MULTI-AGENT OPTIMIZATION (3)

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Multi-Agent Optimization

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- 1. Variational Analysis Tools
- 2. Deterministic Problems
  - foundations & computational schemes
- 3. Stochastic Problems (Walras)
  - foundations & computational schemes

II. Deterministic Models

Outline

1. Economic Equilibirum: Walras Model
2. Stability Analysis
3. A Numerical Procedure

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Classical Arrow-Debreu Model

Pure Exchange:

- Economy $\mathcal{E}$ = exchange of goods $\in \mathbb{R}^n$
- (economic) agents: $i \in \mathcal{I}$, $|\mathcal{I}|$ finite
  consumption by agent i: $x_i \in \mathbb{R}^n$
  endowment: $e_i \in \mathbb{R}^n$, utility: $u_i : \mathbb{R}^n \to [-\infty, \infty)$,
  survival set: $X_i = \operatorname{dom} u_i = \{x_i \mid u_i(x_i) > -\infty\}$
- exchange at market prices: p
- i-budget constraint: $\langle p, x_i \rangle \le \langle p, e_i \rangle$.

Market Equilibrium

Agent-i problem:

$$\text{find } \bar{x}_i(p) \in \operatorname{argmax} \left\{ u_i(x_i) \,\middle|\, \langle p, x_i \rangle \le \langle p, e_i \rangle \right\}$$

Market clearing:

$$s(p) = \sum_{i \in \mathcal{I}} e_i - \sum_{i \in \mathcal{I}} \bar{x}_i(p) \ge 0.$$

Price simplex: $p \in \Delta$,

$$\bar{x}_i(p), s(p) \text{ unchanged when } p \to \alpha p, \alpha > 0$$

Price Equilibrium:

$$\text{find } \bar{p} \text{ so that } s(\bar{p}) \ge 0.$$

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Assumptions

- $u_i : \mathbb{R}^n \to [-\infty, \infty)$ concave; not necessarily differentiable, not strictly concave (in general)
- implies X_i convex, but not necessarily closed
- u_i is increasing, but no monotonicity is assumed
- insatiability: $\forall\, x_i, \exists\, x_i'$ such that $u_i(x_i') > u_i(x_i)$.
- free disposal: w.l.o.g. $\operatorname{int} X_i \neq \emptyset$
- strict survivability: $e_i \in \operatorname{int} X_i$ controversial, but . . .

Basic Properties

Theorem

Under (some of) these assumptions, $p \mapsto \bar{x}_i(p)$ is a osc, closed-, convex-valued mapping such that $\operatorname{dom} \bar{x}_i = \Delta$, and so is $p \mapsto s(p)$; $|\mathcal{I}|$ finite. These mappings are continuous when

Proof. Define

$$f_i^p(x_i) = \begin{cases} u_i(x_i) & \text{when } \langle p, x_i \rangle \leq \langle p, e_i \rangle, \\ -\infty & \text{otherwise} \end{cases}$$

and show that for $p' \to p$ in Δ, $f_i^{p'}$ hypo-converges to f_i^p. Hence, $\operatorname{Limsup}_\nu \bar{x}_i(p') \subset \bar{x}_i(p)$. □

Simplified model

Assumption: $\forall p \in \Delta$, there exists

$$\bar{x}_i(p) = \mathrm{argmax}\left\{u_i(x_i) \,\middle|\, \langle p, x_i\rangle \le \langle p, e_i\rangle\right\}$$

for example, strict concavity of u_i and 'truncation'. Let

$$R_i(p) := X_i \cap \left\{x \,\middle|\, \langle p, x_i - e_i\rangle \le 0\right\}.$$

the set of *feasible trades* of agent i; X_i closed.

Theorem

For all $i \in \mathcal{I}$, the mapping $R_i : \Delta \rightrightarrows \mathbb{R}^n_+$ is closed-, convex-valued and such that for all $p \in \Delta$, $\mathrm{int}\, R_i(p) \neq \emptyset$. Moreover, it's continuous relative to Δ.

Set of feasible trades

Proof. R_i is closed-, convex-valued: clear. $e_i \in \operatorname{int} X_i \implies$

$$\operatorname{int} X_i \cap \{x \mid \langle p, x_i - e_i \rangle < 0\} \neq \emptyset, \forall p \in \Delta,$$

i.e., $\operatorname{int} R_i(p) \neq \emptyset$.

R_i osc on Δ: $\operatorname{Limsup}_{p \to \bar{p}} R_i(p) \subset R_i(\bar{p})$
$p^\nu \to \bar{p}$ in Δ and $x_i^\nu \to \bar{x}$ then

$$\langle p^\nu, x_i^\nu - e_i \rangle \;\to\; \langle \bar{p}, \bar{x}_i - e_i \rangle,$$

$x_i^\nu \in R_i(p^\nu) \implies \bar{x}_i \in R_i(\bar{p})$; X_i closed.

R_i isc on Δ: $\operatorname{Liminf}_{p \to \bar{p}} R_i(p) \supset R_i(\bar{p})$ use
Inner semicontinuity from convexity(a) Theorem. □

Inner semicontinuity from convexity

Recall:

Theorem

Consider a mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $\bar{x} \in \mathbb{R}^n$.

(a) *If S is convex-valued and $\operatorname{int} S(\bar{x}) \neq \emptyset$, then a necessary and sufficient condition for S to be isc relative to $\operatorname{dom} S$ at $\bar{x}$ is that for all $u \in \operatorname{int} S(\bar{x})$ there exists $W \in \mathcal{N}(\bar{x}, u)$ such that $W \cap (\operatorname{dom} S \times \mathbb{R}^m) \subset \operatorname{gph} S$; in particular, S is isc at $\bar{x}$ if and only if $(\bar{x}, u) \in \operatorname{int}(\operatorname{gph} S)$ for every $u \in \operatorname{int} S(\bar{x})$.*

(b) *If S is graph-convex and $\bar{x} \in \operatorname{int}(\operatorname{dom} S)$, then S is isc at $\bar{x}$.*

(c) *If S is isc at $\bar{x}$, then so is $T : x \mapsto \operatorname{con} S(x)$.*

Perturbing the Economy

$$\mathcal{E}^\nu = \{(u_i^\nu, e_i^\nu), i \in \mathcal{I}\} \qquad u_i^\nu : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\},\ \nu \in \mathbb{N},$$

with u_i^ν converging *continuously* to u_i:

$\{x_i^\nu \to x_i\}_{\nu \in \mathbb{N}} \subset X_i, \quad u_i^\nu(x_i^\nu) \to u_i(x_i)$.

Assumption: $X_i^\nu \to X_i, \quad u_i$ same properties as u_i

Application: (stochastic case) $u_i^\nu = u_i + \langle w^\nu, \cdot \rangle$, $w^\nu \to 0$.

Continuous convergence = pointwise-convergence + convergence on the boundary ‘consistent’ with the pointwise convergence on $\operatorname{int}\operatorname{dom} u_i$.

[Proof: relies on epi-convergence of convex functions.]

Excess supply function

Theorem

The demand function $p \mapsto \bar{x}_i(p) : \Delta \to \mathbb{R}^n$ *and the excess supply functions* $p \mapsto s(p)$ *are continuous.*
With $e_i^\nu \in \operatorname{int} X_i^\nu$ *for all* $i \in \mathcal{I}$*, for* $p, p^\nu \in \Delta$*,*

$$\bar{x}_i(p) = \operatorname{argmax} \left\{ u_i(x_i) \,\middle|\, x \in R_i(p) \right\},$$
$$\bar{x}_i^\nu(p^\nu) = \operatorname{argmax} \left\{ u_i^\nu(x_i) \,\middle|\, x \in R_i(p^\nu) \right\}.$$

and $u_i^\nu \xrightarrow{c} u_i$*. Then* $\bar{x}_i^\nu(p^\nu) \to \bar{x}_i(p)$ *for any sequence* $p^\nu \to p$ *in* Δ*; this means that* $\bar{x}_i^\nu \xrightarrow{c} \bar{x}_i$ *relative to* Δ*.*

Proof. Hypo-convergence of

$$v_i^\nu(x_i) = \begin{cases} u_i^\nu(x)_i & \text{when } \langle p^\nu, x_i - e_i^\nu \rangle \le 0 \\ -\infty & \text{otherwise,} \end{cases}$$

The Walrasian

An equilibrium price $\bar{p}$ solves ($s(p)$ set-valued)

$$S(p) \ni 0 \quad \text{where} \quad S : \Delta \rightrightarrows I\!R^n \quad \text{with} \quad S(p) = s(p) - I\!R^n_+;$$

Walrasian: $W : \Delta \times \Delta \to I\!R$ where

$$W(p, q) = \sup\{\langle q, s\rangle \,\big|\, s \in S(p)\}$$

- $\forall\, q \in \Delta :\quad W(\cdot, q)$ is usc,
- $\forall\, p \in \Delta :\quad W(p, \cdot)$ is convex,
- $\forall\, q \in \Delta :\quad W(q, q) \geq 0$.

W is Ky Fan function on a product of compact sets.

$\Longrightarrow$ *exists* $\bar{p}$ maxinf point (equilibirum).

follows pattern in 'Applied Nonlinear Analysis' Aubin/Ekeland

Approximating Economies

$$\left\{\mathcal{E}^\nu = \{(u_i^\nu, e_i^\nu), i \in \mathcal{I}\}\right\} \to \mathcal{E} = \{(u_i, e_i), i \in \mathcal{I}\}$$

Theorem

Suppose $X_i^\nu \to X_i$ for all $i \in \mathcal{I}$, $e_i^\nu \to e_i$, $u_i^\nu \xrightarrow{h} u_i$. Then, $\mathcal{E}^\nu$ and $\mathcal{E}$ have at least one equilibrium price $\bar{p}^\nu$ or $\bar{p}$ in Δ. $\{\bar{p}^\nu\}_{\nu \in \mathbb{N}}$ always has a cluster point and any such cluster point is a market equilibrium price for $\mathcal{E}$.

Proof. Construct the Walrasians W^ν and show that they lop-converge ancillary tight to W. $\square$

An example; Mass-Colell *et al*

$$u_1(x_{11}, x_{21}) := \begin{cases} x_{11} - (1/8)x_{21}^{-8}, & \text{on } [0.3, \infty)^2, \\ -\infty & \text{otherwise}, \end{cases}$$

$$u_2(x_{12}, x_{22}) := \begin{cases} x_{22} - (1/8)x_{12}^{-8}, & \text{on } [0.3, \infty)^2, \\ -\infty & \text{otherwise}, \end{cases}$$

x_{li} amount of good l consumed by the agent i.

$$X_1 = X_2 = [0.3, \infty) \times [0.3, \infty),$$

$$e_1 = (2, r), e_2 = (r, 2), \; r = 2^{8/9} - 2^{1/9}.$$

Equilibirum points and perturbations

$$x_{21} = (p_2/p_1)^{-1/9}, x_{11} = 2 + (r - (p_2/p_1)^{-1/9})p_2/p_1$$
$$x_{12} = (p_1/p_2)^{-1/9}, \;\; x_{22} = 2 + (r - (p_1/p_2)^{-1/9})p_1/p_2.$$

So, from 'supply equals demand' applied to the second good,

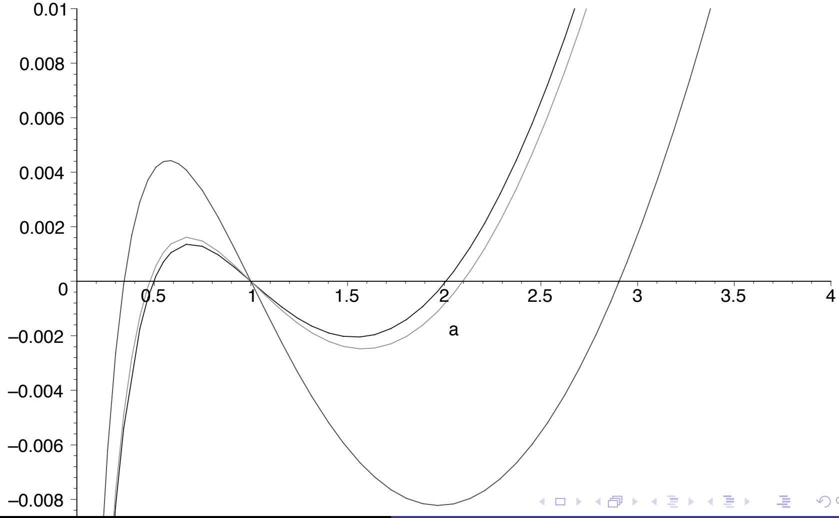
$$(p_2/p_1)^{-1/9} + 2 + (r - (p_1/p_2)^{-1/9})p_1/p_2 = 2 + r.$$

with solutions $p_1/p_2 = 0.5, 1, 2 \quad \Longrightarrow \quad$ 3 equilibrium points.
Perturbations: linear and scaling

case 1: $$u_1^\nu(x_{11}, x_{21}) := \begin{cases} x_{11} - (1/8)x_{21}^{-8} + x_{11}/\nu & \text{on } [0.3, \infty)^2, \\ -\infty & \text{otherwise,} \end{cases}$$

case 2: $$u_1^\nu(x_{11}, x_{21}) := \begin{cases} x_{11} - (1/8)x_{21}^{-8}(1 + 1/\nu) & \text{on } [0.3, \infty)^2, \\ -\infty & \text{otherwise,} \end{cases}$$

Perturbations: $\nu = 10, 100, 1000$



Augmented Walrasian

Aim: $\bar{p}$ maxinf of $W \approx (\bar{p}, \bar{q})$ saddle point of $\tilde{W}_r = \tilde{W}(\cdot, \cdot, r)$

augmenting function: $\sigma : \mathbb{R}^n \to \mathbb{R}$, convex

$$\min \sigma = 0, \quad \operatorname{argmin} \sigma = \{0\}$$

augmented Walrasian: $\tilde{W}_r : \Delta \times \Delta \times (0, \infty) \to \mathbb{R}$,

$$\begin{aligned}\tilde{W}(p, q, r) &= \sup_{y \in \Delta} \left\{ W(p, y) + r\sigma(y) - \langle q, y \rangle \right\} \\ &= \inf_{z \in \mathbb{R}^n} \left\{ W(p, q - z) - r\sigma^*(r^{-1} z) \right\}\end{aligned}$$

with $\sigma = \|\cdot\|$,

$$\tilde{W}_r(p, q) = \inf_z \left\{ W(p, z) \,\middle|\, z \in \mathbb{B}(q, r) \right\}$$

from Variational Analysis

- $\tilde{W}_r$ usc in p
- $\tilde{W}_r$ convex, lsc in (q, r), 'decreasing' in r
- maxinf and saddle points:

$$\sup_{p \in \Delta} \Big(\inf_{q \in \Delta} W(p, q) \Big) = \sup_{p \in \Delta} \Big(\inf_{q \in \Delta, r > 0} \tilde{W}_r(p, q) \Big)$$
$$= \inf_{q \in \Delta, r > 0} \Big(\sup_{p \in \Delta} \tilde{W}_r(p, q) \Big)$$

- $\tilde{W}_r$ lop-converge ancillary tight to W as $r \nearrow \infty$

Iterations

at iteration $k+1$: (p^k, q^k) and scalar r_{k+1} given

$$q^{k+1} = \operatorname*{argmin}_{q \in \Delta} \Big[\min_z \big\{ \langle z, s(p^k) \rangle \,\big|\, z \in \mathbb{B}(q, r_{k+1}) \Big]$$

i.e., minimizing a linear form on a, say $l^\infty -$ ball
reduces to finding the smallest element of $s(p^k)$

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as $r_k \nearrow \infty$, $\; p^k \to \bar{p}$ equilibrium point

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Test Cases

- Cobb-Douglas utility functions

$$u_i(x_i) = \gamma_i \prod_{l=1}^{n} x_{il}^{\beta_{il}} \text{ with } \sum_{l=1}^{n} \beta_{il} = 1, \beta_{il} \geq 0$$

- budget constraints

$$\sum_{l=1}^{n} p_l x_{il} \leq \sum_{l=1}^{n} p_l e_{il}$$

- demand

$$\bar{x}_{il} = (\beta_{il}/p_l)\left(\sum_{l=1}^{n} p_l e_{il}\right), \; l = 1, \ldots, n$$

- experiments: 10 agents, 150 goods (blink!), parallelization?