Interactions between noncommutative algebra and algebraic geometry

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Abstract

This is the final report by the organizers of the workshop on *Interactions between noncommutative algebra and algebraic geometry*, held at the Banff International Research Station September 10 - 15, 2005. The workshop was attended by 36 mathematicians from eight different countries (Australia, Canada, China, France, Great Britain, Israel, Norway, and the United States).

This report is subdivided into three parts. In the first part we introduce the subject matter of the workshop and briefly discuss its history, from the beginning of the 20th century to the present. In the second part we describe some of the currently active research topics which involve the use of algebro-geometric methods in noncommutative algebra or conversely, topics in algebraic geometry (and related mathematical physics), where noncommutative algebra plays an important role. These topics formed the core of the scientific content of the workshop. The third part consists of summaries of lectures given at the workshop.

1 Introduction

Noncommutative phenomena are perhaps as old as mathematics itself; they manifest themselves in the simplest mathematical objects, such as permutations or matrices. Noncommutative algebra developed into a separate subject in the early 20th century. The initial steps, taken by Dickson and Wedderburn, among others, were motivated by attempts to better understand ”hypercomplex numbers”, such as the quaternions, discovered by W. Hamilton in 1843. Subsequent steps, due to E. Artin, R. Brauer, H. Hasse, E. Noether, etc., came in the context of abstract algebra, which was a rapidly developing subject in the 1920s and 30s. The next phase, lasting roughly from the 1930s to the early 1980s and led by A. Albert, S. Amitsur, N. Jacobson, I. Kaplansky, A. Goldie, I. Herstein, among others, focused on developing the structure theory for various types of noncommutative rings.

To illustrate how quickly one encounters open problems in noncommutative algebra and to give the reader a bit of a flavor of the subject, consider the Weyl algebra $A = \mathbf{C}\{x_1, x_2\}$, given by two generators, x_1 and x_2, and one relation,

$$x_1x_2 - x_2x_1 = 1\,.$$

If we replace the 1 on the right hand side by any other complex number $\alpha \neq 0$, we will get an isomorphic algebra; however, if we set $\alpha = 0$ then the resulting algebra will become the commutative polynomial ring

$P = \mathbf{C}[x_1, x_2]$ in two variables. Thus we can think of A as a noncommutative deformation of P; the two have some properties in common, but A is considerably more complicated. One property they have in common is that in both cases we can obtain a skew field by formally inverting all non-zero elements. Recall that a skew field (otherwise known as a division algebra) satisfies all the axioms of a field (i.e., one can perform the four arithmetic operations, and all the usual associative, distributive, etc. rules hold) except that multiplication is not required to be commutative. Of course, if we formally invert all non-zero elements in the polynomial ring $P = \mathbf{C}[x_1, x_2]$, the skew field we obtain will be the rational function $\mathbf{C}(x_1, x_2)$ in two variables. If we do the same thing to A, we will obtain a noncommutative skew field D, called the Weyl skew field. There are many proper skew subfields $S \subset D$ isomorphic to D itself. Given one such S, we can view D as an S-vector space in two different ways, with scalar multiplication on the left or on the right. It is thus natural to ask whether or not these two vector spaces have the same dimension. This seemingly simple question is, in fact, a long-standing open problem. A more general question along the same lines is whether or not the left and the right dimensions of A over B are the same, where A is an arbitrary skew field and B is a skew subfield. This question was posed by E. Artin and settled in the negative by P.M. Cohn [25] and A.H. Schofield [44]. Returning to the Weyl skew field D, note that much is unknown here. If the left and right dimensions of D over S turn out to be the same, what invariants distinguish D from the examples constructed by Cohn and Schofield? On the other hand, if the right and left dimensions are different (for some S), is there an effective way to compute them or to determine whether or not they are the same for a given S?

In last 20 years the subject of noncommutative algebra has been rapidly developing in several different directions. Once common theme has been the increasing penetration of algebro-geometric methods into the subject and conversely the increasing use of noncommutative ring theory within algebraic geometry and related mathematical physics. One important outgrowth of this interaction is an entirely new research area, called noncommutative algebraic geometry. Recall that one of the foundational steps in the early development of (commutative) algebraic geometry was the realization that every commutative ring R can be thought of the the ring of regular functions on a suitably defined space, namely, $X = \mathrm{Spec}(R)$. This dichotomy is of fundamental importance in commutative ring theory: by passing from R to X, one can often translate purely algebraic questions into problems about the geometry of X, in a setting where both geometric tools and geometric intuition are available. Noncommutative algebraic geometry is motivated by an attempt understand noncommutative rings in a similar manner. Curiously, this turned out to be somewhat easier to do with graded rings, using methods derived from projective (rather than affine) algebraic geometry. The reason is that affine algebraic geometry tends to rely on techniques like localisation that are rarely available in the noncommutative setting, whereas the more global and categorical approaches to projective geometry can and have been generalized. Another goal of noncommutative algebraic geometry is to build up and study "noncommutative algebraic varieties" or "noncommutative schemes". In addition to clarifying the structure of noncommutative rings they are of independent interests, and may have interesting and unexpected applications (e.g., in mathematical physics). At this point we are rather far from fully realizing these goals. However, the methods developed in noncommutative projective geometry have already found a number of applications; in particular, they have been used to solve several outstanding open problems in noncommutative ring theory [5, 6, 4, 7, 8].

The purpose of the workshop was to discuss various aspects of the interaction between noncommutative ring theory and algebraic geometry, including the latest developments in noncommutative algebraic geometry. In particular, the following topics were discussed.

2 Areas of recent activity

We will now outline several areas of interaction between algebraic geometry and noncommutative algebra, where there have been interesting new developments in recent years. Most of these developments were discussed during the workshop.

Foundation of noncommutative algebraic geometry

One important question in the field concerns the foundations of noncommutative algebraic geometry. For example, what is "the right" axiomatic definition of a noncommutative space? The approach usually taken in (commutative) algebraic or differential geometry is to first define what a space of the desired type should look like locally, in a sufficiently small open neighbourhood of each point, then specify what kind of transition functions are allowed to ”glue” these local charts together. For example, in a differentiable manifold, a sufficiently small neighbourhood of every point looks like an open ball in $\mathbf{R}^n$, with differentiable transition functions between these local charts. A scheme looks like $\mathrm{Spec}(R)$ in a neighbourhood of each point, with regular transition functions between the charts. As we pointed out above, it is not possible (or at least has not been possible so far) to mimic this approach for noncommutative spaces, because it ultimately relies on the assumption that one can easily pass to a smaller open subset of a given point. In the commutative setting this is done through the technique of localization (i.e., inverting certain elements in a ring), which is usually not available in the noncommutative setting. The successful approaches so far have taken the global point of view from the very beginning. Here is a partial list of papers addressing this subject. M. Artin [3], M. Artin and J.J. Zhang [9], M. Kontsevich and A. Rosenberg [34], Y.I. Manin [37], A. Rosenberg [39, 40, 41], M. Van den Bergh [52], F. Van Oystaeyen and A. Verschoren [54], A.B. Verevkin [55, 56], V. Ginzburg [31], W. Crawley-Boevey, P. Etingof and V. Ginzburg [27]. One purpose of having many different approaches to noncommutative spaces is to understand them from different points of view. The foundations of noncommutative projective geometry that were established by A.B. Verevkin [55, 56] and M. Artin and J.J. Zhang [9] have been largely accepted but this is just the beginning of this theory, and much foundational work remains to be done.

Finite-dimensional division algebras of transcendence degree 2.

Division algebras (or skew fields) that are finite over their centres have been studied since the beginning of the 20th century. These algebras play an important role in algebraic geometry, the theory of algebraic groups, algebraic number theory and algebraic K-theory. Some of the most exciting recent developments in this field have to do with algebras defined over function fields of surfaces. Recall that every finite-dimensional central simple algebra A/K can be written in the form $A = \mathrm{M}_n(D)$, where D is a division algebra with centre K. The index d of A is the degree of D, i.e., $\sqrt{\dim_K(D)}$. The exponent of A is the smallest positive integer e such that $A^{\otimes e}$ is a matrix algebra over K. It is known that $e \leq d$ and that e and d have the same prime divisors. If K is the function field of a surface it has been long conjectured that $e = d$; this is sometimes called the period-index problem. Special cases of this conjecture were proved by M. Artin and J. Tate [2] in the 1980s, but a full solution was obtained only a few years ago by J.A. de Jong [28]. Similar results in the context of arithmetic surfaces were proved earlier by D.J. Saltman [42, 43] (who spoke on this topic at the workshop), and subsequently strengthened by M. Lieblich [36].

Another important open problem in the theory of central simple algebras is the Albert conjecture. Recall that a cyclic algebra of degree n over a field K, containing a primitive nth root of unity ζ_n, is a K-algebra given by two generators, x and y and three relations,

$$x^n \in K\,, \quad y^n \in K\,, \quad \text{and} \quad xy = \zeta_n yx\,.$$

Albert’s conjecture asserts that every division algebra of prime degree $n = p$ is of this form. This conjecture (which might or might not been stated by Albert), has motivated much of the research in the theory of central simple algebras, going as far back as perhaps the 1930s.

In a recent preprint, M. Ojanguren and R. Parimala [38] use and further develop the ideas of M. Artin, D.J. Saltman and J.A. de Jong, to prove Albert’s conjecture for division algebras of prime degree over the function field K of a complex surface. The details of this argument are still being checked by the experts. If the proof holds up, it is believed that a similar method can be used to show that the abelian closure of K has cohomological dimension 1. (Here K is the function field of a complex surface, as above.) Note that the inequality $\mathrm{cd}(K_{ab}) \leq 1$ is currently only known in a few cases; in particular, for K = a number field, or a p-adic field by class field theory and for $K = \mathbf{C}((X))((Y))$ by [26, Theorem 2.2]. For a related conjecture of Bogomolov, see [11, Conjecture 2].

Birational classification of noncommutative surfaces.

Division algebras that are infinite dimensional over their centres appear naturally in noncommutative algebra and noncommutative algebraic geometry. Similarly to the commutative situation, the classification of division algebras of transcendence degree 2 would be equivalent to the birational classification of integral noncommutative projective surfaces. Hence it is important to work out the classification of division algebras of transcendence degree 2. M. Artin proposed a conjectured list of division algebras of transcendence degree

2 in [3]. All algebras on this list are known to be of transcendence degree 2; the conjecture is that there are no others. If Artin's conjecture is proved, it will have many strong consequences in noncommutative ring theory.

Quantum projective spaces.

Quantum $\mathbf{P}^2$s have been classified by M. Artin, W. Schelter, J. Tate and M. Van den Bergh [4, 7, 8]. They are well understood. So it is natural to ask if we can classify quantum $\mathbf{P}^3$s, or more generally quantum $\mathbf{P}^n$s for all $n \geq 3$. Quantum $\mathbf{P}^n$s are fundamental objects in noncommutative algebraic geometry. Many interesting noncommutative spaces can be embedded into some quantum $\mathbf{P}^n$. On the other hand, it is not clear if every noncommutative space can be embedded into quantum $\mathbf{P}^n$s. This problem is not solved even for quantum or noncommutative K3 surfaces and the quantum Calabi-Yau 3-folds. The reason for this is that quantum $\mathbf{P}^n$s are not fully understood.

The complete classification of quantum $\mathbf{P}^n$s is an extremely difficult project. An algebraic approach to constructing quantum $\mathbf{P}^n$s is to form the noncommutative scheme $\mathrm{Proj}\ A$ where A is a noetherian Artin-Schelter regular connected graded algebra of global dimension $n+1$. Therefore the algebraic form of the above mentioned question is the classification of noetherian, Artin-Schelter regular, connected graded algebras. Researchers have been studying many special classes of noetherian Artin-Schelter regular algebras of global dimension 4. One well-studied example is the Sklyanin algebra of dimension 4, introduced by Sklyanin [50, 51]. Artin-Schelter regular algebras of dimension four have been extensively studied by many researchers (S.P. Smith, J.T. Stafford. T. Levasseur, L. Le Bruyn, M. Van den Bergh, J. Tate, M. Vancliff, B. Shelton, K. Van Rompay, L. Willaert, T. Cassidy, D. Stephenson, D.-M. Lu, J. Palmieri, Q.S. Wu and others). in recent years. This gives us hope that a complete classification of quantum $\mathbf{P}^3$'s may be in sight. Note that quantum $\mathbf{P}^3$'s will provide new examples of division algebras of transcendence degree 3. These division algebras are likely to play an important role in noncommutative projective geometry.

Combinatorial noncommutative algebra

The study of finitely generated algebras like the Weyl algebra, enveloping algebras of finite dimensional algebras, Sklyanin algebras is greatly aided by the use of combinatorial techniques that go back to Shirshov, Golod and Shafarevich, Gelfand and Kirillov, and others.

The first real issue is to determine when a finitely generated algebra (or a module over it) is actually finite dimensional. In fact, Golod and Shafarevich found a criterion for infinite dimensionality of algebras involving generators and relations that led to an example of a finitely generated nil algebra (an algebra in which every element is nilpotent) that is infinite dimensional. This settled the Kurosh problem for algebras by showing that not every finitely generated algebra that is algebraic over its base field is finite dimensional. This example also gives immediately a counterexample to the Burnside problem for groups.

Until very recently, the Golod -Shafarevich example was, in some sense, the only such example. These rings all had exponential growth. This past year, Tom Lenagan and Agata Smoktunowicz produced examples of finitely generated nil algebras with polynomial growth.

Let $A = k[V]$ be a finitely generated algebra, where V is a finite dimensional generating subspace of the algebra over the field k, and let $d(n)$ be $\dim(V^n)$, where V^n is the subspace generated by all products of n or fewer elements of V. The Gelfand-Kirillov dimension $\mathrm{GK}(A)$ of A, is defined as

$$\mathrm{GK}(A) = \limsup_{n\to\infty}\ \log_n(d(n))\,.$$

This definition is independent of the choice of the generating set, V. For example, the GK dimension of the commutative polynomial ring in n variables is n; a free algebra has infinite GK dimension; the GK dimension of a finite dimensional algebra is zero; a finitely generated polynomial identity(PI) algebra has finite GK dimension; given any real number, γ, greater than or equal to two, there is a finitely generated PI whose GK dimension is γ. Remarkably, Victor Markov has shown that any finitely generated subalgebra of matrices over a commutative algebra has integral GK dimension.

In general, it 's not known when finitely generated algebras have integral GK dimension even if they're noetherian. An important recent positive step is Smoktunowicz's result that a finitely generated graded integral domain cannot have GK dimension properly between two and three. The conjecture is that finitely generated graded domains all have integral GK dimension.

Noncommutative Iwasawa algebras

Noncommutative Iwasawa algebras form a large and interesting class of complete semilocal noetherian algebras, constructed as completed group algebras of compact $p-$adic analytic groups. Thus, let p be a prime integer, let $\mathbf{Z}_p$ be the $p-$adic integers, and let G be a compact $p-$adic analytic group, so (equivalently - see [29]) G is a closed subgroup of $GL_d(\mathbf{Z}_p)$ for some $d \geq 1$. Then the *Iwasawa algebra of* G is

$$\Lambda_G \quad := \quad \varprojlim \mathbf{Z}_p[G/N],$$

where the inverse limit is taken over the open normal subgroups N of G, (which have finite index in G by the compactness hypothesis). Closely related to Λ_G is its epimorphic image Ω_G, defined as

$$\Omega_G \quad = \quad \varprojlim \mathbf{F}_p[G/N],$$

where $\mathbf{F}_p$ is the field of p elements.

These definitions, and the fundamental properties of these rings, were given in M. Lazard's monumental 1965 paper [35]. In particular, Lazard proved that G contains an open normal subgroup U, nowadays termed a *uniform* subgroup, whose Iwasawa algebra has a particularly smooth form. Thus, for U uniform, Ω_U is the $J-$adic completion of the ordinary group algebra $\mathbf{F}_pU$ by its augmentation ideal J. So Ω_U is filtered by the powers of $J\Omega_U$, and the associated graded algebra is a (commutative) polynomial $\mathbf{F}_p-$algebra. It follows by standard filtered-graded technology that Ω_U is a complete noetherian Auslander-regular scalar local domain. Similar remarks apply to Λ_U, and - thanks to the fact that Ω_G [resp. Λ_G] is a crossed product of Ω_U [resp. Λ_U] by the finite group G/U, similar conclusions can be drawn regarding Ω_G and Λ_G.

In the twenty years from 1970 Iwasawa algebras were little studied. Interest in them has been revived by developments in number theory over the past fifteen years, see for example [24]. Building on the filtered algebra and crossed product techniques outlined above, it's now known when Iwasawa algebras are prime, semiprime, domains, and when they have finite global dimension. Bounds have been found for their Krull dimension, and information obtained about their centres. Details about these results - and much else besides - can be found in the survey article [1].

The emerging picture is of a class of rings which in some ways look similar to the classical commutative Iwasawa algebras, (which are rings of formal power series in finitely many commuting variables over the $p-$adic integers), but which in other respects are very different from their commutative counterparts. And while some progress has been made in understanding these rings, many aspects of their structure and representation theory remain mysterious. A large number of open questions are discussed in [1].

Cluster algebras and cluster categories. Cluster algebras were invented by Fomin and Zelevinsky [32, 33] in 2000 as a tool to approach Lusztig's theory of canonical bases in quantum groups and total positivity in algebraic groups. Since then, cluster algebras have become the center of a rapidly developing theory, which has turned out to be closely related to a large spectrum of other subjects, notably Lie theory, Poisson geometry, Teichmüller theory, integrable system, algebraic combinatorics and polyhedra, and quiver representations. Recent work by many authors has shown that this last link is best understood using the cluster category, which is a triangulated category associated with every Dynkin diagram. A partial list of papers are [10] by Assem, Brüstle, Schiffler and Todorov, [12, 13, 14, 15, 16, 17, 18] by Buan, Marsh, Reineke, Reiten and Todorov, [19, 20] by Caldero, Chapoton and Schiffler, [21, 22] by Caldero and Keller, [30] by Geiss, Leclerc and J. Schröer. The combinatorics of clusters is shown to be tightly related to tilting objects in cluster categories. There have been many new questions motivated by the study of cluster algebras [57] and cluster categories and it is expected that there will be more activities in this direction. Derived categories or triangulated categories have been used more and more in many areas. The recent development of the cluster category is a good example of such.

In the workshop Keller gave a talk on some recent developments and present the cluster multiplication theorem, obtained in his joint work with Caldero [21, 22], which directly links the multiplication of the cluster algebra to the triangles in the cluster category using a Hall algebra approach. Reiten gave a talk based on her recent work with Iyama about algebras of global dimension 3 where its bounded derived category of the finite length modules is Calabi-Yau of dimension 3. This derived category has connections with cluster algebras and the noncommutative crepant resolutions of Van den Bergh.

Noncommutative stacks. The noncommutative phenomena of algebraic stacks (i.e., Artin stacks and Deligne-Mumford stacks) has been observed for many years. Using some ideas from Connes' noncommutative geometry, Chan and Ingalls recently defined a noncommutative coordinate ring associated to a Deligne-Mumford stack with a finite flat scheme cover [23]. This has been extended to the case of Artin stacks by Behrend. There are many moduli problems suggesting that noncommutative algebras are the correct algebraic structure which describe the underlying geometric spaces. The noncommutative crepant resolutions of Van den Bergh [53] is a good example. Most of noncommutative algebras appearing with stacks are finite over their centres.

3 Summaries of selected talks

Speaker: Jacques Alev (Universit'e de Reims)

Title: Poisson trace group of certain quotient varieties.

Summary: Let V be a symplectic space of dimension $2n$, G a finite group of symplectomorphisms of V, $X = V/G$ the quotient variety, A_n the Weyl algebra of index n and A_n^G the invariant algebra which can be seen as "noncommutative functions" over X, hence as a quantization of X. A standard theme is to compare all possible algebro-geometric invariants of the (usually singular) Poisson variety X and of the algebra A_n^G: Poisson (co)homology of X, Hochschild (co)homology of A_n^G, desingularizations of X, etc. Alev presented his computation of $\dim HP_0(X)$ in certain cases and compared it to $\dim HH_0(A_n^G)$.

Speaker: Daniel Chan (University of New South Wales)

Title: Minimal resolutions of canonical orders and McKay correspondence.

Summary: Recently, the Mori program was adapted to orders over surfaces. In particular, there are noncommutative generalisations of discrepancy, canonical singularities and resolutions of singularities. Chan reviewed some of these concepts and showed how minimal resolutions of canonical orders can be written down explicitly. We also discussed McKay correspondence for these canonical orders. This talk was based on joint work with Colin Ingalls and Paul Hacking.

Speaker: William Crawley-Boevey (University of Leeds)

Title: Noncommutative Poisson structures

Summary: This talk described a notion of Poisson structures on noncommutative rings which seems to be better than the straightforward generalization of Poisson brackets to such rings. The speaker also discussed some open problems in this area.

Speaker: Victor Ginzburg (University of Chicago)

Title: Double derivations and cyclic homology.

Summary: Ginzburg described a new construction of cyclic homology of an associative algebra A that does not involve Connes' differential. His approach is based on the complex ΩA, of noncommutative differential forms on A, and is similar in spirit to the de Rham approach to equivariant cohomology. The cyclic homology is defined as the cohomology of the total complex $((\Omega A)[t], d + t \cdot i)$, arising from two anti-commuting differentials, d and ii, on ΩA of degrees $+1$ and -1, respectively. The differential d, that replaces the Connes differential B, is the Karoubi-de Rham differential. The differential i that replaces the Hochschild differential b, is a map analogous to contraction with a vector field. This new map has no commutative counterpart.

Speaker: Ken Goodearl (University of California Santa Barbara)

Title: Quantum matrices and matrix Poisson varieties.

Summary: Goodearl discussed the relations among prime and primitive ideals of the generic quantized coordinate ring $A = \mathcal{O}_q(M_n(\mathbf{C}))$, Poisson prime and Poisson primitive ideals of the classical coordinate ring $R = \mathcal{O}(M_n(\mathbf{C}))$, and symplectic leaves in the Poisson variety $M_n(\mathbf{C})$. The Poisson algebra R is the "semiclassical limit" of A, and so it is conjectured that there should be a bijection between the primitive spectrum of A and the Poisson primitive spectrum of R, hence also a bijection with the space of symplectic leaves in $M_n(\mathbf{C})$. All of these bijections should be equivariant with respect to natural actions of the torus

H of pairs of invertible diagonal matrices. Consequently, the H-invariant prime ideals of A should naturally match up with the H-orbits of symplectic leaves in $M_n(\mathbf{C})$. Specifically: Each H-invariant prime of A is conjectured to be generated by a set of quantum minors, and these quantum minors should match minors defining the closure of a corresponding H-orbit of symplectic leaves in $M_n(\mathbf{C})$. In recent joint work with K.A. Brown and M. Yakimov, Goodearl determined these orbits of symplectic leaves, and described sets of minors defining their closures. These results lead to precise conjectures concerning generating sets for H-invariant prime ideals in A, which were discussed in the talk.

Speaker: Birge Huisgen-Zimmermann (University of California Santa Barbara)

Title: Top-stable degenerations of finite dimensional representations

Summary: Given a finite dimensional representation M of a finite dimensional algebra A, two hierarchies of degenerations of M are analyzed: the poset of those degenerations of M which share the top M/JM with M – here J denotes the radical of the algebra – and the sub-poset of those which share with M the full radical layering $\left(J^l M/J^{l+1}M\right)_{l\geq 0}$. In particular, the speaker addressed the existence of proper top-stable or layer-stable degenerations – more generally, the sizes of the corresponding posets including bounds on the lengths of saturated chains – as well as structure and classification. Here are two sample theorems to indicate the level of detail one can draw from the proposed geometric setting. The most transparent case is that of a squarefree top T. In this situation, two numerical invariants (with quite natural intuitive interpretations) govern the size of the poset of top-stable degenerations of M, namely:

• The difference $t - s$, where t is the number of simple summands in the top of M and s the number of indecomposable summands of M, and

• the difference $m = \dim_K Hom_A(P, JM) - \dim_K Hom_A(M, JM)$, where P is a projective cover of M.

Theorem A. Top-stable degenerations. Suppose $T = M/JM$ is a direct sum of t pairwise non-isomorphic simple A-modules and P a projective cover of T. Write M in the form $M = P/C$ with $C \subseteq JP$.

(1) The lengths of chains of proper top-stable degenerations of M are bounded above by $m + t - s$.

(2) Existence: M has a proper top-stable degeneration if and only if $m + t - s > 0$, if and only if either M fails to be a direct sum of local modules, or else C fails to be invariant under homomorphisms $P \to JP$.

(3) Unique existence: M has a unique proper top-stable degeneration if and only if M is a direct sum of local modules and $m = 1$. If $m = 0$ and $t - s = 1$, M has precisely two distinct proper top-stable degenerations. For all values $m + t - s \geq 2$, there are infinitely many top-stable degenerations in general.

(4) Bases: W.l.o.g., A is a path algebra modulo relations, and P a direct summand of A. If $M' = P/C'$ is a top-stable degeneration of M, then M and M' share a basis consisting of paths in the underlying quiver. That is, there exists a set B of paths such that $\{q + C \mid q \in B\}$ is a basis for M and $\{q + C' \mid q \in B\}$ is a basis for M'.

(5) The maximal top-stable degenerations of M always possess a fine moduli space, classifying them up to isomorphism. It is a projective variety of dimension at most $\max\{0,\, m + (t - s) - 1\}$.

(6) The case $m = 0$: M has only finitely many top-stable degenerations, and the degeneration order coincides with the Ext-order.

As for proper layer-stable degenerations in the case of squarefree top: If M is a direct sum of local modules, there are none. Otherwise, "huge" hierarchies of layer-stable degenerations may arise.

As a by-product, the theory provides a method for computing the top-stable degenerations from quiver and relations of A and a presentation of M. Hence, there is a rich supply of examples. Huisgen-Zimmermann displayed three examples of particular interest and described the conjectural classification in the general situation in terms of these specific instances.

The lecture ended with a sample of the theory for the more involved situation of an arbitrary top:

Theorem B. Suppose $M/JM \cong S_1^{t_1} \oplus \cdots \oplus S_n^{t_n}$, where $S_1, \ldots, S_n$ are the isomorphism types of the simple left A-modules (corresponding to a full set of primitive idempotents e_i of A).

Then M has no proper layer-stable degenerations if and only if

(a) M is a direct sum of local modules, say $M = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{t_i} M_{ij}$, where $M_{ij} = Ae_i/C_{ij}$.

(b) $\dim Hom_A(P, JM) = \dim Hom_A(M, JM)$, and

(c) For each i, the C_{ij} are linearly ordered with respect to inclusion.

Speaker: Tom Lenagan (University of Edinburgh)

Title: Prime ideals and the automorphism group of quantum matrices.

Summary: This talk was based in joint work in progress, in collaboration with Stéphane Launois. In work with Launois and Rigal, the speaker has recently shown that the algebra of quantum matrices is a UFD in the generic case (q is not a root of unity), in the sense that each height one prime is principal, generated by a normal element. The present work starts by establishing a criterion to decide when the algebra of quantum matrices is primitive. This is linked to the description of the height one primes since each height one prime is either invariant under the action of the natural torus that acts on quantum matrices, or it is in the so-called 0-stratum. The algebra of quantum matrices is primitive precisely when there is no height one prime in the 0-stratum, and, in this case, there are only a finite number of height one primes, each one invariant under the torus action. For example, the algebra of 2x3 quantum matrices is primitive. Next, the speaker considered the automorphism group of quantum matrices by studying the action of this group on the prime spectrum, and, in particular on the height one primes. The situation is much more complicated in the non-primitive case, where there are infinitely many height one primes, than in the primitive case, where there are only finitely many primes. In the nonsquare case, Lenagan described the automorphism group. In the square case the situation is not yet fully resolved, but there are partial results.

Speaker: Valery Lunts (Indiana University)

Title: Motivic measures and zeta functions.

Summary: A ”motivic measure” is a ring homomorphism $K[V] \to A$ from the Grothendieck ring of varieties $K[V]$ to an arbitrary ring A. Lunts considered two interesting motivic measures. The first one is related to stable birational geometry of varieties and the second – to derived categories of coherent sheaves. He also discussed a counterexample to a conjecture of Kapranov on the rationality of motivic zeta function. This lecture was based on joint work with Michael Larsen.

Speaker: Daniel Rogalski (University of California San Diego)

Title: Birationally commutative surfaces are naive blow-ups

Summary: The aim of the work presented in this lecture is to classify a wide class of graded rings of GK-dimension 3 in terms of geometry. We say that a connected graded domain A is a birationally commutative if its graded ring of fractions looks like $K[t, t^(- 1); \sigma]$ where K is a commutative field. The main theorem states that if A is such a domain which is noetherian, generated in degree 1, and with $GKdim\ A = 3$, then A can be described as a naive blow-up of some twisted homogeneous coordinate ring of a surface. This is an analog of the commutative result that all surfaces in a given birational class are related by blowing-up. This talk was based on joint work with Toby Stafford.

Speaker: David Saltman (University of Texas at Austin)

Title: Brauer groups of function fields of surfaces.

Summary: The goal is to take a second look at the Brauer group of function fields of surfaces. One aspect is to generalize, in a way, a result proved for p-adic curves. Let $K = F(S)$ be the function field of a regular surface (not necessarily over a field but excellent and Noetherian). Let $\alpha \in Br(K)$ be a Brauer group element of order a prime q unequal to any residue characteristics. Assume K has a primitive q root of one. Then we state a geometric obstruction on the ramification locus of α to its being represented by a division algebra of degree q. Absent this obstruction, we show that there is a cyclic extension of degree q which splits all the ramification of α. In another direction, we recall and redevelop the H^3 obstruction to ramification data coming from a Brauer group element. We want further properties of this obstruction, the ultimate goal being to make it computable. Along the way, we consider the case where $S = Spec(R)$ and R is a regular local domain (of dimension 2 etc.) with henselization R^h. We also consider the function field $K = q(R)$, and the relationship between $Br(K)$ and $Br(K^h)$ for $K^h = q(R^h)$.

Speaker: Paul Smith (University of Washington)

Title: Noncommutative covers of weighted projective varieties.

Summary: Let A be a commutative graded ring generated by a finite number of elements of positive degrees and let $X_{nc} = Proj_{nc}A$ be the Artin-Zhang Proj, and $X = Proj(A)$ the usual commutative weighted projective variety. There is a map $f : X_{nc} - - > X$ in the sense of noncommutative geometry. Moreover, X_{nc} is a quotient stack with coarse moduli space X, and f "is" the natural map of stacks. We study X_{nc} and the map f from the point of view of noncommutative geometry. Often f is a birational isomorphism, and often X can be singular while X_{nc} is smooth so functions as a sort of noncommutative resolution of X. Locally X_{nc} is covered by affine spaces that have coordinate rings that are skew group rings for finite cyclic groups over commutative rings. We describe a sheaf B of noncommutative algebras on X such that $Mod(X_{nc}) = Qcoh(B)$. The case where A is the polynomial ring on two generators of weights 4 and 6 was used to illustrate some of the ideas. This is an important example because then X_{nc} is the compactified moduli stack for pointed elliptic curves. We give an easy proof (in the spirit of noncommutative geometry) of Mumford's result that the Picard group of the uncompactified moduli stack is $Z/12$.

Speaker: Michaela Vancliff (University of Texas at Arlington)

Title: Using an Algebro-Geometric Method to Construct Clifford Quantum $\mathbf{P}^3$s with a Predetermined Finite Point Scheme.

Summary: The classification of generic quantum $\mathbf{P}^3$s (generic regular algebras of global dimension four) has been hindered by the lack of sufficiently generic examples of quantum $\mathbf{P}^3$s on which to formulate and test conjectures. Candidates for generic quantum $\mathbf{P}^3$s are regular algebras of global dimension four that have a finite point scheme and a one-dimensional line scheme, but such algebras are rare in the literature. One possibility for constructing such an algebra is to build it by deforming a regular Clifford algebra of global dimension four that has a finite point scheme.

Speaker: Nikolaus Vonessen (University of Montana)

Title: Group actions on central simple algebras

Summary: Suppose an algebraic group G acts on a central simple algebra A of degree n (and characteristic 0). The goal is to be able to answer the following questions a about the action:

(a) Is A^G a simple algebra, and if so, what is its degree? Its center?

(b) Does A have a G-invariant maximal subfield?

(c) Can the G-action on the center $Z(A)$ be extended to a splitting field L, and if so, what is the minimal possible value of $\operatorname{trdeg}_{Z(A)} L$?

It turns out that under mild assumptions on A and the action, one can obtain much information along these lines by using techniques from birational invariant theory (i.e., the study of group actions on algebraic varieties, up to equivariant birational isomorphisms). The talk illustrated the results with the example of the natural action of GL_m on the universal division algebra $UD(m, n)$ generated by m generic $n \times n$-matrices. In this case the invariants form a division subalgebra of degree n if and only if assuming $n \geq 3$ and $2 \leq m \leq n^2 - 2$. Related methods also make it possible to give an asymptotic estimate of the dimension of the space of SL_m-invariant homogeneous central polynomials $p(X_1, \ldots, X_m)$ for $n \times n$-matrices. This talk was based on joint work with Zinovy Reichstein.

Speaker: Amnon Yekutieli (Ben Gurion University)

Title: Deformation quantization in algebraic geometry.

Summary: The goal is to study deformation quantization of the structure sheaf O_X of a smooth algebraic variety X in characteristic 0. The universal deformation formula of Kontsevich gives rise to an L_∞ quasi-isomorphism between the pullbacks of the DG Lie algebras $T_{poly,X}$ and $D_{poly,X}$ to the bundle of formal coordinate systems of X. Using simplicial sections one obtains an induced twisted L_∞ quasi-isomorphism between the mixed resolutions $Mix(T_{poly,X})$ and $Mix(D_{poly,X})$. If certain cohomologies vanish (e.g. if X is D-affine) it follows that there is a canonical function from the set of gauge equivalence classes of formal Poisson structures on X to the set of gauge equivalence classes of deformation quantizations of O_X. This is the quantization map. When X is affine the quantization map is in fact bijective. This is an algebro-geometric analogue of Kontsevich's celebrated result.

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