Lectures on Mirror Symmetry

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Contents

1. Lecture I. Mirror Symmetry

1.1. Introduction

One of the hallmarks of superstring theory is that it often inpires the right questions about loop spaces, which may have been hard to imagine from the viewpoint of classical geometry alone. The theory of non-linear sigma models is a physical arena where many such questions are inspired, and sometimes even answered. Here, one starts with a manifold as a target, and one is interested in the space of maps from a complex curve into that target. This mapping space is morally the complex analogue of an ordinary loop space, since the latter concerns with mappings from a circle into a manifold. String physics dictates that the most important problem about such a "complex loop space" is to define and compute correlation functions on this space – in other words, to do intersection theory on this space. Of course, this tantalizing idea is begging a whole host of basic mathematical questions. For example, what conditions should be imposed on the target? What kind of curves are allowed? What kind of maps are allowed? How do we make sense of intersection theory? And, of course, can we compute?

The answers to these questions turn out to be surprisingly rich and deep, and these lectures will not presume to even name them all, much less detailing them. Historically, however, the quantitative question – about computation – was first answered by physicists before the other qualitative questions. The answer to the computational question was given early on in a wide variety of examples. The theory behind the computations is what's known as *Mirror Symmetry*.

In the first lecture, we begin with an outline of some of the early historical development. To stay focused, we try to navigate through a narrow path in the vast history of mirror symmetry. This helps us set the stage for the main topics of these lectures:

I. Mirror Calabi-Yau manifolds.
II. Periods and large radius limits.
III. Characteristic numbers on stable map moduli.

All lectures will be expository in nature and the tone will be informal, for the most part. The main purpose is to help acquaint the nonexpert readers with a handful of basic notions and techniques used in some of the recent work. The readers will hopefully find them useful beyond the context of these lectures. Discussions in only a few lectures on such a vast subject have to be prioritized. Global pictures are placed above technical details.

For the details, the readers should refer to the original articles. Many related important topics and far-reaching approaches are not being discussed here, but hopefully will be by other lecturers. For example, G. Tian will discuss his joint work with Y.B. Ruan and J. Li on their approaches to Gromov-Witten theory. B. Dubrovin will discuss his theory of Frobenius manifolds, and N. Hitchin will lecture on a geometric approach to mirror manifolds, as proposed by Stominger-Yau-Zaslow.

1.2. A Tour from Physics to Mirrors

For a more elaborate historical review of early development, the readers should consult [16]. For more recent development, see, for example, the introduction of [36][37] and references therein.

Gepner's correspondence. In 1986, D. Gepner discovered a correspondence between $N = 2$ unitary superconformal field theories (SCFT) and Calabi-Yau manifolds. Mathematically, this connection is surprising because SCFT is understood from the viewpoint of representation theory of an infinite dimensional graded Lie superalgebra – the $N = 2$ super Virasoro algebra. So, this is an unexpected connection between representation theory and complex geometry. From the physics viewpoint, however, this connection is not only expected but natural. It is so because a nonlinear sigma model with a Calabi-Yau target *is* a family of SCFT's to begin with. This family is parameterized by varying a number of "moduli" – complex structure, polarization, vector bundle etc – put on the target manifold. At special points in the sigma model moduli space, physics can manifest itself in different guises. The representation theoretic description is just one such manifestation corresponding to a "renormalization fixed point" in the moduli space. Gepner's observation was a confirmation of this manifestation.

Mirror manifolds. Gepner's correspondence isn't one-to-one. In fact one compares a finite dimensional algebra (called a "chiral ring") in the representation theoretic description of SCFT with the Dolbeault cohomology of a Calabi-Yau manifold. There is an apparent order 2 automorphism of the $N = 2$ super Virasoro algebra. If one twists the chiral ring by this automorphism, one ends up with a different Dolbeault cohomology. In fact, the new Hodge diamond would simply be a mirror reflection of the old Hodge diamond. On the basis of this observation, Dixon, Lerche, Vafa, and Warner hypothesized that Calabi-Yau manifolds should come in pairs where their Hodge diamonds are mirror reflection of one another. Let $h^{p,q}(X)$ be the dimension of the (p, q) Dolbeault cohomology group of a Calabi-Yau threefold X. Then the mirror hypothesis would assert that there would be another Calabi-Yau threefold Y with with

$$h^{p,q}(Y) = h^{3-q,p}(X).$$

This is one of many properties physicists require of a pair of "mirror manifolds".

One example was an SCFT which Gepner found to be corresponding to a quintic threefold in $\mathbf{P}^4$. But its mirror partner was not known. In 1987, Greene and Plesser gave the first construction of a mirror partner of that quintic threefold. Further examples were also found by Roan and others. Candelas, Lynker, and Schimmrigk did a computer search for mirror pairs in weighted projective four spaces and found a near perfect match.

In 1990, Candelas-de la Ossa-Green-Parkes studied the Greene-Plesser mirror threefolds X, Y by deforming them in their respective moduli spaces. Using techniques of special geometry (see the paper of S. Ferrara et al in [16] for references), Candelas et al derived a precise formula relating the Kähler potentials of two special geometries in the "large radius limit". As a consequence, they also proposed an ansatz which gives astounding predictions for the "number" of rational curves in a general quintic.

As shown by Witten, there are actually two distinct topological field theories, known as the A-model and the B-model, obtained by twisting the original sigma model on a given Calabi-Yau target. The comparison of the two special geometries above amounts to comparing the B-model of X with the A-model of Y. In mathematical terms, one is comparing (among other things) the Kodaira-Spencer algebra $H^*(X, \wedge^* TX)$ of X, with the quantum cohomology of the mirror partner Y. The latter is the cohomology algebra of Y, deformed by instanton corrections. (G. Tian will discuss the mathematical theory behind this in his lectures.) To go one step further, Bershadsky-Cecotti-Ooguri-Vafa then consider a sigma model in the presence of "topological gravity". Applying what they call the anomaly equations, together with mirror symmetry, they compute quantum string amplitudes and produce far-reaching enumerative predictions for higher genus curves in a Calabi-Yau manifold.

As examplified in the case of the quintic, mirror symmetry was clearly a prescription capable of remarkable predictions in enumerative geometry. For mathematics, this gave immediate impetus for progress in at least three fronts: to find other Calabi-Yau mirror pairs, to generalize the mirror symmetry prescription to new manifolds, and to set up the arena where those predictions can be put to real mathematical tests.

1.3. From Mirrors to Families

As the quintic example suggest, once we have a candidate mirror pair, we must deform them in their respective moduli spaces, and compare their variations near a "large radius limit". We discuss this notion next.

Throughout the lectures, by a *Calabi-Yau* manifold, we mean a compact Kähler manifold X whose canonical bundle K_X is trivial. The simplest example of a Calabi-Yau manifold is an elliptic curve:

$$y^2 = x^3 + g_2 x + g_3.$$

The easiest way to find Calabi-Yau manifolds is by means of the adjunction formula. Let M be a compact Kähler manifold. Suppose the bundle K_M^{-1} has a nonzero section f such that its zero set $f = 0$ is nonempty and nonsingular. Then by the adjunction formula, we have

$$K_X = (K_M \otimes [X])|_X = 0.$$

In particular, the hypersurface X in M is a Calabi-Yau manifold. For example for $M = \mathbf{P}^n$, we have $K_M^{-1} = \mathcal{O}(n+1)$, and a section of this bundle is a homogeneous polynomial of degree $n+1$ in the coordinates $[z_0, .., z_n]$. Generically a section cuts out a Calabi-Yau manifold in $\mathbf{P}^n$.

Let X be a Calabi-Yau n-fold. Since K_X is trivial, there is a nowhere vanishing holomorphic n-form ω which is unique up to a scalar. From this, we get a map $H_n(X) \to \mathbf{C}$, $\gamma \mapsto \int_\gamma \omega$. We will be interested in the family version of this map.

Let $\mathcal{X} \to \mathcal{M}$ be a flat family whose generic fiber X_z is a Calabi-Yau manifold. There is a flat bundle on $\mathcal{M}$ (away from the singular X_z) whose fiber at z is the vector space $H^n(X_z, \mathbf{C})$. In it, there is a line bundle $\mathcal{L}$ whose fiber is $H^0(X_z, \wedge^n T^* X_z)$. Thus a section of $\mathcal{L}$ gives a family of holomorphic n-forms ω_z. Integrating this against a class $\gamma \in H_n(X_z)$, we get a (multi-valued) function

$$f_\gamma(z) = \int_\gamma \omega_z.$$

This is called the period associated to γ.

•*Example.* Consider the family of cubic curves

$$\sigma_z(x) := x_0^3 + x_1^3 + x_2^3 + z x_0 x_1 x_2 = 0, \quad z \in \mathbf{P}^1$$

defined in $\mathbf{P}^2$. One way to construct the periods is by means of Poincaré residue as follows. Take a meromorphic 2-form Ω_z on $\mathbf{P}^2$ with a first order pole along the curve $\sigma_z(x) = 0$, and take a primitive cycle γ (which is a circle) in the curve. Take a circle bundle Γ over γ inside the normal bundle of $\sigma_z(x) = 0$ in $\mathbf{P}^2$, so that each fiber in Γ goes around the curve once. Now integrate the 2-form Ω_z against Γ, and get a function $f_\gamma(z)$. Explicitly, we can choose

$$\Omega_z = \frac{\sum_i (-1)^i x_i dx_0 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_2}{\sigma_z(x)}.$$

Thus we get the periods

$$f_\gamma(z) = \int_\Gamma \Omega_z.$$

These functions turn out to be a Gauss hypergeometric function.

For a general Calabi-Yau manifold X, a theorem of Bogomolov-Tian-Todorov says that the deformation space of X is unobstructed. In fact, the periods determine the local variation of the complex structure of X completely.

Definition 1.1. *A point $z_0 \in \mathcal{M}$ is called a large radius limit if there is a choice of n-forms ω_z such that the periods have the following property: the subspace $\{\gamma | \, |f_\gamma(z)|$ is bounded near $z_0\} \subset H_n(X)$ is one dimensional.*

Note that near a generic point $z \in \mathcal{M}$, each function $f_\gamma(z)$ is bounded, and so the subspace above will be all of $H_n(X)$. Near a large radius limit, on the other hand, there are as many periods blowing up there as possible. Intuitively, a large radius limit is a point of deepest degeneration in the family. We note that there is an alternative to the above notion, in which one uses the monodromy around z_0 to characterize the deepest degeneration (cf. [40]). In the case we shall discuss in this lecture, the two approaches are essentially equivalent.

•*Example.* In the case of the 1-parameter family $\sigma_z(x) = 0$ of cubic curves above, there are exactly three points on $\mathbf{P}^1$ in the descriminant locus, where the fibers are apparently singular. By computing the periods $f_\gamma(z)$ near those points, it is easy to show that $z = \infty$ is the only large radius limit.

While there is as yet no universally accepted definition for a mirror manifold, there are a number of properties dictated by mirror symmetry. A fruitful viewpoint is to use these properties as the basis for an operational definition. This is the viewpoint we follow here. Let Y be a Calabi-Yau n-fold. A mirror manifold X should have the following property:

1. $h^{n-p,q}(X) = h^{p,q}(Y)$, where $h^{p,q}(Y) := dim\ H^q(Y, \wedge^p T^*Y)$.
2. There is a family $\mathcal{X} \to \mathcal{M}$, with generic fiber diffeomorphic to X, which admits a large radius limit z_0.

Note that according to the property 1, the Euler characteristics of X and Y differ exactly by a sign. Mirror symmetry further dictates that the power series expansions of the periods $f_\gamma(z)$ near z_0 compute the A-model of Y. Thus mirror symmetry is a prescription for computing the A-model Kähler potential (which is the imaginery part of an instanton sum defined by path integral) for one manifold Y, using the B-model of another manifold X near a large radius limit. Mathematically, this should translates into computing the intersection numbers on certain complex loop spaces for Y, using the complex structure variation of a *family* of Calabi-Yau manifolds X.

1.4. lecture plan

In the next lecture, we give a quick introduction to toric geometry. Most of the basic material here can be found in [42][17]. We briefly review the construction of Batyrev [3] and see that a large class of Calabi-Yau manifolds Y in toric varieties do have partners X satisfying condition 1. Here we will deal mostly with the three dimensional cases. (There are still a number of technical loose ends in higher dimensions.) In this construction, pairs of Calabi-Yau manifolds satisfying condition 1 are realized as hypersurfaces in pairs of toric varieties.

For a given Batyrev's pair X, Y, we deform the Calabi-Yau X inside an ambient toric variety, and get a family of manifolds. We then consider the large radius limit problem for this family. We will show that this family has property 2. This part of the lecture is based on my joint paper with S. Hosono and S.T. Yau [29][28].

We will show how to compute the A-model potential of Y using the periods of the family of X, according to the mirror symmetry prescription. As noted before, this prescription computes intersection numbers on certain complex loop spaces for Y. Of course, we have yet to define and formulate a theory for these intersection numbers mathematically. This is deferred till the last lecture. Here, we shall discuss the general problem of studying characteristic numbers of vector bundles on a stable map moduli space for a projective manifold. We discuss some recent progress on this problem for a class of balloon manifolds, and show how this specializes to the mirror symmetry prescription. We also discuss a conjecture for those characteristic numbers for a general projective manifold. This lecture is based on recent joint work with K. Liu and S.T. Yau [36][38].

1.5. Acknowledgements

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2. Lecture II. Mirror Manifolds

2.1. Toric varieties and fans

Almost everything in this section can be found in [42][17]. A toric variety $\mathbf{P}$ is a normal variety equipped with the action of an algebraic torus T having a Zariski dense orbit.

The simplest affine toric varieties are $\mathbf{C}^n$ or $\mathbf{C}^n/\Gamma$ where Γ is a finite subgroup of $T = (\mathbf{C}^\times)^n$ acting on $\mathbf{C}^n$ by the standard linear action. One way to construct a toric variety is to patch together affine toric varieties. The ways in which the patching is done can be classified by combinatorial objects called fans, which we discuss next.

Let N be a rank n lattice (ie. free abelian group), and let

$$N_{\mathbf{R}} = N \otimes \mathbf{R}.$$

Let M denote the dual lattice $Hom_{\mathbf{Z}}(N, \mathbf{Z})$. A cone generated by $v_1, .., v_k$ in $N_{\mathbf{R}}$ is the set $\mathbf{R}_{\geq 0}v_1 + \cdots + \mathbf{R}_{\geq 0}v_k$. All cones in $N_{\mathbf{R}}$ considered here are assumed to be generated by vectors in N, and are assumed to contain no linear subspace of $N_{\mathbf{R}}$, except the origin. (They are called "strongly convex rational polyhedral cones" in the literature.) A cone is uniquely specified by a listing of its generators $\langle v_1, .., v_k\rangle$. This notation will be used below. A *fan in* N is a finite collection Σ of cones with the following properties:

(i) If σ is a cone in Σ, then every face of σ is in Σ.

(ii) If σ, τ are cones in $\mathcal{C}$, then $\sigma \cap \tau$ is a face of both σ and τ.

•*Example 1.* Take $N = \mathbf{Z}$, and let Σ be the collection consisting of the three cones $\langle\ \rangle$, $\langle \pm 1\rangle$, which are respectively the origin and the two half lines in $N_{\mathbf{R}} = \mathbf{R}$. Then Σ is a fan in N.

•*Example 2.* Take $N = \mathbf{Z}^2$, and put $e_1 = (1, 0)$, $e_2 = (0, 1)$. Let Σ be the collection consisting of the seven cones $\langle\ \rangle$, $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle -e_1-e_2\rangle$, $\langle e_1, e_2\rangle$, $\langle e_2, -e_1-e_2\rangle$, $\langle -e_1-e_2, e_1\rangle$ in $N_{\mathbf{R}} = \mathbf{R}^2$. (Draw a picture!) Then Σ is a fan in N.

•*Example 3.* If σ is a cone in $N_{\mathbf{R}}$, then we automatically get a fan Σ consisting of all the proper faces of σ, and the improper face σ itself. By the same token, we can specify a fan by specifying just those cones $\sigma_1, .., \sigma_k$, which are not the proper faces of any other cone in Σ. In this case, we say that $\sigma_1, .., \sigma_k$ *generates the fan* Σ. For instance, in the second example above, the three largest cones $\langle e_1, e_2\rangle$, $\langle e_2, -e_1 - e_2\rangle$, $\langle -e_1 - e_2, e_1\rangle$ generate the fan Σ.

•*Example 4. The fan over a polytope.* A *polytope* Δ in $N_{\mathbf{R}}$ is the convex hull of a finite set of points in the lattice N. We assume that the origin is contained in the interior of Δ. For each proper face f of Δ, let σ_f be the cone over f with apex at the origin. Thus if $v_1, .., v_k$ are the vertices of f, then $\sigma_f = \langle v_1, .., v_k\rangle$. Note that if f is the empty face, then σ_f is the origin, by definition. It is clear that the collection consisting of all the σ_f is a fan in N.

For instance, the fans in Examples 1 and 2 are respectively fans over the polytopes $conv\{1, -1\}$, $conv\{e_1, e_2, -e_1 - e_2\}$.

A fan Σ in N corresponds to a toric variety $\mathbf{P}_\Sigma$ as follows. First, to the zero cone $\langle\ \rangle$ we assign the torus

$$T := U_{\langle\ \rangle} = Hom(M, \mathbf{C}^\times) \cong (\mathbf{C}^\times)^n.$$

To each nonzero cone $\sigma \in \Sigma$, we assign the affine variety

$$U_\sigma := Hom_{sg}(\sigma^\vee \cap M, \mathbf{C}).$$

This is the set of semigroup homomorphisms from the additive semigroup $\sigma^\vee \cap M$ to the multiplicative semigroup $\mathbf{C}$. Here

$$\sigma^\vee := \{u \in M_{\mathbf{R}} | \langle u, v\rangle \geq 0\ \forall v \in \sigma\}$$

is the *dual* of σ. Equivalently, the affine variety U_σ can be described in terms of the its coordinate ring as follows. A semigroup homomorphism $\chi : \sigma^\vee \cap M \to \mathbf{C}$ defines an algebra homomorphism $\chi : R_\sigma \to \mathbf{C}$, from the semigroup algebra $R_\sigma = \mathbf{C}[\sigma^\vee \cap M]$ into $\mathbf{C}$. In turn, algebra homomorphisms $\chi : R_\sigma \to \mathbf{C}$ correspond 1-1 to close points in the affine scheme $Spec(R_\sigma)$.

Note that if τ is a face of σ, then $\sigma^\vee \subset \tau^\vee$, and we have the restriction map $U_\tau \to U_\sigma$. A semigroup homomorphism $\chi : \tau^\vee \cap M \to \mathbf{C}$ is determined by its values on a basis $u_1, .., u_n$ of M. Moreover $\sigma^\vee$ always contains such a basis, for any σ. Thus the restriction map $U_\tau \to U_\sigma$ is injective. We define $\mathbf{P}_\Sigma$ to be the disjoint union of the affine varieties U_σ, modulo the equivalence relation given by the embeddings $U_\tau \to U_\sigma$.

Note that each of the affine variety U_σ contains the torus $T = U_{\langle\ \rangle}$ as an open subset, and has a canonical T-action

$$T \times U_\sigma \to U_\sigma, \quad (t, \chi) \mapsto t' \cdot \chi$$

which extends the product $T \times T \to T$. Here $t : M \to \mathbf{C}^\times$ is a function on M, and t' is its restriction to the subset $\sigma^\vee \cap M$. Note also that the equivalence relation above respects the T-action. Thus the result $\mathbf{P}_\Sigma$ contains T as an open subset, and carries a natural T-action which extends the action of T on itself.

Note that if τ is a proper face of σ, then one can choose a bounding hyperplane $u^\perp$ with $u \in \sigma^\vee \cap M$ such that

$$\tau = \sigma \cap u^\perp.$$

The image of the restriction map $U_\tau \to U_\sigma$ consists of function $\chi : \tau^\vee \cap M \to \mathbf{C}$ such that $\chi(u) \neq 0$. Thus U_τ is embedded in U_σ as an open subset.

2.2. Transition functions

Suppose σ is a *regular cone* of dimension n, ie. it is generated by an integral basis $w_1, .., w_n$ of N. Then $\sigma^\vee$ is a regular cone in M generated by the dual basis $w_1^*, .., w_n^*$. Thus we have a canonical isomorphism

$$\mathbf{C}^n \to U_\sigma = Hom_{sg}(\sigma^\vee \cap M, \mathbf{C})$$

sending $(z_1, .., z_n)$ to χ, defined by $\chi(w_i^*) = z_i$. Continuing with these notations, suppose τ is a regular cone of dimension k, ie. τ is a face of a regular cone σ of dimension n. Say $\tau = \langle w_1, .., w_k \rangle$. Then $\tau^\vee = \mathbf{R}_{\geq 0} w_1^* + \cdots + \mathbf{R}_{\geq 0} w_k^* + \mathbf{R} w_{k+1}^* + \cdots + \mathbf{R} w_n^*$. The canonical isomorphism

$$\mathbf{C}^k \times (\mathbf{C}^\times)^{n-k} \to U_\tau = Hom_{sg}(\tau^\vee \cap M, \mathbf{C})$$

is just the restriction of the isomorphism $\mathbf{C}^n \to U_\sigma$ to the subset $\mathbf{C}^k \times (\mathbf{C}^\times)^{n-k} \subset \mathbf{C}^n$. In particular, regular cones correspond to nonsingular affine varieties.

•In Example 1 above, our fan Σ has two regular cones $\langle \pm 1 \rangle$ in $\mathbf{R}$. Their dual are respectively $\langle \pm 1^* \rangle$. They correspond to two copies of $\mathbf{C}$, glued together along the subset $\mathbf{C}^\times \subset \mathbf{C}$, via the transition function $x \to x^{-1}$. Note that restricting the two maps $U_{\langle \pm 1 \rangle} \tilde{\to} \mathbf{C}$ to $U_{\langle\ \rangle} \subset U_{\langle \pm 1 \rangle}$, we get two maps

$$\mathbf{C}^\times \leftarrow U_{\langle\ \rangle} \to \mathbf{C}^\times.$$

Composing the inverse of the first with the second map, gives the transition function $x \to x^{-1}$. The resulting toric variety is, of course, $\mathbf{P}_\Sigma = \mathbf{P}^1$.

•Example 2 above, our fan Σ has three regular cones $\langle e_1, e_2\rangle,\ \langle e_2, -e_1-e_2\rangle,\ \langle -e_1-e_2, e_1\rangle$ of dimension 2 in $\mathbf{R}^2$. Their duals are respectively $\langle e_1^*, e_2^*\rangle,\ \langle -e_1^*+e_2^*, -e_1^*\rangle,\ \langle -e_2^*, e_1^*-e_2^*\rangle$. They correspond to three copies of $\mathbf{C}^2$, glued together via the transition functions

$$(x,y) \longrightarrow (x^{-1}y, x^{-1}) \longrightarrow (y^{-1}, xy^{-1}).$$

Again, these are maps from $(\mathbf{C}^\times)^2$ onto itself obtained by restricting the isomorphism $U_\sigma \to \mathbf{C}^2$ for each of the three cones σ above, to $U_{\langle\ \rangle} \subset U_\sigma$. You'll recognize instantly that the resulting toric variety is $\mathbf{P}_\Sigma = \mathbf{P}^2$.

•More generally, the fan over the n-simplex with vertices $e_1, e_2, .., e_n, -e_1 - \cdots - e_n$, is the fan for $\mathbf{P}^n$.

•*Example. Type A_k rational double point.* When a cone is not regular, then the corresponding affine variety will be singular. Here is an example. Consider the cone $\sigma = \langle e_2, (k+1)e_1 - ke_2\rangle$ in $N = \mathbf{Z}^2$. The dual is $\sigma^\vee = \langle ke_1^* + (k+1)e_2^*, e_1^*\rangle$. The semigroup $\sigma^\vee \cap M$ has three generators (draw a picture!): $ke_1^* + (k+1)e_2^*,\ e_1^*,\ e_1^* + e_2^*$. From this we find that the semigroup algebra of $\sigma^\vee \cap M$ is

$$R_\sigma = \mathbf{C}[X,Y,Z]/(Z^{k+1} - XY).$$

This is the coordinate ring of U_σ. In other words, U_σ is the affine surface

$$Z^{k+1} = XY.$$

We can resolve the singularity at the origin as follows. The semigroup $\sigma \cap N$ is generated by $e_2, e_1, 2e_1 - e_2, 3e_1 - 2e_2, .., (k+1)e_1 - ke_2$. Any two consecutive generators generate a regular cone. Thus we can *subdivide* $\sigma = \langle e_2, (k+1)e_1 - ke_2\rangle$ into $k+1$ regular cones $\sigma_0, .., \sigma_k$, and get a new fan Σ'. The dual of those cones are

$$\sigma_0^\vee = \langle e_2^*, e_1^*\rangle,\ \sigma_1^\vee = \langle 2e_2^* + e_1^*, -e_2^*\rangle, \cdots, \sigma_k^\vee = \langle (k+1)e_2^* + ke_1^*, -ke_2^* - (k-1)e_1^*\rangle.$$

The toric variety $\mathbf{P}_{\Sigma'}$ is then a chain of $k+1$ copies of $\mathbf{C}^2$ glued together via the transition functions

$$(x,y) \longrightarrow (x^2y, x^{-1}) \longrightarrow (x^3y^2, x^{-2}y^{-1}) \longrightarrow \cdots \longrightarrow (x^{k+1}y^k, x^{-k}y^{-k+1}).$$

There is a proper map $\mathbf{P}_{\Sigma'} \to U_\sigma$ which maps $U_{\sigma_k} \to U_\sigma$ by the canonical restriction.

More generally, any singular toric variety can be resolved equivariantly by successively subdividing the irregular cones in its fan.

We now summarize a few facts about toric varieties. Let $\mathbf{P}_\Sigma$ be a toric variety with fan Σ.

1. (Singularity) An affine subvariety $U_\sigma \subset \mathbf{P}_\Sigma$ is nonsingular iff the cone σ is regular.
2. (Compactness) $\mathbf{P}_\Sigma$ is compact iff the fan Σ is complete, ie. the cones in Σ fill up the space $N_{\mathbf{R}}$.
3. (T-orbits) The set of T-orbits in $\mathbf{P}_\Sigma$ correspond 1-1 with the set Σ. The T-orbit corresponding to $\tau \in \Sigma$ is

$$\mathcal{O}_\tau := Hom(\tau^\perp \cap M, \mathbf{C}^\times) \subset U_\tau.$$

Hence $dim\ \mathcal{O}_\tau = codim\ \tau$. The closure of $\mathcal{O}_\tau$ in $\mathbf{P}_\Sigma$ is the union of orbits $\mathcal{O}_\sigma$ such that τ is a face of σ.
4. (Blow-up) If Σ' is a fan obtained by subdividing one or more cones in Σ, then there is T equivariant map $\varphi : \mathbf{P}_{\Sigma'} \to \mathbf{P}_\Sigma$ which is proper and birational. Every toric variety admits an equivariant resolution by a finite sequence of subdivisions of irregular cones.

Other interesting constructions of toric varieties (some not necessarily normal), can be found in [4][45][14].

2.3. Line bundle and cohomology

We record here some useful facts about compact toric varieties. Recall that for the projective space $\mathbf{P}^n$, the section space of any holomorphic line bundle can be described in terms of the ring $\mathbf{C}[x_0, .., x_n]$. Namely

$$H^0(\mathbf{P}^n, \mathcal{O}(k))$$

consists of all degree k elements in $\mathbf{C}[x_0, .., x_n]$. It turns out that there is a very similar description for any toric variety (see [14]). Throughout this subsection, let Σ be a complete regular fan in N.

Recall that we have a group homomorphism

$$Pic(\mathbf{P}_\Sigma) \xrightarrow{c_1} A_{n-1}(\mathbf{P}_\Sigma).$$

This is injective because $\mathbf{P}_\Sigma$ is normal. An important fact about compact toric varieties is that this homomorphism is also surjective. Recall that the k dimensional T-orbit closures in $\mathbf{P}_\Sigma$ are labelled by the $(n-k)$ dimensional cones in Σ. Denote by D_ρ the T invariant divisor corresponding to the one dimensional cone ρ. Then one has an exact sequence

$$0 \to M \to \oplus \mathbf{Z} D_\rho \to A_{n-1}(\mathbf{P}_\Sigma) \to 0.$$

Here the second arrow is given by $\mu \mapsto \sum \langle \mu, \rho \rangle D_\rho$, where $\langle \mu, \rho \rangle$ is the pairing between μ with the integral generator of the one-cone ρ. Let S be the polynomial algebra in the variables x_ρ, one for each one-cone ρ. This algebra is graded by the abelian group $A_{n-1}(\mathbf{P}_\Sigma)$. Namely, the monomial degree is given by

$$deg\ x^a = \sum a_\rho [D_\rho]$$

for any vector $a = (a_\rho)$ with $a_\rho \in \mathbf{Z}_{\geq 0}$. Then for any line bundle $\mathcal{L} = \sum a_\rho [D_\rho]$ on $\mathbf{P}_\Sigma$, one has [14]

$$H^0(X, \mathcal{L}) \cong S_{\mathcal{L}}$$

where $S_{\mathcal{L}}$ is the space of degree $\mathcal{L}$ polynomials in S. Thus $S_{\mathcal{L}}$ is spanned by the monomials x^b with

$$\sum b_\rho [D_\rho] = \mathcal{L}, \quad b_\rho \geq 0.$$

This is equivalent to the condition that there is a vector $\mu \in M$ such that

$$b_\rho - a_\rho = \langle \mu, \rho \rangle$$

for all ρ. Thus the solutions $b_\rho \geq 0$ to these equations correspond 1-1 with the integral points in the polytope

$$\Delta_{\mathcal{L}} := \{\mu \in M_{\mathbf{R}} | \langle \mu, \rho \rangle \geq -a_\rho\}.$$

For example if $\mathcal{L} = \sum [D_\rho]$, then $\Delta_{\mathcal{L}}$ is nothing but the polar dual (see below) of the polytope $conv\{generators\ of\ one-cones\ \rho\}$ in N.

Let Σ be a regular complete fan in N. Then the cohomology ring of $\mathbf{P}_\Sigma$ coincides with the Chow ring $A_*(\mathbf{P}_\Sigma)$, which has the following simple description. It is a ring generated by $A_{n-1}(\mathbf{P}_\Sigma)$, with the relations among the classes $[D_\rho]$ generated by the monomials

$$\prod_{\rho \in C} [D_\rho]$$

where C is any set of one-cones which does not generate a cone in Σ. The total Chern class of $\mathbf{P}_\Sigma$ has the following simple expression:

$$c(T\mathbf{P}_\Sigma) = \prod (1 + [D_\rho]).$$

In particular,

$$c_1(T\mathbf{P}_\Sigma) = \sum [D_\rho].$$

2.4. Reflexive polytopes

As in Example 4 above, let Δ be an integral polytope in N, ie. the convex hull of finitely many points in N containing the origin in the interior. The polar dual ∇ of Δ is defined to be the set

$$\nabla := \{\mu \in M_{\mathbf{R}} | \langle \mu, \nu \rangle \geq -1, \ \forall \nu \in \Delta\}.$$

This is a polytope in M bounded by the planes

$$P_\nu = \{\mu \in M_{\mathbf{R}} | \langle \mu, \nu \rangle = -1\},$$

one for each vertex ν of Δ. Note that ∇ contains the origin of M, and that the polar dual of ∇ is Δ. Note, however, that ∇ need not be integral, ie. its vertices need not be in M. In fact, ∇ is integral iff there are no integral points except the origin, in the interior of Δ.

Definition 2.1. *An integral polytope Δ is called reflexive if its polar dual ∇ is also integral.*

•*Example 1.* Let $\Delta = conv\{e_1, e_2, -e_1 - e_2\}$ in $\mathbf{R}^2$, and $\nabla = conv\{-e_1^* + 2e_2^*, -e_1^* - e_2^*, 2e_1^* - e_2^*\}$. This is a dual pair of polytopes which are reflexive.

•Given a reflexive polytope Δ, we can consider the fan Σ over Δ, as in Example 4. Thus we get a compact toric variety $\mathbf{P}_\Sigma$. But $\mathbf{P}_\Sigma$ is typically singular. For example, the polytope ∇ above yields a singular toric variety. In this case, each of the three cones over the codimension 1 faces of ∇ are not regular. The resulting toric variety has six singular points corresponding to the six points on the faces of ∇. Thus this toric variety can be resolved by blowing up 6 points, ie. by subdividing *each* of the three cones.

On the other hand, given a complete fan Σ, we get a polytope as follows. Each one dimensional cone ρ is generated by a unique primitive lattice point. Let $\Delta(\Sigma)$ be the convex hull of all these generators.

Theorem 2.2. *[29] Suppose Σ is a complete regular fan. Then $\Delta(\Sigma)$ is reflexive iff $c_1(\mathbf{P}_\Sigma) \geq 0$.*

This raises the following question. *Given a reflexive polytope Δ, is there a regular fan Σ such that $\Delta = \Delta(\Sigma)$?* It can be shown that in dimension 3 or less, the answer is affirmative, but not so in dimension 4 or higher. (For examples, see [28].)

2.5. Batyrev's construction

The construction in [3] yields a pair of "mirror" Calabi-Yau varieties for every pair of dual reflexive polytopes. But the resulting spaces may be singular. To ensure smoothness, we restrict ourselves to dimension four or less. For simplicity, in this section we will always assume that the toric varieties $\mathbf{P}_\Sigma$ we work with are nonsingular and that $c_1(\mathbf{P}_\Sigma) \geq 0$.

Fix such a toric variety $\mathbf{P}_\Sigma$. A generic anticanonical section $f \in H^0(\mathbf{P}_\Sigma, K^{-1})$ cuts out a Calabi-Yau hypersurface. The construction in [3] compares these Calabi-Yau hypersurfaces in pairs of toric varieties $\mathbf{P}_\Sigma$, $\mathbf{P}_\Phi$.

Theorem 2.3. *[3] If $\Delta(\Sigma)$ and $\Delta(\Phi)$ are reflexive polytopes polar dual to each other, then the Calabi-Yau hypersurfaces in $\mathbf{P}_\Sigma$ and $\mathbf{P}_\Phi$ are mirror to each other. That is, their Hodge diamonds are mirror reflections of one another.*

It turns out that the Hodge numbers are given by explicit formulas which enumerate points on various faces of the polytopes $\Delta(\Sigma)$, $\Delta(\Phi)$ (see [3]).

There are also conjectural generalizations of the construction above to complete intersections [5].

3. Lecture III. The Large Radius Limit Problem

Throughout this lecture, we will stick to the notations introduced in the last two lectures. Let Φ, Σ be two complete regular fans in M, N respectively. We shall always assume that the associated polytopes $\Delta(\Phi)$, $\Delta(\Sigma)$ is a dual reflexive pair, and that the associated toric varieties are projective. In this lecture, we will discuss the following main theorem:

Theorem 3.1. *[29] The moduli space of Calabi-Yau hypersurfaces in $\mathbf{P}_\Phi$ admits a large radius limit.*

•Strictly speaking, the LRL lies in some resolution of the moduli space. We will say what we mean by "the moduli space" later.

•Since $\Delta(\Sigma)$ is the polar dual of $\Delta(\Phi)$, we have

$$\Delta(\Sigma) = \{\nu | \langle \mu, \nu \rangle \geq -1, \ \forall \mu \in \Delta(\Phi)\}.$$

Observe that this coincides with the polytope $\Delta_{\mathcal{L}}$, introduced earlier, with $[\mathcal{L}] = \sum [D_\rho]$. We will describe the LRL problem starting from the pair of reflexive polytopes $\Delta(\Phi)$, $\Delta(\Sigma)$.

•Recall that our Calabi-Yau hypersurfaces X in $\mathbf{P}_\Phi$ are cut out by sections of the anticanonical bundle K^{-1} on $\mathbf{P}_\Phi$. Thus they are parameterized by the projective space $\mathbf{P}H^0(\mathbf{P}_\Phi, K^{-1})$. We have seen that $c_1(K^{-1}) = c_1(T\mathbf{P}_\Phi) = \sum [D_\rho]$, and that

$$H^0(\mathbf{P}_\Phi, K^{-1}) = S_{K^{-1}}$$

has dimension given by the number $|\Delta(\Sigma) \cap N|$.

•*Example.* Take Φ to be the fan generated by the cones $\langle e_1, e_2 \rangle$, $\langle e_2, -e_1 - e_2 \rangle$, $\langle -e_1 - e_2, e_1 \rangle$ in $M_{\mathbf{R}} = \mathbf{R}^2$. In this case, $\mathbf{P}_\Phi = \mathbf{P}^2$. Note that $\Delta(\Phi) = conv\{e_1, e_2, -e_1 - e_2\}$ is a reflexive polytope, with polar dual $\nabla = conv\{-e_1^* + 2e_2^*, -e_1^* - e_2^*, 2e_1^* - e_2^*\}$. We know that $K^{-1} = \mathcal{O}_{\mathbf{P}^2}(3)$, hence $H^0(\mathbf{P}_\Phi, K^{-1})$ is a ten dimensional vector space spanned by the degree 3 monomials in three variables. Those monomials correspond 1-1 with the lattice points in $\nabla \cap N$, which are

$$-e_1^* + 2e_2^*, \ -e_1^* + e_2^*, \ -e_1^*, \ -e_1^* - e_2^*, \ -e_2^*, \ e_1^* - e_1^*, \ 2e_1^* - e_2^*, \ e_1^*, \ e_2^*, \ 0.$$

These vectors form a complete regular fan Σ in N, with $\nabla = \Delta(\Sigma)$.

•Now return to the general case. The $T = Hom(N, \mathbf{C}^\times)$ action on $\mathbf{P}_\Phi$ induces a linear action on the projective space $\mathbf{P}H^0(\mathbf{P}_\Phi, K^{-1})$. Two sections related by T obviously define the same Calabi-Yau hypersurface. One could define our moduli space to be the GIT quotient:

$$\mathbf{P}H^0(\mathbf{P}_\Phi, K^{-1})/T.$$

Note that this is itself a toric variety. We have the following

Conjecture 3.2. *The quotient above is a normal toric variety whose fan is the Grobner fan $\mathcal{G}$ of the toric ideal $\mathcal{I} := \langle y^{l^+} - y^{l^-} | l \in L\rangle \subset \mathbf{C}[y_0, ..., y_p]$.*

Notations. We list the points in $\Delta(\Sigma) \cap N$ as $\{\nu_0 = 0, \nu_1, .., \nu_p\}$. Then $L \subset \mathbf{Z}^{p+1}$ is defined to be the lattice of vectors $l = (l_0, .., l_p)$ satisfying the relation $0 = \sum l_i(1, \nu_i) \in \mathbf{Z} \oplus N$. Here $y^{l^+} := \prod_{l_i>0} y_i^{l_i}$, and likewise for y^{l-}. The Grobner fan $\mathcal{G}$ is a complete fan in the lattice L^*. The cones in this fan correspond to certain Grobner bases of the ideal $\mathcal{I}$. For a recent study of this fan, see [45].

•For us, even without affirming the above conjecture, we can proceed by *taking*

$$\mathcal{M} := \mathbf{P}_\mathcal{G}$$

to be our moduli space of Calabi-Yau hypersurfaces in $\mathbf{P}_\Phi$, and study the periods of the hypersurfaces as functions on $\mathcal{M}$. We will see that, for mirror symmetry, this turns out to be the right thing to do. Note that $\mathcal{M}$ is typically singular.

•*Problem: Let $\mathcal{M}'$ be any equivariant resolution of $\mathcal{M}$. Describe the large radius limits in $\mathcal{M}'$, if any.*

•Key idea: A large radius limit in $\mathcal{M}'$ should be a point of deepest degeneration. This should be some torus fixed point. So let's try to describe the those fixed points in $\mathcal{M}'$.

3.1. Geometric description of LRL

Recall that we have a natural exact sequence

$$0 \to M \to \oplus_{i=1}^p \mathbf{Z} D_i \to A_{n-1}(\mathbf{P}_\Sigma) \to 0$$

where the D_i are the T invariant divisors in $\mathbf{P}_\Sigma$ labelled by the one-cones $\langle \nu_1 \rangle, ..., \langle \nu_p \rangle$ in Σ. It is more convenient to work with the following equivalent sequence

$$0 \to \mathbf{Z} \oplus M \to \oplus_{i=0}^p \mathbf{Z} D_i \to A_{n-1}(\mathbf{P}_\Sigma) \to 0$$

where D_0 is a formal symbol. Here the second arrow is $(k, \mu) \mapsto \sum_{i=0}^p (k + \langle \mu, \nu_i \rangle) D_i$, and the third arrow sends D_0 to the class $-\sum_{i=1}^p [D_i]$. On the other hand, by definition of L, we also have the exact sequence

$$0 \to L \to \mathbf{Z}^{p+1} \to \mathbf{Z} \oplus N \to 0$$

where the third arrow is $l \mapsto \sum_i l_i(1, \nu_i) \in \mathbf{Z} \oplus N$. Dualizing this gives

$$0 \to \mathbf{Z} \oplus M \to \mathbf{Z}^{p+1^*} \to L^* \to 0.$$

If we identify $\mathbf{Z}^{p+1^*}$ with $\oplus_{i=0}^p \mathbf{Z} D_i$ in an obvious way, then it follows that

$$A_{n-1}(\mathbf{P}_\Sigma) = L^*.$$

Thus with this identification, the Grobner fan of the ideal $\mathcal{I}$ is now a complete fan in $A_{n-1}(\mathbf{P}_\Sigma)$.

Theorem 3.3. *[29] The Kähler cone $\mathcal{K} \subset A_{n-1}(\mathbf{P}_\Sigma) \otimes \mathbf{R}$ of $\mathbf{P}_\Sigma$ is the interior of a cone in the Grobner fan $\mathcal{G}$ in L^*.*

•The proof is somewhat technical. It uses Sturmfels' theory of toric ideals. See [29].

•Note that since $\mathbf{P}_\Sigma$ is assumed to be projective, K_Σ has the maximal dimension. Recall now that maximal dimensional cones in a fan corresponds 1-1 the torus fixed points. Thus the closure of the Kähler cone corresponds uniquely to a torus fixed point p_0 in the $\mathcal{M}$. This point is singular in general.

Theorem 3.4. *Let $\varphi : \mathcal{M}' \to \mathcal{M}$ be any equivariant resolution at $p_0 \in \mathcal{M}$. Then every torus fixed point in $\varphi^{-1}(p_0)$ is a large radius limit.*

•The proof first reduces the problem to a PDE problem on $\mathcal{M}'$. Namely, the periods of the Calabi-Yau hypersurfaces X in $\mathbf{P}_\Phi$ are solutions to a system of linear PDE, known as the Picard-Fuchs equations. It turns out that these PDEs have been studied in a long series of papers by Gel'fand-Graev-Kapranov- Retakh-Zelevinsky. They call these PDEs generalized hypergeometric equations, now known as GKZ systems. (See below.) They are systems of linear PDEs with regular singularities.

•We use the combinatorial description of the cone K_Σ, which we derived from the theory of toric ideals, to show that at a fixed point $p \in \varphi^{-1}(p_0)$, the GKZ system admits exactly one bounded solution. And this solution comes from a period of Calabi-Yau hypersurfaces X in $\mathbf{P}_\Phi$. (See below.)

•The entire construction of LRL above can be readily generalized to the case of Calabi-Yau complete intersections in toric variety.

3.2. Remarks: some open problems

The most significant part of our result is not so much the proof of existence, but the explicit construction of the LRL. Namely they are torus fixed points corresponding to the Kähler cone of the mirror manifold. This cone turns out to be rather easy to compute in practice. This allows us to compute the periods at this point, and to carry out the *mirror symmetry prescription* in full to compute the A-model of a Calabi-Yau manifold.

Conjecture 3.5. *Every LRL in $\mathcal{M}'$ is a fixed point in $\varphi^{-1}(p_0)$.*

•Important progress has been made recently by Strominger-Yau-Zaslow for a geometric construction of general mirror manifolds. One important open problem is to understand the LRLs from their point of view.

3.3. The mirror prescription

Recall that the anticanonical section space $H^0(\mathbf{P}_\Phi, K^{-1})$ is spanned by the set of $A_{n-1}(\mathbf{P}_\Phi)$-graded homogeneous monomials of degree $[K^{-1}] = \sum[D_\rho]$. The monomials are labelled by the set $\Delta(\Sigma) \cap N = \{\nu_0, .., \nu_p\}$. Let $a = (a_0, .., a_p)$ be the coordinates on $H^0(\mathbf{P}_\Phi, K^{-1})$ relative this monomial basis. We would like to find explicitly the periods in these coordinates near a large radius limit, for our Calabi-Yau hypersurfaces X in $\mathbf{P}_\Phi$.

First, there is a holomorphic top form on X which is the Poincare residue of a meromorphic n-form on $\mathbf{P}_\Phi$ with a pole of order 1 along X. There is an n-cycle on $\mathbf{P}_\Phi$ given by a real torus in T. Integrating that meromorphic n-form against this real torus, we get (up to an overall constant)

$$\omega_0(t) = \sum_{l_1,..,l_p \geq 0,\ \sum l_i \nu_i = 0} \frac{(l_1 + \cdots + l_p)!}{l_1! \cdots l_p!} e^{l_1 t_1} \cdots e^{l_p t_p}$$

where $e^{t_i} := -a_i/a_0$ (see [29] for details). With a bit of formal manipulation, we can write

$$\omega_0(t) = \sum_l \frac{\langle l, [\mathcal{L}]\rangle!}{\prod_i \langle l, [D_i]\rangle!}\ e^{\langle l, t\rangle} \tag{3.1}$$

where the $[D_i]$ are the Chow classes represented by the T invariant divisors $[D_i]$ in $\mathbf{P}_\Sigma$, and $[\mathcal{L}] := \sum[D_i]$. Note that the vector $(l_1, .., l_p)$ has been identified with an element of the lattice $L = A_1(\mathbf{P}_\Sigma)$. It is easy to verify that ω_0 is a solution to the GKZ system:

$$\Box_l f(a) = 0, \quad \sum_i \langle \mu, \nu_i \rangle \theta_i f(a) = 0, \quad l \in L,\ \mu \in M.$$

Here $\Box_l := \prod_{l_i > 0}(\frac{\partial}{\partial a_i})^{l_i} - \prod_{l_i < 0}(\frac{\partial}{\partial a_i})^{-l_i}$ and $\theta_i := a_i \frac{\partial}{\partial a_i}$.

In the case when p is small, it is not hard to solve the above GKZ system completely. Details of many explicit examples have been worked out in [28] (see also references therein). In [28], the following formula for solutions to the GKZ system was given (cf. (3.1)):

$$\omega(t) := \sum_{l \in \Lambda} \frac{\prod_{m=1}^{\langle l, [\mathcal{L}]\rangle}([\mathcal{L}] + m) \times \prod_{\langle l, D_i\rangle < 0} \prod_{m=0}^{-\langle l, [D_i]\rangle - 1}([D_i] - m)}{\prod_{\langle l, D_i\rangle \geq 0} \prod_{m=1}^{\langle l, [D_i]\rangle}([D_i] + m)}\ e^{\langle l, t\rangle + t}.$$

Here

$$t := \sum t_i [D_i], \qquad \Lambda := \bar{\mathcal{K}}^\vee \cap L,$$

which is a semigroup in the lattice $L = A_1(\mathbf{P}_\Sigma)$. (Note: The formula in [28] has been written in terms of the Γ-function. The formula here differs from that by an irrelevant overall constant invertible factor in $A_*(\mathbf{P}_\Sigma)$.)

Note that to ensure convergence of $\omega(t)$, we must restrict t so that

$$Re\langle l, t\rangle < 0$$

for all but finitely many $l \in \Lambda$. But since Λ is a semigroup, this is equivalent to having the same condition for all nonzero $l \in \Lambda$. This means that

$$t \in -\mathcal{K} + \sqrt{-1}A_{n-1}(\mathbf{P}_\Sigma) \otimes \mathbf{R}.$$

The set of the right hand side is called complexified Kähler cone of $\mathbf{P}_\Sigma$.

Consider now the case when $\mathbf{P}_\Phi, \mathbf{P}_\Sigma$ are four dimensional toric varieties. Following [12], and numerous other computations, we proposed in [28] the following mirror symmetry prescription. That the A-model Kähler potential Ψ of the Calabi-Yau hypersurface Y in $\mathbf{P}_\Sigma$ is given by the explicit formula

$$\frac{1}{\omega_0(t)} \int_{\mathbf{P}_\Sigma} [\mathcal{L}] \cdot \omega(t) = 2\Psi(T) - \sum_k T_k \frac{\partial \Psi(T)}{\partial T_k}$$

where the $T_k = T_k(t)$ are functions defined by the asymptotic form

$$\omega(t) = \omega_0(t) + \sum_k T_k(t) H_k + (deg \geq 2 \ terms).$$

Here $H_1, .., H_m$ is a fixed basis of Kähler classes in $A_{n-1}(\mathbf{P}_\Sigma)$. Note that

$$T_k(t) = \langle H_k^*, t\rangle + exponential \ term.$$

Even though the A-model potential $\Psi(T)$ was not known exactly before the advent of mirror symmetry, on general ground physicists knew it must have the form

$$\Psi(T) = \frac{1}{3!} \sum K_{ijk} T_i T_j T_k + (deg\ 2\ polynomial) + O(e^T)$$

where the K_{ijk} are the triple intersection numbers of of the classes $H_1, .., H_m$ restricted to Y. The term of order $O(e^T)$ is known as worldsheet instanton correction. Later, we shall ignore the polynomial part of $\Psi(T)$ and consider only the instanton sum.

This completes our exposition of the mirror symmetry prescription for Calabi-Yau manifolds in toric varieties. In the next lecture, we will give an alternative mathematical definition for the Kähler potential Ψ (the instanton correction), and discuss the formula above from an entirely different viewpoint. We will show that this formula is a special case of a general phenomenon we call the *Mirror Principle*, which we discuss next.

4. Lecture IV. Intersection Numbers on Stable Moduli

4.1. Set-up

Let X be a projective n-fold, and $d \in H^2(X, \mathbf{Z})$. Let $M_{0,k}(d,X)$ denote the moduli space of k-pointed, genus 0, degree d, stable maps $(C, f, x_1, .., x_k)$ on X [32]. Note that our notation is without the bar. By the work of [33](cf. [7][19]), each nonempty $M_{0,k}(d,X)$ admits a Chow class $LT_{0,k}(d,X)$ of degree $dim\ X + \langle c_1(X), d\rangle + k - 3$. This cycle plays the role of the fundamental class in topology, hence $LT_{0,k}(d,X)$ is called the virtual fundamental class.

Let V be a convex vector bundle on X. (ie. $H^1(\mathbf{P}^1, f^*V) = 0$ for every holomorphic map $f : \mathbf{P}^1 \to X$.) Then V induces on each $M_{0,k}(d,X)$ a vector bundle V_d, with fiber at $(C, f, x_1, .., x_k)$ given by the section space $H^0(C, f^*V)$. Let b be any multiplicative characteristic class [26]. (ie. if $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles, then $b(E) = b(E')b(E'')$.) The problem we study here is to compute the characteristic numbers

$$K_d := \int_{LT_{0,0}(d,X)} b(V_d)$$

and their generating function:

$$\Phi(t) := \sum K_d\ e^{d \cdot t}.$$

There is a similar and equally important problem if one starts from a concave vector bundle V [36]. (ie. $H^0(\mathbf{P}^1, f^*V) = 0$ for every holomorphic map $f : \mathbf{P}^1 \to X$.) More generally, V can be a direct sum of a convex and a concave bundle. Important progress made on these problems has come from mirror symmetry. All of it seems to point toward the following general phenomenon [12], which we call *the Mirror Principle.* Roughly, it says that the function $\Phi(t)$ can be computed by a change of variables in terms of certain explicit special functions, loosely called generalized hypergeometric functions.

When X is a toric manifold with $c_1(X) \geq 0$, b is the Euler class, and V is a line bundle, there is a general formula derived from mirror symmetry in [28]. The formula computes $\Phi(t)$ in terms of generalized hypergeometric functions. Similar hypergeometric functions were later used in [21] in studying equivariant quantum cohomology based on a series of axioms.

4.2. Main Ideas

We now sketch our main ideas for computing the classes $b(V_d)$.

Step 1. Localization on the linear sigma model. Consider the moduli spaces $M_d(X) := M_{0,0}((1,d), \mathbf{P}^1 \times X)$. The projection $\mathbf{P}^1 \times X \to X$ induces a map $\pi : M_d(X) \to M_{0,0}(d,X)$. Moreover, the standard action of S^1 on $\mathbf{P}^1$ induces an S^1 action on $M_d(X)$. We first study a slightly different problem. Namely consider the classes $\pi^* b(V_d)$ on $M_d(X)$, instead of $b(V_d)$ on $M_{0,0}(d,X)$. First, there is a canonical way to embed fiber products (see below)

$$F_r = M_{0,1}(r,X) \times_X M_{0,1}(d-r,X)$$

each as an S^1 fixed point component into $M_d(X)$. Let $i_r : F_r \to M_d(X)$ be the inclusion map. Second, there is an evaluation map $e : F_r \to X$ for each r. Third, suppose that there is a projective manifold W_d with S^1 action, that there is an equivariant map $\varphi : M_d(X) \to W_d$, and embeddings $j_r : X \to W_d$, such that the diagram

$$\begin{array}{ccc} F_r & \xrightarrow{i_r} & M_d(X) \\ e\downarrow & & \downarrow\varphi \\ X & \xrightarrow{j_r} & W_d \end{array}$$

commutes. Let α denotes the weight of the standard S^1 action on $\mathbf{P}^1$. Then applying the localization formula [2], this diagram allows us to recast our problem to one of studying the S^1-equivariant classes

$$Q_d := \varphi_! \pi^* b(V_d)$$

defined on W_d. Moreover we can expand the class

$$A_d := \frac{j_0^* Q_d}{e_{S^1}(X_0/W_d)}$$

on X in powers of α^{-1}, and find that it is of order α^{-2}.

The spaces W_d in the commutative diagram above are called the linear sigma model of X. They have been introduced in [41] following [46] when X is a toric manifold,

Step 2. Gluing identity. Consider the vector bundle $\mathcal{U}_d := \pi^* V_d \to M_d(X)$, restricted to the fixed point components F_r. A point in (C,f) in F_r is a pair $(C_1, f_1, x_1) \times (C_2, f_2, x_2)$ of 1-pointed stable maps glued together at the marked points, ie. $f_1(x_1) = f_2(x_2)$. From this, we get an exact sequence of bundles on F_r:

$$0 \to i_r^* \mathcal{U}_d \to U_r' \oplus U_{d-r}' \to e^* V \to 0.$$

Here $i_r^*\mathcal{U}_d$ is the restriction to F_r of the bundle $\mathcal{U}_d \to M_d(X)$. And U'_r is the pullback of the bundle $U_r \to M_{0,1}(d,X)$ induced by V, and similarly for U'_{d-r}. Taking the multiplicative characteristic class b, we get the identity on F_r:

$$e^* b(V) b(i_r^*\mathcal{U}_d) = b(U'_r) b(U'_{d-r}).$$

This is what we call the *gluing identity.* This may be translated to a similar quadratic identity, via Step 1, for Q_d in the equivariant cohomology groups $H^*_{S^1}(W_d)$. The new identity is called the Euler data identity.

Step 3. Linking theorem. The construction above is functorial, so that if X comes equipped with a torus T action, then the entire construction becomes $G = S^1 \times T$ equivariant and not just S^1 equivariant. In particular, the Euler data identity is an identity of G-equivariant classes on W_d. Our problem is to first compute the G-equivariant classes Q_d on W_d satisfying the Euler data identity, and with the property that $A_d \sim \alpha^{-2}$. Note that the restrictions $Q_d|_p$ to the T fixed points p in $X_0 \subset W_d$ are polynomials functions on the Lie algebra of G. Suppose that X is a balloon manifold. Then it can be shown that (with a mild assumption on $e_G(X_0/W_d)$) the classes are uniquely determined by the values of the $Q_d|_p$, when α is some scalar multiple of a weight on the tangent space T_pX. These values of $Q_d|_p$ can be computed explicitly by exploiting the structure of a balloon manifold.

Once these values are known, it is often easy to manufacture explicit G-equivariant classes $\tilde{Q}_d$ with the restrictions $\tilde{Q}_d|_p$ having the above same values, and satisfying the Euler data identity. In this case, we say that the data $\tilde{Q}_d$ are linked to the data Q_d. By a suitable change of variables, one can also arrange that $\frac{j_0^*\tilde{Q}_d}{e_{S^1}(X_0/W_d)} \sim \alpha^{-2}$. By the preceding discussion, we get $Q_d = \tilde{Q}_d$.

Step 4. Computing $\Phi(t)$. Once the classes $Q_d = \varphi_! \pi^* b(V_d)$ are determined, we can unwind the many maps use in Step 1. The preceding computations can be done simply in the form of power series. This finally computes the generating function $\Phi(t)$.

For a class of balloon manifolds (defined later), we will prove that the 4-step procedure above yield a formula for $\Phi(t)$ as stated in the Conjecture at the end of this lecture. For a general projective manifold without T action, we expect that this formula continues to hold.

In this lecture, for clarity, we restrict ourselves to the case when the tangent bundle of X is convex. We prove that the Conjecture holds whenever X is a balloon manifold having a linear sigma model W_d such that $e_G(X_0/W_d)$ satisfies a nondegeneracy condition.

In the nonconvex case, we must replace $M_{0,k}(d,X)$ by Li-Tian's virtual fundamental cycle [33] for the purpose of localization and integration. A soon-to-appear paper, Mirror Principle III, joint with K. Liu and S.T. Yau will be devoted entirely to dealing with the added technicality arising from this replacement. All the results in this paper can be generalized with only slight modifications resulted from this replacement. By the equivalence, established in [34], of symplectic GW theory and algebraic GW theory for projective manifolds, we also expect that the results in this paper can be readily generalized to the symplectic case [43][44].

4.3. Equivariant localization

We first discuss some basic facts about localization. The key technique of our proof is the equivariant localization formula, due to Atiyah-Bott [1][11][2], and Berline-Vergne [10]. For an orbifold version of the localization formula, see [31]. The spirit of the localization we'll use is closer to the Bott residue formula. We first explain this formula.

Let X and Y be two spaces, by which we mean compact manifolds or orbifolds, with a torus T-action. When an orbifold is involved, the integral and localization formulas should be taken in the orbifold sense. Let $\{F\}$ be the components of the fixed point set. Let $H_T^*(\cdot)$ denote the equivariant cohomology group with complex coefficient, and $i_F : F \to X$ the inclusion map. We say that equivariant localization holds on X, if the two maps

$$i_F^* : \quad H_T^*(X) \to H_T^*(F), \quad i_{F!} : \quad H_T^*(F) \to \; H_T^*(X).$$

which are respectively the pull-back, and the Gysin map, are such that the following formulas holds: given any equivariant cohomology class ω on X, we have

$$\omega = \sum_F i_{F!}\left(\frac{i_F^*\omega}{e_T(F/X)}\right).$$

This formula is equivalent to the integral version of the localization formula

$$\int_X \omega = \sum_F \int_F \frac{i_F^*\omega}{e_T(F/X)}.$$

An important fact about equivariant theory is that, if V is an equivariant vector bundle on an orbifold X, then any characteristic class of V has an equivariant extension. Let $T = S^1$ for simplicity. If c_{2k} is a characteristic class of degree $2k$, then its equivariant extension can be represented by the form

$$c = c_{2k} + c_{2k-2} + \cdots + c_0$$

in the equivariant cohomology of X.

4.4. Functorial localization formula

In this subsection, we derive a formula which is often used here. Let X, Y be two T-spaces, and

$$f: \quad X \to Y$$

be an equivariant map. Let E be a fixed point component in Y, and $F := f^{-1}(E)$ be a fixed component in X. Let g be the restriction of f to F, and $j_E : E \to Y$, $i_F : F \to X$ be the inclusion maps. Thus we have the commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{i_F} & X \\ g \downarrow & & \downarrow f \\ E & \xrightarrow{j_E} & Y. \end{array}$$

Lemma 4.1. *Given any class $\omega \in H_T^*(X)$, we have the equality on E:*

$$\frac{j_E^* f_!(\omega)}{e_T(E/Y)} = g_! \left(\frac{i_F^* \omega}{e_T(F/X)} \right)$$

Proof: Let us consider localization of $\omega f^* j_{E!}(1)$ on X,

$$\omega f^* j_{E!}(1) = i_{F!} \left(\frac{i_F^*(\omega f^* j_{E!}(1))}{e_T(F/X)} \right).$$

Note the contributions from fixed components other than F vanish. Applying the push-forward $f_!$ to both sides, we get

$$f_!(\omega) j_{E!}(1) = f_! i_{F!} \left(\frac{i_F^*(\omega f^* j_{E!}(1))}{e_T(F/X)} \right).$$

Now $f\circ i_F = j_E\circ g$ which, implies

$$f_! i_{F!} = j_{E!} g_!,\quad i_F^* f^* = g^* j_E^*.$$

Thus we get

$$f_!(\omega) j_{E!}(1) = j_{E!} g_!\left(\frac{i_F^*(\omega)\ g^* e_T(E/Y)}{e_T(F/X)}\right).$$

Applying j_E^* to both sides, we then arrive at

$$j_E^* f_!(\omega)\ e_T(E/Y) = e_T(E/Y)\ g_!\left(\frac{i_F^*(\omega)\ g^* e_T(E/Y)}{e_T(F/X)}\right) = e_T(E/Y)^2\ g_!\left(\frac{i_F^*(\omega)}{e_T(F/X)}\right).$$

Since $e_T(E/Y)$ is invertible, our assertion follows. □

4.5. Balloon manifolds

By a *balloon manifold*, we mean a complex projective T-manifold X with the following properties. There are only finite number of T-fixed points. At each fixed point p, the T-weights on the isotropic representation T_pX are pairwise linearly independent. This class of manifolds were introduced by Goresky-Kottwitz-MacPherson [22]. (We refer the reader to [24] for an excellent exposition.) Throughout this paper, we assume that TX is convex.

One important property of a balloon n-fold is that at each fixed point p, there are exactly n balloons, ie. T-invariant $\mathbf{P}^1$, each balloon connecting p to one other fixed point q. The induced action on each balloon is the standard rotation with two fixed points p and q. (see [22][25]). We *denote by pq the balloon connecting the fixed points p,q.* The graph with vertices X^T and edges given by the balloons is called the GKM graph. Toric manifolds, compact homogeneous spaces are examples of balloon manifolds.

We shall fix an integral basis $H_1,..,H_m$ of $H^2(X,\mathbf{Z})$ consisting of Kähler classes on X. We denote by the same notations their T-equivariant analogues. For $\omega\in H^2(X)$ and $d\in H_2(X)$, we denote their pairing by $\langle\omega,d\rangle$.

For convenience, we introduce the following notations:

$$H = (H_1,..,H_m)$$
$$H\cdot\zeta = H_\zeta = H_1\zeta_1+\cdots+H_m\zeta_m$$
$$H(p) = H|_p = (H_1(p),..,H_m(p))$$
$$H_\zeta(p) = H_1(p)\zeta_1+\cdots+H_m(p)\zeta_m.$$

Here $\zeta = (\zeta_1, .., \zeta_m)$ are formal variables. We denote by $K^\vee \subset H_2(X)$ the set of points in $H_2(X, \mathbf{Z})_{free}$ in the dual of the closure of the Kähler cone of X. Since $K^\vee$ is a semigroup in $H_2(X)$, it defines a partial ordering $\succ$ on the lattice $H_2(X, \mathbf{Z})_{free}$. That is, $d \succeq r$ iff $d - r \in K^\vee$. Let $\{H_j^\vee\}$ be the basis dual to the $\{H_j\}$ in $H_2(X)$. If $d \succeq r$ for two classes d, $r \in H_2(X)$, then $d - r = d_1 H_1^\vee + \cdots + d_m H_m^\vee$ for nonnegative integers $d_1, \cdots, d_m$.

We also consider a balloon manifold as a symplectic manifold with a symplectic structure given by $\omega = H_\zeta$ for some generic ζ. By the convexity theorem of Atiyah [1] and Guillemin-Sternberg [23], the image of the moment map μ_ζ in the dual Lie algebra $\mathcal{T}^*$ is a convex polytope, known as the moment polytope. When X is a toric manifold, the moment polytope is known as a Delzant polytope [15]. In this case, it is well-known that the normal fan of this polytope is the defining fan of X.

We shall assume throughout this paper that $c(p) = c(q)$ for all $c \in H_T^2(X)$, then $p = q$. This condition is also equivalent to the statement that the moment map with respect to $\omega = H_\zeta$ and the T action is injective to the set of vertices of the moment polytope, when restricted to the fixed points X^T. Note that this implies that the moment map embeds the GKM graph into $\mathcal{T}^*$ as the 1-skeleton of the moment polytope. Toric manifolds and compact homogeneous manifolds are examples satisfying this condition.

When X is a toric n-fold, we have $N = m + n$ T-invariant divisors in X. Let $D_a = c_1(L_a)$, $a = 1, .., N$, be the equivariant first Chern classes of the corresponding equivariant line bundles. These T divisors correspond 1-1 with the one-cones of the defining fan of X [42]. Moreover the fixed points correspond 1-1 with the n-cones. Labelling the n-cones by $p \in X^T$, we have a balloon [42] pq in X iff the n-cones p, q intersect in a codimension one subcone. Since X is smooth, hence the n-cones are regular, there are exactly n balloons pq for each fixed q. One can give a dual description of all these by using the Delzant polytope.

4.6. Sigma models

Let X be balloon manifold with a fixed T-equivariant embedding $X \to \mathbf{P}(n)$, as discussed above. We write

$$M_d(X) := M_{0,0}((1, d), \mathbf{P}^1 \times X).$$

Since X is assumed to be convex, $M_d(X)$ is an orbifold [8]. The standard S^1 action on $\mathbf{P}^1$ together with the T action on X induce a $G = S^1 \times T$ action on $M_d(X)$.

Here is a description of some S^1 fixed point components F_r, labelled by $0 \preceq r \preceq d$, inside of $M_d(X)$. Let F_r be the fiber product

$$F_r := M_{0,1}(r, X) \times_X M_{0,1}(d-r, X)$$

More precisely, consider the map

$$ev_r \times ev_{d-r} : M_{0,1}(r, X) \times M_{0,1}(d-r, X) \to X \times X$$

given by evaluations at the marked points; and

$$\Delta : \ X \to X \times X$$

the diagonal map. Then

$$F_r = (ev_r \times ev_{d-r})^{-1} \Delta(X).$$

Note that $F_d = M_{0,1}(d, X)$ by convention. The set F_r can be identified with an S^1 fixed point component of $M_d(X)$ as follows. Consider the case $r \neq 0, d$ first. Given a point $(C_1, f_1, x_1) \times (C_2, f_2, x_2)$ in F_r, we get a new curve C by gluing C_1, C_2 to $\mathbf{P}^1$ with x_1, x_2 glued to $0, \infty \in \mathbf{P}^1$ respectively. The new curve C is mapped into $\mathbf{P}^1 \times X$ as follows. Map $\mathbf{P}^1 \subset C$ identically onto $\mathbf{P}^1$, and collapse C_1, C_2 to $0, \infty$ respectively; then map C_1, C_2 into X with f_1, f_2 respectively, and collapse the $\mathbf{P}^1$ to $f(x_1) = f(x_2)$. This defines a point (C, f) in $M_d(X)$. For $r = 0$, we glue (C_1, f_1, x_1) to $\mathbf{P}^1$ at x_1 and 0. For $r = d$, we glue (C_2, f_2, x_2) to $\mathbf{P}^1$ at x_2 and ∞. We will identify F_r as a subset of $M_d(X)$ as above, and let

$$i_r : F_r \to M_d(X)$$

denotes the inclusion map. Clearly, we also have an evaluation map

$$e_r : F_r \to X$$

which sends a pair in F_r to the value at the marked point. In the following, we will simply write e_r as e without causing any confusion.

We call a compact manifold or orbifold W_d with $G = S^1 \times T$ action a *linear sigma model* of degree d for X, if the following conditions are satisfied:

1. The S^1 action on W_d has fixed point components given by X_r, labelled by $0 \preceq r \preceq d$, and each X_r is T-equivariantly isomorphic to X.

2. There is a G-equivariant birational map φ from $M_d(X)$ to W_d, such that $\varphi|_{F_r} = e$, and $\varphi^{-1}(X_r) = F_r$.
3. All equivariant cohomology classes in $H_G^2(W_d)$ are lifted from $H_T^2(X)$, and the lift $\hat{D} \in H_G^2(W_d)$ of $D \in H_T^2(X)$ restricts to $D + \langle D, r\rangle\alpha$ on X_r.
4. The G-equivariant Euler class of the normal bundle of X_0 in W_d has the form

$$e_G(X_0/W_d) = \prod_a \prod_{m_a} (D_a - m_a\alpha)$$

where the m_a's are positive integers and the D_a's are classes in $H_T^2(X)$, such that at a given T fixed point p in X, the nonzero $D_a(p)$'s are multiples of distinct weights of T_pX.

Note W_d need not be unique. We identify X_r with X by assumption 1, and denote by

$$j_r : X_r \to W_d$$

the inclusion map.

We call a balloon manifold X a *admissible* if it has a linear sigma model W_d for each d, and and that $H_\varsigma(p) \neq H_\varsigma(q)$ for any two distinct fixed points p, q in X. The main result in this paper is to show that the mirror principle holds for any admissible balloon manifold.

Remark 4.2. *Condition 4 is actually assuming more than what we need. This condition can be replaced by the following weaker, but more technical condition. For each fixed point p and for any d, as a function of α, $e_G(X_0/W_d)|_p$ has possible zero only at either 0 or a multiple of a weight λ on T_pX. In addition if $[pq]$ is a balloon and $d = \delta[pq]$, then λ/δ is at worse a simple zero. For example, the following form*

$$e_G(X_0/W_d) = \frac{\prod_a \prod_{m_a} (D_a - m_a\alpha)}{\prod_b \prod_{n_b} (D_b - n_b\alpha)}$$

with the m_a, n_b nonzero integers, is more general than condition 4 and is more than adequate for our purpose.

Example 1: Projective space $\mathbf{P}^n$ with $W_d = \mathbf{P}^{(n+1)d+n}$ is admissible. (For a construction of the map φ in this case, see [36].) The lifted hyperplane class κ has the required property that

$$j_r^*\kappa = H + \langle H, r\rangle\alpha = H + r\alpha.$$

The equivariant Euler class

$$e_G(\mathbf{P}_0^n/W_d) = \prod_{i=0}^{n}\prod_{m=1}^{d}(H - \lambda_i - m\alpha)$$

where λ_i's denote the weights of the torus T action on $\mathbf{P}^n$. Clearly the equivariant classes $\{H - \lambda_i\}$ has the required property.

Example 2: More generally for $\mathbf{P}(n)$, we can take $N_d(\mathbf{P}(n))$ to be W_d. In fact the S^1 fixed point components on $N_{k,l}$ are exactly $k+1$ copies $\mathbf{P}_r^l$, $r = 0,..,k$, of $\mathbf{P}^l$. Each $\mathbf{P}_r^l$ consists of $l+1$ tuples of monomials, each being a scalar multiple of $w_0^r w_1^{k-r}$. Similarly the S^1 fixed point components on $N_d(\mathbf{P}(n))$ are copies $\mathbf{P}(n)_r$, $0 \preceq r \preceq d$, of $\mathbf{P}(n)$. All equivariant cohomology classes in $H_G^2(N_d(\mathbf{P}(n))$ are lifted from $H_T^2(\mathbf{P}^n)$ (cf [36]). Let κ_i be the lift of the hyperplane class H_i, of the ith factor $\mathbf{P}^{n_i}$. Then

$$j_r^*\kappa_i = H_i + \langle H_i, r\rangle\alpha$$

where j_r denotes the inclusion of $\mathbf{P}(n)_r$ in $N_d(\mathbf{P}(n))$. By using the formula in [36], it is easy to show the equivariant Euler class $e_G(\mathbf{P}(n)_0/N_d(\mathbf{P}(n)))$, which is a product of e_G's in last example, has the required property.

Example 3: Let $N_{k,l}$ be the space of $l+1$ tuples $[f_0,..,f_l]$ of degree k polynomials $f_i(w_0,w_1)$, modulo scalar. Thus $N_{k,l} \cong \mathbf{P}^{(l+1)k+l}$. It is called the linear sigma model for $\mathbf{P}^l$. (See [36].) Let X be a balloon manifold with a T equivariant embedding

$$X \to \mathbf{P}(n) := \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}.$$

Let

$$N_d(\mathbf{P}(n)) := N_{d_1,n_1} \times \cdots \times N_{d_m,n_m}.$$

Recall that we have a collapsing map $\varphi : M_k(\mathbf{P}^l) \to N_{k,l}$, which is $G := S^1 \times T$ equivariant. By taking composite with the projection from $M_d(\mathbf{P}(n))$ to each $M_{d_j}(\mathbf{P}^{n_j})$, we obtain a G-equivariant map

$$M_d(\mathbf{P}(n)) \to N_d(\mathbf{P}(n))$$

which we also denote by φ. Note that $M_d(X)$ can be viewed as a cycle in $M_d(\mathbf{P}(n))$. We denote the image cycle $\varphi_!(M_d(X))$ in $N_d(\mathbf{P}(n))$ φ by $N_d(X)$.

If $N_d(X)$ is a manifold or an orbifold, then Properties 1-3 are automatically satisfied, if furthermore $e_G(X_0/W_d)$ has property 4, then we can simply take $W_d = N_d(X)$ as the linear sigma model.

Example 4: Convex toric varieties. In this case W_d is a toric n-fold, as introduced by Witten [46]and used first by Morrison-Plesser [41]to study quantum cohomology. Recall that a toric manifold X can be realized as the GIT quotient $\mathbf{C}^N//T^m_{\mathbf{C}}$ where $T^m_{\mathbf{C}}$ is a m-dimensional complex torus acting on $\mathbf{C}^N$. Here $m = rank\ H^2(X, \mathbf{Z})$, $N = n + m$. Let $[z_1, \cdots, z_N]$ denote the coordinates on $\mathbf{C}^N$. Then each z_j can be viewed as a section of a line bundle L_j on X [14][41]. Modulo the induced action by T^m_C from $\mathbf{C}^N$, a map from $\mathbf{P}^1$ into X is uniquely represented by an N-tuple of polynomials

$$[f_1(w_0, w_1), \cdots, f_N(w_0, w_1)]$$

where f_j is a section of the line bundle $O(l_j)$ over $\mathbf{P}^1$ with $l_j = \langle c_1(L_j), d\rangle$. Let $\mathbf{C}^N(d)$ be the vector space of N-tuple of polynomials of degree $(d_1, \cdots, d_N)$ as above. Then as described in [41], W_d is the GIT quotient by the induced action of T^m_C on it:

$$W_d = \mathbf{C}^N(d)//T^m_C.$$

Let $M^o_d(X)$ denote the set of points (f, C) in $M_d(X)$ such that $C \simeq \mathbf{P}^1$. We call $M^o_d(X)$ the smooth part of $M_d(X)$. We can define a map φ_o from $M^o_d(X)$ to W_d in the following way: each (f, C) gives a map from $\mathbf{P}^1$ to X, and modulo the induced T^m_C action, uniquely determines N-tuple of polynomials as above, therefore gives a point in W_d, which we define to be the image of (f, C) under φ_o. This is clearly a canonical identification.

It is not difficult to see that the S^1-fixed components in W_d can be described as GIT quotient,

$$X_r \simeq \{[a_1 w_0^{\langle D_0, r\rangle} w_1^{\langle D_0, d-r\rangle}, \cdots, a_N w_0^{\langle D_N, r\rangle} w_1^{\langle D_N, d-r\rangle}] | a \in \mathbf{C}^N\}//T^m_C.$$

The equivariant Euler class of its normal bundle in W_d is

$$e_G(X_r/W_d) = \prod_{a=1}^{N} \prod_{k=0, k\neq\langle D_a, r\rangle}^{\langle D_a, d\rangle} (D_a + \langle D_a, r\rangle\alpha - k\alpha).$$

Here $D_a = c_1(L_a)$ is the equivariant first Chern class of the line bundle L_a corresponding to the ath component in the coordinates of X. The lift $\hat{D}_a$ of D_a to W_d clearly has the property

$$j_r^* \hat{D}_a = D_a + \langle D_a, r\rangle \alpha.$$

As pointed out in [41], the cohomology of W_d are generated by the $\hat{D}_a$. Thus W_d has properties 1, 3, and 4. It can be shown that the map φ_o extends to a regular G-equivariant map φ from $M_d(X)$ to W_d, with property 2 (see [38]). So, for a toric X we can take its linear sigma model to be W_d as constructed above. It follows that X is an admissible balloon manifold.

4.7. The Gluing Identity

Returning to the general case, we let X be an admissible balloon manifold from now on. In this section, we apply the functorial localization formula to the linear sigma model. The argument used here is modelled on the one used in [36], except that the T-action is *not* used here. Thus all the results in this section hold for manifolds without T action.

Recall that we have the commutative diagram:

$$\begin{array}{ccc} F_r & \stackrel{i_r}{\longrightarrow} & M_d(X) \\ e \downarrow & & \downarrow \varphi \\ X_r & \stackrel{j_r}{\longrightarrow} & W_d. \end{array}$$

We also have the natural forgetting map $\rho : M_{0,1}(d, X) \to M_{0,0}(d, X)$, and the projection map $\pi : M_d(X) \to M_{0,0}(d, X)$. Note that we have a commutative diagram

$$\begin{array}{ccc} M_d(X) & & \\ \pi \downarrow & \nwarrow i_0 & \\ M_{0,0}(d, X) & \stackrel{\rho}{\longleftarrow} & M_{0,1}(d, X). \end{array}$$

Let $\varphi : M_d(X) \to W_d,\ e : F_r \to X_r$ play the respective roles of $f : X \to Y,\ g : F \to E$ in the first localization lemma. Then it follows that

Lemma 4.3. *Given any G-equivariant cohomology class ω on $M_d(X)$, we have the following equality on X_r for $0 \preceq r \preceq d$:*

$$\frac{j_r^* \varphi_!(\omega)}{e_G(X_r/W_d)} = e_! \left(\frac{i_r^*(\omega)}{e_G(F_r/M_d(X))} \right).$$

Let L_r denote the line bundle on $M_{0,1}(r, X)$ whose fiber at $(f, C; x)$ is the tangent line at the marked point $x \in C$. Let π_1 denote the projection from $\mathbf{P}^1 \times X$ to $\mathbf{P}^1$.

The normal bundle of F_r in $M_d(X)$ can be computed just as in [36]. For $r \neq 0, d$, we have

$$N(F_r/M_d(X)) = H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1) + T_{x_1}C_0 \otimes L_r + T_{x_2}C_0 \otimes L_{d-r} - A_{C_0}.$$

Here we have used the notations as in [36]: a point (f_1, C_1, x_1) in $M_{0,1}(r, X)$ and a point (f_2, C_2, x_2) in $M_{0,1}(d-r, X)$ is glued to $C_0 \simeq \mathbf{P}^1$ at 0 and ∞ respectively to get the point (f, C) in $M_d(X)$ with $C \simeq C_1 \cup C_0 \cup C_2$. Since x_1 and x_2 are mapped to the same point in X under the projection $\pi_2 : \mathbf{P}^1 \times X \to X$, so this point can be considered as a point in F_r by gluing together (f_1, C_1, x_1) and (f_2, C_2, x_2) at the marked points. Similarly, for $r = 0, d$, we have

$$N(F_0/M_d(X)) = H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1) + T_{x_1}C_0 \otimes L_d - A_{C_0}$$

and

$$N(F_d/M_d(X)) = H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1) + T_{x_2}C_0 \otimes L_d - A_{C_0}.$$

In the above $H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1)$ corresponds to the deformation of C_0; $T_{x_1}C_0 \otimes L_r$ and $T_{x_2}C_0 \otimes L_{d-r}$ correspond respectively to the deformations of the nodal points x_1 and x_2; A_{C_0} denotes the automorphism group to be quotiented out.

The equivariant Euler classes of the normal bundles above are computed as in [36], to which we refer the readers for details. For $r \neq 0, d$, the equivariant Euler classes are:

$$e_G(F_r/M_d(X)) = -\alpha(-\alpha + c_1(L_{d-r})) \cdot \alpha(\alpha + c_1(L_r))$$

where the two factors on the right hand side are pullbacked to F_r from $M_{0,1}(d-r, X)$, $M_{0,1}(r, X)$ respectively. For $r = 0, d$, we have

$$e_G(F_0/M_d(X)) = -\alpha(-\alpha + c_1(L_d)),\ e_G(F_d/M_d(X)) = \alpha(\alpha + c_1(L_d))$$

respectively. Combining this with the preceding lemma, we get the following equality on $X = X_0$:

$$\frac{j_0^* \varphi_!(\omega)}{e_G(X_0/W_d)} = ev_! \left(\frac{i_0^*(\omega)}{\alpha(\alpha - c_1(L_d))} \right).$$

Here we have dropped the subscript from ev_d. In particular, if ψ is a class on $M_{0,0}(d, X)$, then for $\omega = \pi^* \psi$, we get $i_0^*(\omega) = i_0^*(\pi^* \psi) = \rho^* \psi$. This yields

Lemma 4.4. *Given any T-equivariant cohomology class ψ on $M_{0,0}(d,X)$, we have the following equality on X:*

$$\frac{j_0^*\varphi_!(\pi^*\psi)}{e_G(X_0/W_d)} = ev_!\left(\frac{\rho^*\psi}{\alpha(\alpha - c_1(L_d))}\right).$$

Fix a T-equivariant multiplicative class b_T. Fix a T-equivariant bundle of the form $V = V^+ \oplus V^-$, where $V^\pm$ are respectively the convex/concave bundles. (cf. [36].) We call such a V a mixed bundle. We assume that

$$\Omega := \frac{b_T(V^+)}{b_T(V^-)}$$

is a well-defined invertible class on X. By convention, if $V = V^\pm$ is purely convex/concave, then $\Omega = b_T(V^\pm)^{\pm 1}$. Recall that the bundle $V \to X$ induces the bundles

$$V_d \to M_{0,0}(d,X), \quad U_d \to M_{0,1}(d,X), \quad \mathcal{U}_d \to M_d(X).$$

Moreover, they are related by $U_d = \rho^* V_d$, $\mathcal{U}_d = \pi^* V_d$, Throughout this section, we denote

$$Q: \quad Q_d := \varphi_!(\pi^* b_T(V_d)).$$

If ω is a class on W_d, we write

$$i_r^*\omega^v := \frac{j_r^*\omega}{e_G(X_r/W_d)}$$

which is a class on $X = X_r$.

Lemma 4.5. *For $0 \preceq r \preceq d$,*

$$\Omega \ i_r^* Q_d^v = \overline{i_0^* Q_r^v} \ i_0^* Q_{d-r}^v.$$

Proof: For simplicity, let's consider the case $V = V^+$. The general case is entirely analogous.

Recall that a point (f, C) in $F_r \subset M_d$ comes from gluing together a pair of stable maps $(f_1, C_1, x_1), (f_2, C_2, x_2)$ with $f_1(x_1) = f_2(x_2) = p \in X$. From this, we get an exact sequence over C:

$$0 \to f^*V \to f_1^*V \oplus f_2^*V \to V|_p \to 0.$$

Passing to cohomology, we have

$$0 \to H^0(C, f^*V) \to H^0(C_1, f_1^*V) \oplus H^0(C_2, f_2^*V) \to V|_p \to 0.$$

Hence we obtain an exact sequence of bundles on F_r:

$$0 \to i_r^*\mathcal{U}_d \to U_r' \oplus U_{d-r}' \to e^*V \to 0.$$

Here $i_r^*\mathcal{U}_d$ is the restriction to F_r of the bundle $\mathcal{U}_d \to M_d(X)$. And U_r' is the pullback of the bundle $U_r \to M_{0,1}(d, X)$, and similarly for U_{d-r}'. Taking the multiplicative characteristic class b_T, we get the identity on F_r:

$$e^*b_T(V)b_T(i_r^*\mathcal{U}_d) = b_T(U_r')b_T(U_{d-r}').$$

This is what we call the *gluing identity.*

Now put

$$\omega = \frac{b_T(U_r)}{e_G(F_r/M_r(X))} \times \frac{b_T(U_{d-r})}{e_G(F_0/M_{d-r}(X))}.$$

Consider the commutative diagram

$$\begin{array}{ccc} F_r & \xrightarrow{\Delta_0} & M_{0,1}(r, X) \times M_{0,1}(d-r, X) \\ e \downarrow & & \downarrow ev_r \times ev_{d-r} \\ X & \xrightarrow{\Delta} & X \times X \end{array} \tag{4.1}$$

where Δ is the diagonal map, and Δ_0 is the inclusion induced by Δ. For any class ω on $M_{0,1}(r, X) \times M_{0,1}(d-r, X)$, we get

$$\Delta^*(ev_r \times ev_{d-r})_!(\omega) = e_!\Delta_0^*(\omega).$$

On one hand is

$$\begin{aligned} \Delta^*(ev_r \times ev_{d-r})_!(\omega) &= (ev_r)_!\frac{b_T(U_r)}{e_G(F_r/M_r(X))} \cdot (ev_{d-r})_!\frac{b_T(U_{d-r})}{e_G(F_0/M_{d-r}(X))} \\ &= (ev_r)_!\frac{\rho^*b_T(V_r)}{e_G(F_r/M_r(X))} \cdot (ev_{d-r})_!\frac{\rho^*b_T(V_{d-r})}{e_G(F_0/M_{d-r}(X))} \\ &= \overline{i_0^*Q_r^v}\ i_0^*Q_{d-r}^v, \end{aligned}$$

the last equality being a consequence of Lemma 4.4. On the other hand, applying the gluing identity, we have

$$\begin{aligned} e_!\Delta_0^*(\omega) &= e_!\left(\frac{b_T(U'_r)}{\alpha(\alpha+c_1(L_r))}\ \frac{b_T(U'_{d-r})}{\alpha(\alpha-c_1(L_{d-r}))}\right) \\ &= e_!\frac{e^*b_T(V)i_r^*b_T(\mathcal{U}_d)}{e_G(F_r/M_d(X))} \\ &= b_T(V)\ e_!\frac{i_r^*b_T(\mathcal{U}_d)}{e_G(F_r/M_d(X))} \\ &= b_T(V)i_r^*Q_d^v, \end{aligned}$$

the last equality being a consequence of Lemma 4.3. This proves our assertion. □

If we take V to be the trivial line bundle, and b_T to be the total Chern class, then the preceding lemma reduces to

Lemma 4.6. *For $0 \preceq r \preceq d$, we have the following equality on X:*

$$e_G(X_r/W_d) = \overline{e_G(X_0/W_r)}e_G(X_0/W_{d-r}).$$

In particular, we have

$$e_G(X_d/W_d) = \overline{e_G(X_0/W_d)}.$$

4.8. Euler Data

Notations: We denote by κ_i the G-equivariant class on W_d with the property that $j_r^*\kappa_i = H_i + \langle H_i, r\rangle\alpha$. By the localization theorem, κ_i is determined by these restriction conditions, and is a class in the localized equivariant cohomology of W_d. More generally a class $\phi \in H_T^2(X)$ has a G-equivariant lift $\hat{\phi} \in H_G^2(W_d)$ determined by $j_r^*\hat{\phi} = \phi + \langle\phi, r\rangle\alpha$. We denote by $\langle H_T^2(X)\rangle$ the ring generated by $H_T^2(X)$, and by R_d the ring generated by their lifts $\hat{\phi}$. We put $\mathcal{R} = \mathbf{Q}(\mathcal{T}^*)[\alpha]$, where $\mathbf{Q}(\mathcal{T}^*)$ is the rational function field on the Lie algebra of T. For convenience, we introduce the notations

$$\kappa\cdot\zeta = \kappa_\zeta := \kappa_1\zeta_1 + \cdots + \kappa_m\zeta_m$$
$$i_r^*\omega^v := \frac{j_r^*\omega}{e_G(X_r/W_d)}$$

where ω is a class on W_d.

It is often necessary to work over a larger field than $\mathbf{C}$ for coefficients of cohomology groups. For example when we consider the case of the equivariant Chern polynomial c_T, a formal variable x is introduced. In this case we replace everywhere the scalars $\mathbf{C}$ by $\mathbf{C}(x)$. This will be implicit in all of the discussion below.

Recall the localization formula:

$$\int_{W_d} \omega = \sum_{0 \preceq r \preceq d} \int_X \frac{j_r^*(\omega)}{e_G(X_r/W_d)}.$$

We shall often apply the following version:

$$\int_{W_d} \omega \ e^{\kappa_\zeta} = \sum_{0 \preceq r \preceq d} \int_X i_r^* \omega^v \ e^{H_\zeta + \langle H_\zeta, r\rangle \alpha}.$$

Definition 4.7. *Fix an invertible class $\Omega \in H_T^*(X)^{-1}$. A list $P: \ P_d \in H_G^*(W_d)^{-1}$, $d \succ 0$, is a Ω-Euler data if on X,*

$$\Omega \ i_r^* P_d^v = \overline{i_0^* P_r^v} \ i_0^* P_{d-r}^v$$

(called Euler data identity) for all $r \preceq d$, and the $\int_{W_d} P_d \cdot \omega$ are polynomial in α for all $\omega \in R_d$. By convention we set $P_0 = \Omega$.

Example 0. In the last section we have proved, using the *gluing identity*, that the data $Q: \quad Q_d = \varphi_!(\pi^* b_T(V_d))$ associated with a mixed bundle V and a multiplicative class b_T satisfies the Euler data identity. This indicates that the gluing identity is really the geometric origin motivating our definition of Euler data. Note that since Q_d is the equivariant push-forward of a class in $H_G^*(M_d(X))$, the polynomial condition on Q is automatic.

Example 1. Let L be any equivariant line bundle with $c_1(L) \geq 0$. Let $\hat{L}$ be the G-equivariant lift of $c_1(L)$.

$$P_d = \prod_{k=0}^{\langle c_1(L), d\rangle} (\hat{L} - k\alpha)$$

is an Ω-Euler data where $\Omega = c_1(L)$.

Example 2. Let L be any equivariant line bundle with $c_1(L) < 0$. Let $\hat{L}$ be the G-equivariant lift of $c_1(L)$.

$$P_d = \prod_{k=1}^{-\langle c_1(L), d\rangle - 1} (\hat{L} + k\alpha)$$

is an Ω-Euler data where $\Omega = c_1(L)^{-1}$.

Example 3. If P, P' are Ω, Ω'-Euler data respectively, then $P \cdot P'$ is a $\Omega\Omega'$-Euler data as shown in [36].

Example 4. Let L be as in Example 1, and x be a formal variable. Then

$$P_d = \prod_{k=0}^{\langle c_1(L), d\rangle} (x + \hat{L} - k\alpha)$$

is an Ω-Euler data where $\Omega = c_T(L)$ denotes the Chern polynomial.

In each of Examples 1-4 above, the Euler data identity follows immediately from the algebraic identity $\Omega\ j_r^* P_d = \overline{j_0^* P_r}\ j_0^* P_{d-r}$, and Lemma 4.6.

Strictly speaking, in the examples above, we must require that $c_1(L)$ be an invertible class. This requirement can be easily met by twisting L by a trivial line bundle on which T acts by a suitable weight. In the end, we will only be interested in the nonequivariant limit of an Euler data. Thus the choice of twisting is of no consequence at the end. Alternatively, we can consider the Chern polynomial or the total Chern class (which is automatically invertible) instead of the first Chern class.

4.9. An algebraic property

Let $\mathcal{S}$ denotes the set of sequences $B: \quad B_d \in H_G^*(X)^{-1}, \quad d \succeq 0$. By convention, we set $B_0 = \Omega$.

Definition 4.8. *Given any $B \in \mathcal{S}$, define the formal series*

$$HG[B](t) := e^{-H \cdot t/\alpha} \left(\Omega + \sum_{d \succ 0} B_d\ e^{d \cdot t} \right).$$

Note that $e^{H \cdot t/\alpha} HG[B](t)$ takes value in the ring $H_G(X)^{-1}[[K^\vee]]$. (Notations: if R is a ring, hen $\mathcal{R}[[K^\vee]] := \{\sum_{d \in \Lambda} a_d e^{d \cdot t} | a_d \in \mathcal{R}\}$. We use the notations $e^{d \cdot t} = e^{\langle H_t, d\rangle}$ interchangeably.)

Let P be an Euler data, and let B be the list with $B_d := i_0^* P_d^v$. By the localization formula and the Euler data identity, we have

$$\begin{aligned}
\int_{W_d} P_d\ e^{\kappa_\zeta} &= \sum_{r \preceq d} \int_X i_r^* P_d^v\ e^{H_\zeta + \langle H_\zeta, r\rangle \alpha} \\
&= \sum_{r \preceq d} e^{-d \cdot \tau} \int_X \Omega^{-1} \overline{\left[e^{-H_t/\alpha} i_0^* P_r^v\ e^{r \cdot t}\right]}\ \left[e^{-H_\tau/\alpha} i_0^* P_{d-r}^v\ e^{(d-r) \cdot \tau}\right].
\end{aligned}$$

Here $t = \zeta\alpha + \tau$. Note that $\bar{\zeta} = -\zeta$, $\bar{\alpha} = -\alpha$, and all other variables are invariant under the "bar" operation. Now multiply both sides by $e^{d\cdot\tau}$ and sum over $d \in K^\vee$, we get the formula:

$$\sum_d e^{d\cdot\tau} \int_{W_d} P_d \, e^{\kappa\zeta} = \int_X \Omega^{-1} \, \overline{HG[B](\zeta\alpha+\tau)} \, HG[B](\tau). \tag{4.2}$$

By definition, the coefficient of $e^{d\cdot\tau}$ on the right hand side is a power series in ζ with coefficients which are polynomial in α, ie. the series lies in $\mathcal{R}[[e^\tau, \zeta]]$.

Conversely, given $B \in \mathcal{S}$ such that

$$\int_X \Omega^{-1} \, \overline{HG[B](\zeta\alpha+\tau)} \, HG[B](\tau) \in \mathcal{R}[[e^\tau, \zeta]],$$

there exists a unique Euler data $P : \; P_d$ satisfying (4.2). Namely, P_d is defined by the conditions

$$j_r^* P_d = \Omega^{-1} e_G(X_r/W_d) \, \overline{B_r} \, B_{d-r}.$$

Thus an Euler data P gives rise to a list $B \in \mathcal{S}$ in a canonical way. Abusing the terminology, *we shall call such a B an Euler data.*

4.10. Linking theorem

Definition 4.9. *Two Euler data A, B are linked if for every balloon pq in X and every $d = \delta[pq] \succ 0$,*

$$(A_d - B_d)|_q$$

is regular at $\alpha = \frac{\lambda}{\delta}$ where λ is the weight on the tangent line $T_q(pq)$.

Theorem 4.10. *[38]Suppose A, B are linked Euler data satisfying the following properties: for $d \succ 0$,*

(i) If $q \in X^T$, the only possible poles of $(A_d - B_d)|_q$ are scalar multiples of a weight on T_qX.

(ii) $deg_\alpha(A_d - B_d) \le -2$.

Then $A = B$.

Remark 4.11. *In our applications later, the situation is better than the conditions (i)-(ii) demand. We will have two Euler data A, B such that A_d, B_d separately, rather than* *just $A_d - B_d$, will satisfy both conditions (i)-(ii) at the outset. In this situation, to prove that $A = B$, it suffices to prove that they are linked.*

Throughout this section, we fix an invertible class Ω on X, and will denote by $\mathcal{A}$ the set of Ω-Euler data.

Definition 4.12. *A map $\mu : \mathcal{A} \to \mathcal{A}$ is called a mirror transformation if it preserves linking. In other words, $\mu(A)$ and A are linked for any $A \in \mathcal{A}$. We call $\mu(A)$ a mirror transform of A.*

We now consider a construction of mirror transformations, as motivated by the classic example of [12]. Consider a transformation $\mu : \mathcal{S} \to \mathcal{S}$, $B \to \tilde{B}$, of the type

$$\tilde{B}_d = B_d + \sum_{r \prec d} a_{d,r} B_r \tag{4.3}$$

where the $a_{d,r} \in H_G^*(X)^{-1}$ are a given set of coefficients. This transformation is obviously invertible, and preserves $B_0 = \Omega$.

Given a power series $f \in \mathcal{R}[[K^\vee]]$ with no constant term, we have an invertible transformation $\mu_f : \mathcal{S} \to \mathcal{S}$, $B \mapsto \tilde{B}$, such that

$$e^{f/\alpha}\ HG[B](t) = HG[\tilde{B}](t).$$

In fact, we have

$$\tilde{B}_d = B_d + \sum_{r \prec d} f_{d-r} B_r$$

where $e^{f/\alpha} = \sum_{s \succeq 0} f_s e^{s \cdot t}$, $f_s \in \mathcal{R}[\alpha^{-1}]$. This is clearly a transformation of type (4.3).

Given power series $g = (g_1, .., g_m)$, $g_j \in \mathcal{R}[[K^\vee]]$ with no constant term, we have an invertible transformation $\nu_g : \mathcal{S} \to \mathcal{S}$, $B \mapsto \tilde{B}$, such that

$$HG[B](t + g) = HG[\tilde{B}](t).$$

In fact since

$$HG[B](t + g) = e^{-H \cdot t/\alpha} e^{-H \cdot g/\alpha} \sum_{d \succeq 0} B_d\ e^{d \cdot t} e^{d \cdot g},$$

if we write $e^{d \cdot g} = \sum_{s \succeq 0} g_{d,s} e^{s \cdot t}$, $g_{d,s} \in \mathcal{R}$ and $e^{-H \cdot g/\alpha} = \sum_{s \succeq 0} \hat{g}_s e^{s \cdot t}$, $\hat{g}_s \in \mathcal{R}[H/\alpha]$, then

$$\tilde{B}_d = B_d + \sum_{r \prec d} a_{d,r} B_r$$

where the $a_{d,r} \in H_G^*(X)^{-1}$ are quadratic expressions in the $g, \hat{g}$. Thus we obtain another transformation $\mathcal{S} \to \mathcal{S}$ of type (4.3).

Theorem 4.13. *[38]The transformations $\mu_f, \nu_g : B \mapsto \tilde{B}$ above each defines a mirror transformation. That is, if B is a Euler data then $\mu_f(B)$ and $\nu_g(B)$ are both Euler data linked to B.*

Remark 4.14. *All mirror transformations we will use later will be of the type μ_f, ν_g as above. Moreover, all Euler data we will encounter will have property (i) of Theorem 4.10. The transformations μ_f, ν_g clearly preserve this property.*

Theorem 4.15. *Suppose that A, B have property (i) of Theorem 4.10, and that A, B are linked. Suppose that A is an Euler data with $deg_\alpha A_d \leq -2$ for all $d \prec 0$, and that there exists power series $f \in \mathcal{R}[[K^\vee]]$, $g = (g_1, .., g_m)$, $g_j \in \mathcal{R}[[K^\vee]]$, all without constant term, such that*

$$e^{f/\alpha} HG[B](t) = \Omega - \Omega \frac{H \cdot (t+g)}{\alpha} + O(\alpha^{-2}) \tag{4.4}$$

when expanded in powers of α^{-1}. Then

$$HG[A](t+g) = e^{f/\alpha}\ HG[B](t).$$

Proof: By Theorem 4.13, f, g define two mirror transformations μ_f, ν_g, with

$$\begin{aligned} HG[\tilde{B}](t) &= e^{f/\alpha} HG[B](t) \\ HG[\tilde{A}](t) &= HG[A](t+g) \end{aligned} \tag{4.5}$$

where $\tilde{B} = \mu_f(B)$, $\tilde{A} = \nu_g(A)$. Now both $\tilde{B}, \tilde{A}$ have property (i) of Theorem 4.10. (See remark after Theorem 4.13.)

Since $deg_\alpha A_d \leq -2$, $HG[\tilde{A}](t)$ has the same asymtotic form as $HG[\tilde{B}](t)$ in eqn. (4.4) *mod* $O(\alpha^{-2})$. It follows that

$$e^{H \cdot t/\alpha}\ HG[\tilde{A} - \tilde{B}](t) \equiv O(\alpha^{-2}),$$

or equivalently $deg_\alpha(\tilde{A}_d - \tilde{B}_d) \leq -2$. Thus $\tilde{A}, \tilde{B}$ satisfy condition (ii) of Theorem 4.10. Since A is linked to B, it follows that $\tilde{A}$ is linked to $\tilde{B}$. By Theorem 4.10, we conclude that $\tilde{A} = \tilde{B}$. Now our assertion follows from eqns. (4.5). □

Remark 4.16. *The preceding theorem says that one way to compute A (or Q) is by first finding an explicit Euler data B linked to A, and then relate A and B via mirror transformations.*

4.11. From stable map moduli to Euler data

Fix an admissible balloon manifold with $c_1(X) \geq 0$. Fix a T-equivariant multiplicative class b_T. Its nonequivariant limit is denoted by b. Fix a T-equivariant bundle of the form $V = V^+ \oplus V^-$, where $V^\pm$ are respectively the convex/concave bundles. As before, we write

$$\Omega = \frac{b_T(V^+)}{b_T(V^-)}.$$

Let V_d be the bundle induced by V on the 0-pointed degree d stable map moduli of X. Throughout this section, we denote

$$\begin{aligned}
Q: \quad Q_d &:= \varphi_!(\pi^* b_T(V_d)) \\
K_d &:= \int_{M_{0,0}(d,X)} b(V_d) \\
\Phi &:= \sum K_d \ e^{d \cdot t} \\
A: \quad A_d &:= i_0^* Q_d^v.
\end{aligned}$$

Note that all these objects depend on the choice of b_T and V, though the notations do not reflect this.

Theorem 4.17. *[38](i) $deg_\alpha A_d \leq -2$.*

(ii) If for each d the class $b_T(V_d)$ has homogeneous degree the same as the degree of $M_{0,0}(d, X)$, then in the nonequivariant limit we have

$$\int_X e^{-H \cdot t/\alpha} A_d = \alpha^{-3}(2 - d \cdot t) K_d$$

$$\int_X \left(HG[A](t) - e^{-H \cdot t/\alpha} \Omega \right) = \alpha^{-3}(2\Phi - \sum t_i \frac{\partial \Phi}{\partial t_i}).$$

Theorem 4.18. *[38]More generally suppose b_T is an equivariant multiplicative class of the form*

$$b_T(V) = x^r + x^{r-1} b_1(V) + \cdots + b_r(V), \quad rk\ V = r$$

where x is a formal variable, b_i is a characteristic class of degree i. Suppose $s := rk\ V_d - dim\ M_{0,0}(d,X) \geq 0$ is independent of $d \succ 0$. Then

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X e^{-H\cdot t/\alpha} A_d = \alpha^{-3} x^{-s}(2 - d\cdot t)K_d$$

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(HG[A](t) - e^{-H\cdot t/\alpha}\Omega\right) = \alpha^{-3}x^{-s}(2\Phi - \sum t_i \frac{\partial \Phi}{\partial t_i}).$$

4.12. Linking theorem for A

Now consider a mixed bundle $V = V^+ \oplus V^-$ on X. Fixed a choice of equivariant multiplicative class b_T. We assume that V has the following property: there exists nontrivial T-equivariant line bundles $L_1^+, .., L_{N^+}^+; L_1^-, .., L_{N^-}^-$ on X with $c_1(L_i^+) \geq 0$ and $c_1(L_j^-) < 0$, such that for any balloon $pq \cong \mathbf{P}^1$ in X we have

$$V^{\pm}|_{pq} = \oplus_{i=1}^{N^{\pm}} L_i^{\pm}|_{pq}.$$

Note that $N^{\pm} = rk\ V^{\pm}$. We also require that

$$b_T(V^+)/b_T(V^-) = \prod_i b_T(L_i^+)/\prod_j b_T(L_j^-).$$

In this case we call the list $(L_1^+, .., L_{N^+}^+; L_1^-, .., L_{N^-}^-)$ the splitting type of V. Note that V is not assumed to split over X. Given such a bundle V and a choice of multiplicative class b_T, we obtain an Euler data $Q:\ Q_d = \varphi_!(\pi^* b_T(V_d))$ (or A) as before.

Theorem 4.19. *Let $b_T = e_T$ be the equivariant Euler class. Let pq be a balloon, $d = \delta[pq] \succ 0$, and λ be the weight on the tangent line $T_q(pq)$. Then at $\alpha = \lambda/\delta$, we have*

$$j_0^*(Q_d)|_q = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} \left(c_1(L_i^+)|_q - k\lambda/\delta\right) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle - 1} \left(c_1(L_j^-)|_q + k\lambda/\delta\right).$$

In particular Q is linked to

$$P:\quad P_d = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (\hat{L}_i^+ - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle - 1} (\hat{L}_j^- + k\alpha).$$

Proof: We first consider one positive line bundle L. As in [36], we consider a point $(f, C) \in M_d(X)$ where f is δ-cover from $C = \mathbf{P}^1$ to the balloon $pq \simeq \mathbf{P}^1$. For $\alpha = \lambda/\delta$, this map can be written as

$$f: \; C \to \mathbf{P}^1 \times pq \subset \mathbf{P}^1 \times X$$

where the second map is the inclusion. In terms of coordinates we can write the first map as

$$f: \; [w_0, w_1] \to [w_1, w_0] \times [w_0^\delta, w_1^\delta].$$

Note that the T-action induces standard rotation on $pq \simeq \mathbf{P}^1$ with the weights λ_1, λ_2 and $\lambda = \lambda_1 - \lambda_2$. It is now easy to see that this point (f, C) is fixed by the subgroup of G with $\alpha = \lambda/\delta$. On the other hand as argued in [36], $(\pi_2 \circ f, C)$ is then a smooth fixed point in $M_{0,0}(d, X)$ under the T-action. The restriction $j_0^* Q_d|_p$ with $\alpha = \lambda/\delta$ is equal to the value of $e_T(\mathcal{U}_d)$ at (f, C). This, in turn, is equal to the restriction of $e_T(V_d)$ at $(\pi_2 \circ f, C)$ in $M_{0,0}(d, X)$.

Assume the restriction of L to $pq \simeq \mathbf{P}^1$ is $\mathcal{O}(l)$ with $l = \langle c_1(L), [pq] \rangle$. We compute that the equivariant Euler class restricted to this point $(\pi_2 \circ f, C)$. As in [36], we get

$$e_T(U_d) = \prod_{m=0}^{l\delta} (l\lambda_1 - m\frac{\lambda}{\delta}).$$

Also note that $c_1(L)(p) = l\lambda_1$ and $d = \delta[pq]$, this implies that $Q_d = \varphi_!(\pi^* e_T(V_d))$ is linked to

$$P_d = \prod_{m=0}^{\langle c_1(L), d \rangle} (c_1(\hat{L}) - m\alpha).$$

Similarly for a concave line bundle L, if its restriction to the balloon pq is $\mathcal{O}(-l)$ with $-l = \langle c_1(L), [pq] \rangle$, then

$$e_T(U_d) = \prod_{m=-}^{l\delta-1} (-l\lambda_1 + m\frac{\lambda}{\delta})$$

which implies the formula that in this case Q_d is linked to

$$P_d = \prod_{m=1}^{-\langle c_1(L),d\rangle -1} (c_1(\hat{L}) + m\alpha).$$

The general case is just a product of these cases. □

Similarly we can prove the following formula for the Chern polynomial.

Theorem 4.20. *Let $b_T = c_T$ be the equivariant Chern polynomial. Let pq be a balloon, $d = \delta[pq] \succ 0$, and λ be the weight on the tangent line $T_q(pq)$. Then at $\alpha = \lambda/\delta$, we have*

$$j_0^*(Q_d)|_q = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} \left(x + c_1(L_i^+)|_q - k\lambda/\delta\right) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle -1} \left(x + c_1(L_j^-)|_q + k\lambda/\delta\right).$$

In particular Q is linked to

$$P: \quad P_d = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + \hat{L}_i^+ - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle -1} (x + \hat{L}_j^- + k\alpha).$$

4.13. Applications

For simplicity, we apply our results to the case of direct sum of line bundles on an n dimensional admissible balloon manifold X. For more general case, see [36],[38]. Let

$$V = V^+ \oplus V^-, \quad V^+ := \oplus L_i^+, \quad V^- := \oplus L_j^-$$

satisfying $c_1(V^+) - c_1(V^-) = c_1(X)$, where the $L_i^\pm$ are respectively convex/concave line bundles on X. Let

$$\Omega = B_0 := c(V^+)/c(V^-) = \prod_i (x + c_1(L_i^+)) / \prod_j (x + c_1(L_j^-))$$

$$B_d := \frac{1}{e_G(X_0/W_d)} \times \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + c_1(L_i^+) - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle -1} (x + c_1(L_j^-) + k\alpha)$$

$$Q_d := \varphi_! \pi^* c_T(V_d)$$

$$A_d := i_0^* Q_d^v$$

$$HG[B](t) := \sum B_d e^{d\cdot t}$$

$$\Phi(t) := \sum K_d e^{d\cdot t}.$$

Suppose that $deg_\alpha e_G(X_0/W_d) \geq \langle c_1(X), d\rangle$. Then we see that $HG[B](t)$ has the asymptotic form

$$HG[B](t) = \Omega\left(F_0 - \alpha^{-1}H\cdot(F_0 t + F) + \alpha^{-1}G) + O(\alpha)^{-2}\right)$$

where F_0, F, G are some functions independent of α. Set $f := \alpha log F_0 - G/F_0$ and $g := F/F_0$. Then we get

$$e^{f/\alpha}HG[B](t) \equiv HG[A](t+g) \;\; mod\; O(\alpha^{-2}).$$

It follows from Theorems 4.15 and 4.18 that

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0}\int_X \left(e^{f/\alpha}HG[B](t) - e^{-H\cdot\tilde{t}/\alpha}\Omega\right) = \alpha^{-3}x^{-s}(2\Phi(\tilde{t}) - \sum_i \tilde{t}_i\frac{\partial\Phi(\tilde{t})}{\partial\tilde{t}_i}).$$

where $s := rk\; V^+ - rk\; V^- - (n-3)$, $\tilde{t} := t + g$.

Precursors to the above general formula have been many examples [12][39][27][6][13][9]. Here are some explicit examples.

4.14. A complete intersection in $\mathbf{P}^1\times\mathbf{P}^2\times\mathbf{P}^2$

The complete intersection of degrees $(1,3,0)$, $(1,0,3)$ in this 5-dimensional toric manifold X has been studied in [27] using mirror symmetry, and in [30] computing some of the intersection numbers K_d for the Euler class $b = e$ in terms of modular forms.

From our point of view, that complete intersection correspond to the following choice of convex bundle:

$$V = \mathcal{O}_1(1)\otimes\mathcal{O}_2(3)\oplus\mathcal{O}_1(1)\otimes\mathcal{O}_3(3)$$

where $\mathcal{O}_i(l)$ denotes the pullback of $\mathcal{O}(l)$ from the ith factor. The Kähler cone of X is abviously generated by the hyperplanes H_1, H_2, H_3 from the three factors of X, and hence $K^\vee$ can be identified with the set of $d = (d_1, d_2, d_3) \in \mathbf{Z}^3_{\geq 0}$. We consider intersection numbers K_d for the Euler class $b = e$ as before. Thus we set $\Omega = e(V) = (H_1 + 3H_2)(H_1 + 3H_3)$. The Euler data P we need is given by

$$P_d = \prod_{k=0}^{d_1+3d_2}(\kappa_1 + 3\kappa_2 - k\alpha)\times\prod_{k=0}^{d_1+3d_3}(\kappa_1+3\kappa_3 - k\alpha)$$
$$j_0^*(P_d) = \prod_{k=0}^{d_1+3d_2}(H_1 + 3H_2 - k\alpha)\times\prod_{k=0}^{d_1+3d_3}(H_1+3H_3 - k\alpha).$$

The linear sigma model is $W_d = N_d(\mathbf{P}(n)) = N_{d_1,1} \times N_{d_2,2} \times N_{d_3,2}$. The equivariant Euler class, after taking nonequivariant limit with respect to the T action, is given by

$$e_G(X_0/W_d) = \prod_{m=1}^{d_1} (H_1 - m\alpha)^2 \prod_{m=1}^{d_2} (H_2 - m\alpha)^3 \prod_{m=1}^{d_3} (H_3 - m\alpha)^3.$$

Now we can easily write down the hypregeometric series and all the K_d can be computed at once using the obvious intersection form on X, given by the relations:

$$\int_X H_1 H_2^2 H_3^2 = 1, \quad H_1^2 = 1, \ H_2^3 = 1, \ H_3^3 = 1.$$

Once we have the hypergeometric series, the corresponding Picard-Fuchs equation can be easily written down as given in [27].

4.15. $V = \mathcal{O}_1(-2) \otimes \mathcal{O}_2(-2)$ *on* $\mathbf{P}^1 \times \mathbf{P}^1$

Here we denote by $\mathcal{O}_i(l)$ the pullback of $\mathcal{O}(l)$ from the ith factor of $X = \mathbf{P}^1 \times \mathbf{P}^1$. Our bundle V has $rk\ V^+ - rk\ V^- = n - 3 = -1$. Thus we can apply our formula with $x = 0$. We put $\Omega = \frac{1}{H_1 H_2}$. The Euler data P that compute the K_d is now given by:

$$P_d = \prod_{k=1}^{2d_1-1} (-2\kappa_1 + k\alpha) \times \prod_{k=1}^{2d_2-1} (-2\kappa_2 + k\alpha).$$

The corresponding equivariant Euler class, after taking the nonequivariant limit with respext to the T-action is

$$e_G(X_0/W_d) = \prod_{m=1}^{d_1} (H_1 - m\alpha)^2 \prod_{m=1}^{d_2} (H_2 - m\alpha)^2.$$

4.16. A General Mirror Formula

Many of our results so far are proved for projective manifolds without T-action. Here we first discuss a formula for computing the numbers

$$K_d = \int_{M_{0,0}(d,X)} b(V_d)$$

for a general convex projective n-fold X *without T-action.* For simplicity, let's focus on the case when the multiplicative class b is the Chern polynomial c, and V is a direct sum of line bundles on X. There is a similar formulation in the general case. We fix a projective embedding $X \to \mathbf{P}(n)$, as before. Note that the map $\varphi : M_d(X) \to N_d(\mathbf{P}(n))$ is now only S^1-equivariant. Recall that the subvariety $W_d := \varphi(M_d(X)) \subset N_d(\mathbf{P}(n))$ contains as S^1 fixed point components copies of X: X_r, $0 \preceq r \preceq d$. We assume that the localization formula holds on it.

We denote by $e_{S^1}(X_0/W_d)$ the equivariant Euler class of the normal bundle of X_0 in W_d. Let

$$V = V^+ \oplus V^-, \quad V^+ := \oplus L_i^+, \quad V^- := \oplus L_j^-$$

satisfying $c_1(V^+) - c_1(V^-) = c_1(X)$, and $rk\ V^+ - rk\ V^-(n-3) \geq 0$. where the $L_i^\pm$ are respectively convex/concave line bundles on X. Let

$$\Omega = B_0 := c(V^+)/c(V^-) = \prod_i (x + c_1(L_i^+))/\prod_j (x + c_1(L_j^-))$$

$$B_d := \frac{1}{e_{S^1}(X_0/W_d)} \times \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + c_1(L_i^+) - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle - 1} (x + c_1(L_j^-) + k\alpha).$$

$$HG[B](t) := \sum B_d e^{d\cdot t}$$

$$\Phi(t) := \sum K_d e^{d\cdot t}.$$

Conjecture 4.21. *There exist unique power series $f(t), g(t)$ such that the following formula holds:*

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(e^{f/\alpha} HG[B](t) - e^{-H\cdot\tilde{t}/\alpha}\Omega\right) = \alpha^{-3}x^{-s}(2\Phi(\tilde{t}) - \sum_i \tilde{t}_i \frac{\partial\Phi(\tilde{t})}{\partial\tilde{t}_i}).$$

where $s := rk\ V^+ - rk\ V^- - (n-3)$, $\tilde{t} := t + g$. Moreover, f, g are determined by the condition that the integrand on the left hand side is of order $O(\alpha^{-2})$.

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