidean space is regularly homotopic to a hypersurface whose principal directions have a prescribed topological type, and whose principal curvatures are prescribed to within an arbitrary small error. Further he described how to construct regular homotopies which control the principal curvatures and directions of hypersurfaces. These results have been obtained in his recent joint work with Marek Kossowski, and generalize theorems of Gluck and Pan on positively curved surfaces in 3-space, which had been proved by explicit constructions. Also they are somewhat reminiscent of the classical continuity method used to obtain some of the recent results on locally convex hypersurfaces with boundary.

2 Fully nonlinear equations in conformal geometry.

Nonlinear partial differential equations have played a role in the problem of uniformization of conformal structures since the work of Poincare. In dimensions greater than two, the analysis of conformal structure is approached from two perspectives. The first deals with conformally flat structures where the theory of Kleinian groups provide a large family of examples and hyperbolic geometry provided the basic tools for analysis. The second perspective deals with general conformal structures by studing the metrics from the view point of partial differential equations. The well known Yamabe equation, is among a family of conformally covariant partial differential equations which are geometrically natural and analytically interesting. The curvature tensor Rm associated to a Riemannian metric g may be decomposed as

$$Rm = W + \frac{1}{2} A \odot g$$

where $\odot$ indicates the Kulkarni-Nomizu product of bilinear forms, where W is the Weyl tensor, $A = \frac{1}{n-2}(Rc - \frac{R}{2(n-1)}g)$ is the Schouten tensor, and R the scalar curvature. We say A belongs to the k-positive cone if the k-th symmetric function $\sigma_k(A)$ of the eigenvalues of A is positive, and that A may be joined to the identity matrix by a path of matrices A_t along which $\sigma_k(A_t) > 0$. We say that a metric belongs to the k-positive cone, if the Schouten tensor belongs to the k-positive cone at each point. Under a conformal change of metric $g' = v^{-2}g$, the Weyl tensor transforms by scaling, while the Schouten tensor transforms by a complicated expression involving the hessian as well as the gradient of the conformal factor:

$$A' = A + \frac{1}{v}\{\nabla^2 v - \frac{\Delta v}{n} g\} + \frac{1}{2n(n-1)}\{R + 2(n-1)\frac{\Delta v}{v} - n(n-1)\frac{|\nabla v|^2}{v^2}\}g.$$

- The symmetric functions σ_k of the eigenvalues of the Schouten tensor may be expressed then in terms of a fully nonlinear expression involving up to two derivates of the conformal factor e^w. When $k = 1$ the equation is known as the Yamabe equation, which is a semi-linear equation with a critical nonlinearity. A great deal is understood about this equation. When $k \geq 2$ the equation is fully nonlinear thus the study of this equation requires techniques coming from two previously disjoint area in nonlinear partial differential equations. There are several motivation to strive for an understanding of the fully nonlinear equations originating from geometry as well as analysis. The most important reason is that the higher degree equation yields stronger control of the Ricci tensor, and hence stronger control of the geometry. The first evidence in this direction is provided by the work of Gursky-Viaclovsky [11] in which they characterize the space forms in 3-D as the critical points of the functional $\int \sigma_2(A_g)dV$ among the metrics of normalized volume. This work points to the possibility to understand the three dimensional geometrization problem via the study of the σ_2 equation. As a strong confirmation of this possibility, the work of Chang-Gursky-Yang ([1, 2, 3]) provided the necessary and sufficient condition in four dimensions for the existence of metrics with Schouten tensor A_g to lie in the 2-positive cone, as well as solvability of the the equation to prescribe $\sigma_2(A_g)$, and as a consequence a sphere theorem in 4-D characterizing the 4-sphere by two conformally invariant conditions in terms of the positivity of two conformally covariant linear operators. The key that ties the topology to the analysis is provided through the Chern-Gauss-Bonnet formula, in which the expression $\sigma_2(A_g)$ plays the leading role.

Following this work, there has been a lot of activity in extending this result in two directions. The first is to provide similar sufficient conditions for existence of metrics with Schouten tensor belonging to the k-positive cone in which they provided an alternative argument to the main result of [1] without appealing to the regularization procedure via higher order equations, and actually yielded further criteria for positivity of the Paneitz operator. Matt Gursky spoke about this new result at the workshop. The second main direction is directed at the solvability of the σ_k equations once the k-positive cone is known to be nonempty. In this direction, there is a lot of work in the past two years. For general conformal structures, Gursky-Viaclovsky [13] proved the necessary apriori estimates in dimensions three and four for nearly all of the prescribed σ_k equations, thus nearly answering almost completely the solvability question, once the k-positive cone is non-empty. At the workshop Viaclovsky spoke about this work, and related questions dealing with the case where $-A$ belongs to the k-positive cone.

On the other hand, for conformally flat structures in all dimensions, there is rather satisfactory progress in obtaining apriori estimates for all the σ_k equations. This is the result of a large collaborative effort by several competing groups of analysts including P. Guan, G. Wang, C.S. Lin, Y.Y. Li, A. Li in which the main technical tools are (i) the local estimate of Guan-Wang [8], (ii), the conformal fully nonlinear flow, (iii) the method of moving planes coupled with some inspired computations deriving largely from previous experience with the Yamabe equation as well as the fully nonlinear equations of Monge-Ampere type. In particular, when the k-positive cone is non-empty, there is a complete existence theory ([9, 15]). Y.Y. Li was invited to present the current state of the problem, but unfortunately was not able to make the workshop.

Among the conformally flat structures, the ones arising from Kleinian groups provide a potential area for application of the newly developed fully nonlinear equations. Traditionally, the study of Kleinian groups had proceeded either through the study of Beltrami equations and thus restricted to the case of 2-D, or by doing the analysis on the hyperbolic structures which the Kleinian quotient bounds. The condition that the Schouten tensor lies in the k-positive cone, i.e. $\sigma_j(A) > 0$ for $j = 1, ..., k$ places strong restriction on the Ricci part of the curvature tensor. In particular, the well known work of Schoen-Yau says that the conformally flat structures with $\sigma_1(A) > 0$ are Kleinian quotients, with limit set whose Hausdorff dimension is bounded by $(n-2)/2$, and hence certain homotopy groups as well as homology groups of the Kleinian quotient vanishes. In recent development, this result has its counterpart for the metrics in the k-positive cone. Guofang Wang presented a joint work with P.F. Guan and C.S. Lin [10, 6] on certain further vanishing of cohomology, and G. Del Mar presented her forthcoming thesis work [16] on further bounds for the Hausdorff dimension of the limit set, resulting in further vanishing of homotopy groups. A consequence of these work is the beginning of a classification theory of Kleinian groups whose Kleinian quotients have metrics belonging to certain k-positive cones.

There is further anticipated development of these conformally invariant equation that relates to the recently developing scattering theory of convex co-compact hyperbolic manifolds, as well as the more general conformally compact manifolds that is playing an increasing role in the holographic principle as predicted by the mathematical physicists. Alice

Chang presented the recent joint work with J. Qing and P. Yang [4] on a generalized Gauss-Bonnet formula for conformally compact manifolds in which the principat term involves the renormalized volume, a subtle quantity that is directly related to the scattering pole of the conformally compact manifold. This presents possibilities for further interaction of the conformally invariant nonlinear pdes with the scattering theory, and the theory of conformal invariants that is under active development by C. Fefferman and his students.

3 Analysis on subelliptic Monge-Ampère equations

A higher dimensional partial Legendre transform and regularity of degenerate Monge-Ampère equations.

Eric Sawyer presented his joint work with Rios and Whedeen on the C^∞-regularity of degenerate Monge-Ampère equations in higher dimensions. In general, we know (under some reasonable conditions) that solution to degenerate elliptic Monge-Ampère equations is $C^{1,1}$. This regularity is sharp. But, in differential geometry, one needs to known when such a solution is in fact C^∞. In 2-case, there exists a previous result of P. Guan says that if u is $C^{1,1}$, $u_{11} \geq c > 0$, $\det(u_{ij} \geq 0$ is of finite type, then $u \in C^\infty$. Sawyer-Rios-Whedeen generalized that result to higher dimensions under stronger assumption that u is $C^{2,1}$. The important part of their result is the higher dimensional partial Legendre transform. They were able to relate the partial Legendre transform in a way similar to Cauchy-Riemann type equation. They work relies on the important work of Sawyer-Whedeen on regularity of degenerate quasilinear elliptic equations.

Convex functions in the subelliptic setting and applications.

In the past decade, research for fully nonlinear equations in Euclidean spaces has made considerable progress. We refer to the two monographs in this direction by Caffarelli-Cabré and by Gutierrez. The simplest example is the so-called Monge-Ampère equation which for smooth function u is given by

$$det(D^2u) = f.$$

In considering solutions to the above equation, the notion of convex functions plays a crucial role. The notions of generalized weak solutions and viscosity solutions which were introduced by A.D. Aleksandrov rely on the properties of convex functions.

Convex functions in Euclidean space can be characterized as universal viscosity subsolutions of homogeneous fully non-linear degenerate elliptic equations of second order.

Motivated by the role that convex functions play in the theory of fully nonlinear equations, Guozhen Lu and his collaborators formulate several notions of convexity in the subelliptic structure on the Heisenberg group towards the aim of developing an instrinsic theory of subelliptic fully non-linear equations. We will discuss the notion of group convexity, horizontal convexity and viscosity convexity. These definitions can be considered in the general case of Hörmander vector fields. Their proofs strongly use the viscosity theory for subelliptic equations. They study convex functions defined by requiring that their symmetrized horizontal second derivatives are non-negative in the viscosity sense. They call these functions *v-convex*. The main result is to establish their local Lipschitz continuity. They introduced the notion of *horizontal convexity*, which has many interesting properties.

The upper-semicontinuous horizontally convex functions are v-convex, and therefore Lipschitz continuous. These two notions of convexity are equivalent. The notion of convexity independently defined by Cabré and Caffarelli is equivalent to horizontal convexity. Another approach of horizontal convexity is also considered by Danielli-Garofalo-Nhieu.

Using the fact that symmetrized second order derivatives of convex functions are signed measures, the recent results of Gutierrez-Montanari of Monge-Ampere operators on H^1, the gradient estimates for convex functions], and the weak differentiability of convex functions by Ambrosio-Magnani, they derived the second order differentiability a.e. of convex functions on H^1. Thus generalizing the well-known Alexandrov's theorem to the subelliptic setting. This proof works in general Carnot groups as long as one can show the second commutators of convex functions are signed measures (thus each mixed second order derivatives are signed measures and then we can apply the theorem of Ambrosio-Magnani to get weak differentiability of second order).

Fully nonlinear subelliptic theory is at the beginning stage now. It is desirable to develop a satisfactory theory of existence, uniqueness and regularity theory for the Monge-Ampere operator recently discovered.

Boundary value problems for complex Monge-Ampere equation and characterization of the ball using Kahler–Einstein metric

In the complex analysis, one of major problem is classifying domains under biholomorphism. Riemann mapping theorem in complex analysis of one variable play the exactly role. However, when dimension is greater than one, the problem become more complicated and very interesting. Strongly related problem is the Fefferman mapping problem: Every biholomorphism between two smoothly bounded pseudoconvex domain in complex Euclidean space must be a diffeomorphism between the closures of the domains. Many progress has been made by many excellent mathematicians started with C. Fefferman in 1974. In general, the problem is still open. Boundary regularity of complex Monge-Ampere equtions is stronly related to the Fefferman mapping problem. In part one of Song-Ying Li's talk, he gave a sharp regularity for degeneate complex Monge-Ampere equation on pseudoconvex domain of finite type in n-demsional complex Euclidean space.

Bergman metric and Kahler-Einstein metric are very important metrics in complex analysis and complex geometry, they have certain boundary behavior near boundary of a strictly pseudoconvex domain in n-dimensional complex Euclidean space. A general open question was asked by S. T. Yau: To classify the domains have Bergman Kahler-Einstein metric. Only very special case was known. The second part of his talk was to present a theorem to characterize the unit ball in n-dimensional complex Euclidean space by using Kahler-Einstein metric, which is the first step of approaching Yau's question in a long term project.

4 Monge-Ampère equation and mass transportation

Geometric properties of the set of probability densities of prescribed second moments.

The set of probability densities can be endowed with the so-called Wasserstein metric. The geodesic between two probability densities can be explicitly written using the convex function that appear in the Monge-Ampere equation involving these densities. Motivated by applications in kinetic theory, Wilfrid Gangbo and Eric Carlen analyze the induced geometry of the set of densities satisfying the constraint on the variance and means, and they determine all of the geodesics on it. It turns out, for example, that the entropy is uniformly strictly convex on the constrained manifold, though not uniformly convex without the constraint.

Semigeostrophic fluid flow in an elliptical ocean basin.

Robert McCann's talk describes joint work with Adam Oberman concerning exact solutions to the semigeostrophic model for atmospheric and oceanic flows. This rotating fluid model involves an active scalar transport problem, much like the vorticity formulation of Euler's equation, but with Monge-Ampere instead of Poisson's equation relating stream function to advected scalar. Affine invariance of the determinant provides exact solutions in which both pressure and stream function remain quadratic but evolve nonlinearly. The finite dimensional dynamics are expressed canonically in Hamiltonian form.

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