

## **Problems presented at Directions in Combinatorial Matrix Theory**

Several questions were presented at the open problems sessions during the workshop, and some of the participants were kind enough to provide writeups of their open problems.

Here you will find problems posed by Chi-Kwong Li on graph isomorphisms and Charlie Johnson / Dale Olesky on totally positive matrices. Richard Anstee / Atilla Sali also discuss a conjecture (which appears elsewhere) on forbidden configurations, and Rob Craigen gives a writeup of his exercise on tournament matrices. Finally, Michael Doob gives a ‘two-napkin proof’ of one theorem on graph spectra, and challenges us to find such a proof of another known result in the area.

## A problem on graph isomorphisms

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Let  $A_1$  and  $A_2$  be the adjacency matrices of the simple graphs  $G_1$  and  $G_2$ , respectively. It is well-known that the two graphs are isomorphic if there is a permutation matrix  $P$  such that

$$A_1 = P^t A_2 P,$$

and thus the two matrices  $A_1$  and  $A_2$  have the same eigenvalues (counting multiplicities). Also, it is well-known that there are non-isomorphic graphs whose adjacency matrices have the same eigenvalues. Here are several related questions. Suppose  $A = (a_{ij})_{1 \leq i, j \leq n}$ . Denote by

$$D(A) = \{a_{11}, \dots, a_{nn}\}$$

the multi-set of the diagonal entries of  $A$ , and

$$\text{diag}(A) = (a_{11}, \dots, a_{nn})$$

the vector of diagonal entries of  $A$ .

### Problems

1. Suppose  $D(A_1^k) = D(A_2^k)$  for all  $k = 1, 2, \dots$ . Are  $G_1$  and  $G_2$  isomorphic?
2. Suppose there is a permutation matrix  $P$  such that  $\text{diag}(A_1^k) = \text{diag}(P^t A_2^k P)$  for all  $k = 1, 2, \dots$ . Are  $G_1$  and  $G_2$  isomorphic?

Evidently, the hypothesis on  $A_1$  and  $A_2$  in Problem 2 is stronger than that in Problem 1. Both of them imply that  $A_1^k$  and  $A_2^k$  have the same traces for  $k = 1, 2, \dots$ . It follows that  $A_1$  and  $A_2$  have the same eigenvalues.

Even if the answers to Problems 1 and 2 are negative, it is interesting to study the following.

3. Construct counter-examples with the minimum of vertices.
4. Identify a set of graphs, say, the set of trees or the set of regular graphs, such that  $G_1$  and  $G_2$  from the set are isomorphic if  $D(A_1^k) = D(A_2^k)$  or  $\text{diag}(A_1^k) = \text{diag}(A_2^k)$  for all  $k = 1, 2, \dots$

These problems arose in some discussion with Philippe Barbe (CNRS, France). The solutions of these problems may have potential applications in computational statistics.

## SUMS OF TOTALLY POSITIVE MATRICES

C.R. Johnson and D.D. Olesky

An  $m \times n$  matrix is totally positive (TP) if all of its minors are positive. We have recently shown that an arbitrary  $m \times n$  positive matrix can be written as the sum of at most  $\min\{m, n\}$  totally positive matrices, and that this is in general the best possible value for the number of summands.

The following positive matrix  $A$  is the sum of two TP matrices:

$$A = \begin{bmatrix} 2 & 4.1 & 5.3 \\ 2.7321 & 5.5651 & 6.9848 \\ 4 & 8.1 & 10.01 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1.7321 & 2 & 2.025 \\ 3 & 4 & 4.01 \end{bmatrix} + \begin{bmatrix} 1 & 3.1 & 4.3 \\ 1 & 3.5651 & 4.9598 \\ 1 & 4.1 & 6 \end{bmatrix},$$

even though all minors of  $A$  of orders 2 and 3 are negative. On the other hand, it can be shown that

$$B = \begin{bmatrix} 1 & 10 & 1000 \\ 10 & 1 & 10 \\ 1000 & 10 & 1 \end{bmatrix}$$

cannot be written as the sum of two TP matrices.

**Question.** For any fixed  $k$  ( $2 \leq k < \min\{m, n\}$ ), what characterizes the sum of  $k$  TP matrices? (The above examples indicate that even the case  $k = 2$  is interesting.)

Let  $J_n$  denote the  $n \times n$  matrix with each entry equal to 1. Then

$$J_3 = \begin{bmatrix} 1/2 & 1/3 & 1/5 \\ 2/3 & 1/2 & 1/3 \\ 4/5 & 2/3 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 & 2/3 & 4/5 \\ 1/3 & 1/2 & 2/3 \\ 1/5 & 1/3 & 1/2 \end{bmatrix}$$

is a sum of two TP matrices.

**Question.** Can the rank 1 matrix  $J_n$  (for each  $n \geq 4$ ) be written as a sum of two TP matrices? (Note that if this is possible, then any rank 1 matrix can be so written because of diagonal scaling.)

## A Conjecture for Forbidden Configurations

Richard Anstee (UBC) and Attila Sali (Rényi Institute, Hungary)

We define a *simple* matrix as a  $(0,1)$ -matrix with no repeated columns. Such a matrix can be thought of a set of subsets of  $\{1, 2, \dots, m\}$  with the columns encoding the subsets and the rows indexing the elements. Assume we are give a  $k \times l$   $(0,1)$ -matrix  $F$ . We say that a matrix  $A$  has a *configuration*  $F$  if there is a submatrix of  $A$  which is a row and column permutation of  $F$ . We are interested in  $A$  which have no configuration  $F$  ( $F$  is referred to as a *forbidden configuration* in this case).

We define  $\text{forb}(m, F)$  as the smallest function so that if  $A$  is an  $m \times n$  simple matrix with  $n > \text{forb}(m, F)$  then  $A$  has  $F$  as a configuration. It seems hopeful to establish the asymptotic behaviour of  $\text{forb}(m, F)$  through the following conjecture which asserts that certain constructions establish the asymptotic bounds.

The building blocks are the square matrices  $I$ ,  $I^c = J - I$ , and the triangular matrix

$$T = \begin{bmatrix} 1 & & & 1's \\ & 1 & & \\ & & \ddots & \\ 0's & & & 1 \end{bmatrix}.$$

We put the three building blocks together using a simple product construction. Let  $A_i$  be an  $m_i \times n_i$  simple matrix for  $1 \leq i \leq k$ . Denote  $A_1 \times A_2 \times \dots \times A_p$  as the  $(\sum m_i) \times (\prod n_i)$  simple matrix whose columns are formed in all possible ways by putting a column of  $A_1$  in the first  $m_1$  rows and putting a column of  $A_2$  in the next  $m_2$  rows etc. If each  $m_i$  is  $\Theta(m)$  (e.g.  $m_i = m/p$ ) then  $A_1 \times A_2 \times \dots \times A_k$  has  $\Theta(m)$  rows and  $\Theta(m^p)$  columns.

Let  $F$  be a  $(0,1)$ -matrix. Let  $X(F)$  be the smallest  $p$  so that  $F$  is a configuration in  $A_1 \times A_2 \times \dots \times A_p$  for every choice of  $A_i$  as either  $I_{m/p}$ ,  $I_{m/p}^c$  or  $T_{m/p}$  (we assume  $m$  is a large multiple of  $p$ ).

Note that the definition of  $X(F)$  ensures  $\text{forb}(m, F)$  is  $\Omega(m^{X(F)-1})$  since the definition of  $X(F)$  gives a construction of a simple matrix with  $\Theta(m)$  rows and  $\Theta(m^{X(F)-1})$  columns and no configuration  $F$  (for  $X(F) = 1$  more care must be taken).

**Conjecture**[AS]  $\text{forb}(m, F)$  is  $\Theta(m^{X(F)-1})$ .

Our results show this conjecture is true for any  $F$  with either 1,2 or 3 rows ([AGS],[AFS],[AS]) and also for  $k = 4$  in the case that the  $F$  is required to be simple. Computing  $X(F)$  is non trivial and we have yet to make a direct connection of our proofs of asymptotic bounds for  $\text{forb}(m, F)$  with the derivation of  $X(F)$ .

## Bibliography

- [AGS] R.P. Anstee, J.R. Griggs, A. Sali, Small Forbidden Configurations, *Graphs and Combinatorics* **13** (1997), 97-118.
- [AFS] R.P. Anstee, R. Ferguson, A. Sali, Small Forbidden Configurations II, *Electronic J. Combin.* **8**(2001), R4 (25pp)
- [AS] R.P. Anstee, A. Sali, Small Forbidden Configurations IV, *Combinatorica*, submitted.

## Tournament matrices and $(0, 1)$ incidence matrices

Rob Craigen

A *tournament* is a  $(0, 1)$ -matrix  $B$  such that  $B + B^t = J - I$ , where  $J$  is the matrix of all 1's.

**THEOREM 0.1.** *A  $(0, 1)$ -matrix  $A$  is the incidence matrix of a directed path if and only if  $I - A$  is invertible and  $A(I - A)^{-1}$  is a tournament.*

*Proof.* First assume that  $A$  is a directed path. There is exactly one directed path between any pair of distinct vertices. Since  $A^k$  is the incidence graph of the graph whose edges are directed paths of length  $k$ , it follows that  $B = A + A^2 + A^3 + \dots$  is a tournament (This sum is actually finite, since  $A^k = 0$  for large enough  $k$ .) Now,  $(I + B)(I - A) = I$ , so  $I - A$  is invertible, and its inverse is  $I + B$ , so  $A(I - A)^{-1} = A(I + A + \dots) = B$ , as required.

Conversely, suppose  $I - A$  is invertible and  $A(I - A)^{-1} = A + A^2 + A^3 + \dots$  is a tournament. Regarding  $A$  as the incidence matrix of some graph  $G$  we see that there is exactly one path connecting every pair of distinct vertices in  $G$ . I claim that i)  $G$  is path connected; ii)  $G$  can contain no cycles; iii)  $G$  can have at most two pendant vertices.

i) is immediately clear.

If  $G$  contains a cycle then not all edges can be oriented in the same direction around this cycle, else there are at least two paths between any pair of vertices. Thus, there is a vertex  $v_a$  on the cycle with outedges to both neighbors and a vertex  $v_b$  on the cycle with inedges to both neighbors (by the pigeonhole principle). Without loss of generality,  $v_a$  and  $v_b$  can be chosen so that there is a  $v_a \rightarrow v_b$  path. Let  $v_c$  the vertex adjacent to  $v_a$  that is not located on this path. There is no  $v_c \rightarrow v_b$  path, else we have two  $v_a \rightarrow v_b$  paths. Thus there must be a  $v_b \rightarrow v_c$  path. But then we can go from  $v_a$  to  $v_c$  by  $v_a \rightarrow v_b \rightarrow v_c$  or by the single arc  $v_a \rightarrow v_c$ , a contradiction. ii) follows.

If  $G$  has three pendant vertices, by the pigeonhole principle, two of them are connected either by inedges or outedges. In either case, there is no path between these vertices. iii) follows.

By ii),  $G$  is a directed forest. By i), then, it is a directed tree. Now iii) implies that  $G$  is a directed path.  $\square$

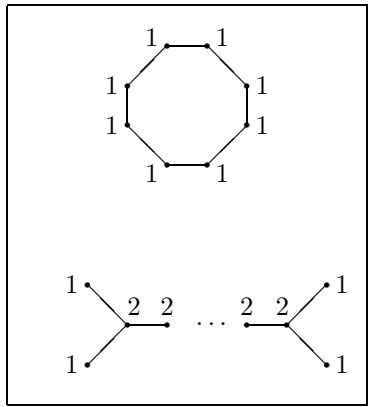
# Looking for a two napkin solution

Michael Doob

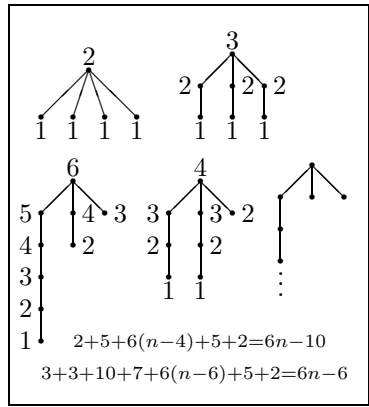
When flying to or from a conference, there often comes a time when the flight attendant offers you a cup of coffee or a soft drink. If you're using your laptop computer, it's a good time to close it, for you might hit some unexpected turbulence and spill your beverage into your computer. Even if you are working with a pencil or pen on a notepad, you wouldn't want it to be stained, so it's a good time to set it aside. However, all is not lost! There is still the little napkin you get with the drink, and it is perfect for little scribbles or short proofs. If the essence of a proof can be so completed, it is a one napkin solution. If you ask nicely, you can get a second napkin, and this allows for a two napkin solution.

Here is an example of a one napkin solution. The essence of the argument was given by J. Smith in [1].

In our mind's eye we have a few facts. If we consider an  $n \times n$  matrix  $A$  which has an orthonormal basis of eigenvectors  $\{x_1, \dots, x_n\}$ , and  $x$  is any vector, then a trivial consequence of (bi)linearity is that the Rayleigh quotient  $\frac{\langle Ax, x \rangle}{\langle x, x \rangle}$  is a convex combination of the corresponding eigenvalues. Also, the Perron-Frobenius theorem tells us that an irreducible matrix with nonnegative entries has a real eigenvalue of maximum modulus with a corresponding eigenvector with all components positive.



The front of the napkin



The back of the napkin

If in fact  $A$  is the adjacency matrix of a connected graph  $G$ , then it satisfies both of these properties, and, in addition, all the eigenvalues are real. This

means that any Rayleigh quotient is a lower bound for the largest eigenvalue. Now suppose that  $x$  is the eigenvector from the Perron-Frobenius theorem for the graph  $G$  and we join a pair of previously unjoined vertices with an edge. If we compare the Rayleigh quotient for the new graph using the same  $x$ , we have two new positive entries in the numerator and the denominator is unchanged. Hence the Rayleigh quotient increases, and since it is a lower bound for the largest eigenvalue of the new graph, we have a useful conclusion. *Observation: Adding an edge to a connected graph increases the largest eigenvalue.*

In our mind's eye we note also the following graph labelling property: The rows and columns of  $A$  correspond to the vertices  $\{v_1, \dots, v_n\}$ . If we want to evaluate  $Ax$ , we label  $v_k$  with  $x_k$  for  $k = 1, \dots, n$ , and then the action of  $Ax$  is obvious:  $(Ax)_k$  is simply the sum of the labels on the vertices adjacent to  $v_k$ . If particular,  $x$  is an eigenvector with corresponding eigenvalue  $\lambda$  if and only if (for any vertex) the sum of the labels adjacent to a vertex is  $\lambda$  times the label of the vertex itself.

Now let's look at our napkin.

The front of the napkin shows that 2 is an eigenvalue of the two given graphs. The top graph actually an abbreviation an arbitrary cycle. Since a path  $P_n$  is contained in these graphs, we see from our previous observation that the largest eigenvalue of  $P_n$  is less than 2.

Now suppose that  $G$  is a connected graph with largest eigenvalue less than 2. From the front of the napkin, it can't contain a cycle and so must be a tree. Note that the four labelled graphs on the back of the napkin all have 2 as eigenvalue. The first one implies that  $G$  has no vertex with degree greater than three, and the bottom graph on the front of the napkin implies at most one vertex is of degree three. If there are no vertices of degree three, then the graph is obviously  $P_n$ . Finally, if there is one vertex of degree three, we may think of the graph as three paths emanating from the one vertex. The upper right graph on the back of the napkin implies that not all of these paths have two or more edges, that is, some path has only one edge. The middle graph on the bottom shows that the two remaining graph can not have three or more edges each, and the lower left graph shows that if one path has two edges, the last one can not have five or more edges. The final case is when two of the paths have one edge, and this is given as the lower right graph. Since it is contained in the bottom graph on the front of the napkin, this last example in fact has largest eigenvalue less than 2. Could it be cospectral with  $P_n$ ? If it were, it would have the same number of closed walks of length 4 ( $= \text{tr}(A^4)$ ). A quick count of them is given on the bottom of the napkin:  $P_n$  has  $6n - 10$  while the last graph on the napkin (assuming  $n$  vertices) has  $6n - 6$ .

So we have a one napkin proof that the  $P_n$  is determined by its spectrum (in truth, there are the subgraphs of the labelled graphs on the back of the napkin to be considered, but there are only three of them and they cause no problem). We derived this characterization without even computing the spectrum itself.

Here's a challenge problem: take the complement of a path  $P_n$ . One would conjecture that this graph is characterized by its spectrum also, and in fact that conjecture is true. Given the ease of the one napkin proof for  $P_n$ , one would

hope for an easy proof for the complement. The challenge is to find a two napkin proof of the characterization of the complement of  $P_n$  by its spectrum.

[1] J. H. Smith, Some properties of the spectrum of a graph. *Combinatorial Structures and their Applications*, (Eds. R. Guy, *et al.*), Gordon and Breach, (1970), 403–406.