Infinite dimensional Lie algebras and local von Neumann algebras in CFT

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May 6 - May 20, 2006

The workshop set up interesting relations between Infinite dimensional Lie algebras and local conformal nets of von Neumann algebras. Each participant has given one of more talks and stimulated discussions on certain topics. A detailed list of the talks is the following.

1 Bojko Bakalov

1.1 Representations of Vertex Algebras

Bakalov gave a series of three lectures on representations of vertex algebras. After giving the definition of a vertex algebra, several consequences of the Borcherds identity were discussed, which led to several equivalent definitions of the notion of a module. Next, the introduction of various gradings gave three different variants of modules. The second lecture was devoted to Zhus C2-finiteness condition and its consequences; in particular its relationship to the notions of rationality and regularity. The third lecture discussed the correspondence between the representations of a vertex algebra and of its associative Zhu algebra. A recent induced-module construction of Dong and Jiang was reviewed.

1.2 Vertex Algebras in Higher Dimensions

Bakalov gave a talk on his recent joint works with Nikolov and with Nikolov-Rehren-Todorov on vertex algebras in higher-dimensional spacetime, which provide an algebraic framework for investigating axiomatic quantum field theory with global conformal invariance. The talk presented an introduction to these algebras, as well as new results. The latter included generalizations of the Borcherds identity and of the n-th product identity. Two examples were discussed: the complex massless scalar free field and the complex scalar bilocal field.

2 Sebastiano Carpi

2.1 From Vertex Algebras To Conformal Nets I

The unitary structure on a (local) Vertex Operator Algebra (VOA) is considered and its uniqueness up to automorphisms explained. The possible relations between the automorphism group of a unitary VOA and the correseponding unitary subgroup are discussed. In particular it is shown that the automorphism group is finite if and only if the unitary subgroup is and that in this case they coincide. It is explained how to associate a covariant net on the circle starting from a unitary VOA with (polynomial) energy bounds. A unitary VOA with energy bounds is sayd to be strongly local if the associated net is local. (Joint work with Y. Kawahigashi, R. Longo and M. Weiner)

2.2 From Vertex Algebras To Conformal Nets II

The one-to-one correspondece between vertex subalgebra of a strongly local VOA and covariant subnets of the corresponding local net of von Neumann algebra is explained. As a result sub VOA of a strongly local VOA are strongly local. It is shown that a unitary VOA with energy bounds is strongly local if is generated by a family of strongly local fields. As a consequence a VOA generated by currents and Virasoro elements is is strongly local. The coincidence of the unitary automorphism group of a strongly local VOA and the one of the corresponding net is explained. Some applications are discussed. (Joint work with Y. Kawahigashi, R. Longo and M. Weiner)

2.3 Short talk in the final joint informal discussion

Some open problems concerning the relation of VOA modules and representations of conformal nets of von Neumann algebras are outlined.

3 Alberto De Sole

3.1 Quantum and classical W algebras

We first reviewed the notions of vertex algebra and Poisson vertex algebra, and the relation between them via the so called quasi-classical limit. We also reviewd the definition of the Zhu algebra of a vertex algebra, which controls its representations, both in the quantum and in the classical case. We then introduced the definition of the affine and finite W algebras, obtained by the method of quantum Hamiltonian reduction, and we discussed the relaion among them.

4 Victor Kac

4.1 Basics on vertex algebras and principal W algebras

In my talks I explained the basics of the theory of vertex algebras in relation to common grounds with the theory of local conformal nets. Also, I discussed in some detail the principal W-algebras, their minimal models and unitary minimal models, and stated some related open problems.

5 Yasuyuki Kawahigashi

5.1 Representation theory of local conformal nets and complete rationality

I presented representation theory of local conformal nets based on the Doplicher-Haag-Roberts theory and introduced braiding structure. Next I presented the definition of complete rationality and explained various related conditions. Then I brielfy explained modular invariants and how to classifiy extensions of local conformal nets.

5.2 α-induction and modular invariants

As a continuation of the above talk, I explained more detailed propertoes of alpha-induction, which is an induction procedure for representations of a local conformal net to those of its extension using a braiding. Graphical method to prove modular invariance property was presented in detail. A comparison to representation theory of a vertex operator algebra was also given.

5.3 Realization of a modular tensor category as a representation category of a local conformal net

An open problem to reaized a given modular tensor category as a representation category of a local conformal net was explained. Then I explained in more detail the special case of modular tensor categories arising as quantum doubles.

6 Roberto Longo

6.1 Real Hilbert subspaces and conformal nets

This introductory talks explains the part of the structure associated with a local Möbius covariant net that depends only on the Hilbert space structure or, equivalently, on the Möbius group representation. In a sense, this is the universal structure associated with a net and can be descibed by the Tomita-takesaki modular theory (joint work with R. Brunetti and D. Guido). Borchers theorem and the Wisbrock-Araki-Zsido theorem on half-sided modular inclusions have a version in the real Hilbert subspace setting.

6.2 Jones index and Doplicher-Haag-Roberts superselection sectors

This talk was aimed to give an introduction to the fundamental relation between Jones index theory of subfactors and the DHR sector theory in algebraic QFT.

6.3 An analogue of the Kac-Wakimoto formula, black hole entropy, and boundary conformal QFT

We have explained a local version of a formula by Kac and Wakimoto tha can be proved in full generality in the framework of local conformal nets. This formula gives an expression for the statistical dimension that is related to the incremental black hole entropy. Finally we have explined the basic structure of boundary conformal QFT from the operator algebraic viewpoint (this last topic is taken by a joint work with H. Rehren).

7 Mihály Weiner

7.1 Examples of unitary VOAs that integrate to local conformal nets

This introductory talk aimed to establish the necessary background for the new results achieved in a joint work and presented later in the workshop by S. Carpi.

The passage from unitary VOAs to local conformal nets involves several technical difficulties. This is mainly due to the fact the vertex operators, in general, are unbounded.

For a set of densely defined, closed operators $\{A_\alpha\}$ there is a minimal von Neumann algebra $\mathcal{M}$ such that A_α is affiliated to $\mathcal{M}$ for every α. We may call $\mathcal{M}$ the "generated" von Neumann algebra, and denote it by $\{A_\alpha\}''$.

Let $\{A_\alpha\}$ and $\{B_\beta\}$ are two sets of densely defined, closed operators with a common invariant core $\mathcal{D}$. Suppose that $[A_\alpha, B_\beta] = 0$ (on $\mathcal{D}$) for every α and β. Does it follow that $\{A_\alpha\}''$ and $\{B_\beta\}''$ are commuting von Neumann algebras?

It is well known, that the answer, in general, is "no". Yet this is exactly what we should check (in our particular case), in order to ensure the locality of the generated net of von Neumann algebras.

In my talk I gave an overview of existing methods of dealing with such questions using certain estimates both in the general setting — e.g. by using Nelson's commutator theorem — and in the particular case of vertex operators, where it is implemented through the use of certain "energy bounds". Moreover, I explained how to find some energy bounds in the affine and in the Virasoro case.

In the mentioned examples, these energy bounds, for a generating set of fields, turn out to be linear (or better, than linear). Thus in these fortunate examples — at least for this generating set of fields — the mentioned commutator theorem of Nelson can be employed.

Finally, I gave the following conjecture on energy bounds: a (quasi) primary field of conformal dimension $d > 1$ should admit an energy bound of degree $d - 1$. This lead to several discussions after the talk. In particular, following an idea of B. Bakalov, we have managed to give such bounds for an infinite set of fields in the $W_{1+\infty}$ model. This infinite set contains a quasi primary field for *every* conformal dimension $d > 1$. (Before this, there was no known examples for such bound for a field of conformal dimension $d \geq 3$.) Hence this example seems to give some support to this conjecture.

8 Feng Xu

8.1 Some arithmetic properties of chiral quantities

We sketch a proof of some arithmetic properties of chiral quantities such as congruence subgroup properties in completely rational nets.

8.2 Mirror extensions of local nets

In this talk we give examples of new rational nets which are obtained as ”mirrors” of exotic extensions such as conformal inclusions.