Trends in Applied Harmonic Analysis

Tony Chan (National Science Foundation),
Charles Chui (Stanford University),
Rong-Qing Jia (University of Alberta)

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Introduction

Over the past two decades, two research areas related to the applied and computational aspects of mathematical analysis have been introduced and rapidly developed. One area is wavelet analysis and the other may be coined mathematics of imaging (or image science). At the very early stage of their advancement, while wavelet analysis created a common link among mathematicians, electrical engineers, and scientists from various disciplines, the research area of image science has attracted the attention of several leading applied mathematicians whose significant impacts have helped in bringing this investigation to an attractive and highly respectable field of interdisciplinary research. One of the common mathematical ingredients of these two rapidly developed research areas is harmonic analysis. For this and other reasons, the combined development could be considered within the field of Applied Harmonic Analysis.

The main objective of this workshop is to identify the trends in the advancement of Applied Harmonic Analysis, in a very broad sense. Indeed the research interests of the three co-organizers, as well as the list of participants of the workshop, well represent the communities of image science, wavelets, approximation theory, learning theory, harmonic analysis, and partial differential equations.

In the following, we summarize the highlights of the lectures during this workshop. It is noted that in addition to the formal presentations by all of the 37 participants, there were plenty of opportunities for answers and questions as well as discussions among small groups. Furthermore, there was a two-hour formal discussion session led by one of the co-organizer, Charles Chui, on Friday morning after a good breakfast.

1. Wavelet bases and applications

Riesz bases of wavelets are generalizations of orthogonal and biorthogonal wavelets. Because of their flexibility, Riesz bases of wavelets are more suitable for representations of functions on bounded domains with possibly nonuniform or irregular meshes. In this workshop, Rong-Qing Jia introduced the projection method for construction of wavelet bases. His two Ph. D. students Wei Zhao and Hanqing Zhao discussed applications of wavelet bases to image compression and numerical solutions of partial differential equations. Song Li presented his recent results on Riesz bases of multiwavelets.

Riesz bases of wavelets are constructed on the basis of multilevel decompositions of a Hilbert space. Given a nested family of closed subspaces of a Hilbert space, one may consider projections onto these subspaces. The kernel spaces of the projection operators are wavelet spaces. In his talk, Rong-Qing Jia gave necessary and sufficient conditions on the projections such that the combination of Riesz bases of the wavelet spaces forms a Riesz basis of the whole space. Moreover, by means of multilevel decompositions, he developed a general theory for wavelet bases of Sobolev spaces. Under the guidance of the general theory, he introduced discrete wavelets on intervals and discussed their applications to image compression. He also investigated wavelet bases of splines for Sobolev spaces on bounded domains and their applications to numerical solutions of partial differential equations.

In his talk, Hanqing Zhao gave more details on applications of discrete wavelets to image compression. He proposed a new approach to constructing a family of wavelets which are discrete sequences. Compared to the traditional wavelets, the discrete wavelets have simple expressions and short supports. Being discrete, those wavelets are naturally designed for discrete mathematical models, and can be easily adapted to bounded intervals. In comparison with the well-known biorthogonal $9/7$ wavelets, the performance of the discrete wavelets is comparable, but the quality of the boundary parts of the compressed images is much better. Moreover, the computational cost is about one half of the time consumption of the $9/7$ wavelets. In light of their simplicity and flexibility, discrete wavelets are expected to have the potential of wide usage in many applications.

Many practical problems in elasticity and fluid dynamics are modeled by biharmonic equations. In his talk, Wei Zhao illustrated the wavelet method for numerical solutions of biharmonic equations. Inner wavelets and boundary wavelets based on cubic B-splines were constructed. Then these wavelets were adapted to the corresponding bounded domain. He indicated many advantages of the wavelet method over the traditional methods. First, the condition number of the stiffness matrix is quite small and uniformly bounded. Second, a high order convergence rate is achieved. Third, the transition and subdivision matrices are applied so that the algorithm becomes optimal. Fourth, the wavelet bases can be used to solve general elliptic equations of fourth order with variable coefficients. He provided numerical results to demonstrate these advantages.

In his talk, Song Li presented his recent results on Riesz bases of multiwavelets. Given two vectors of compactly supported functions in a certain Sobolev space, he provided a characterization for the affine families generated from these vectors of functions to form Riesz sequences in the space of square integrable functions. Furthermore, starting with a pair of compactly supported biorthogonal refinable vectors of functions, he gave a general principle for constructing Riesz bases of multiwavelets.

2. Subdivision schemes

Subdivision schemes are iteration schemes performed on multi-levels. The convergence of a subdivision scheme and the properties of its limit function have been extensively studied. In this workshop, Serge Dubuc presented his joint work with Jean-Louis Merrien on the de Rham transform of subdivision schemes and applications to computer graphics. Gitta Kutyniok introduced a new class of bivariate subdivision schemes and discussed their connections with shearlet multiresolution analysis.

In his talk, Serge Dubuc generalized the Chaikin - de Rham corner cutting and defined the de Rham transform for any subdivision scheme and even for any Hermite subdivision scheme. In order to study the smoothness of the limit functions, spectral properties of subdivision matrices were analyzed. Spectral theory is the link of his talk with harmonic analysis. By defining the de Rham transform, he was able to generate more examples of subdivision schemes and, in many cases, to obtain smoother limit functions.

Gitta Kutyniok in her talk introduced the representation system of shearlets. Shearlets are an affine system which precisely detects orientations of singularities, in the sense of resolving the wavefront set, while providing optimally sparse representations. The main new idea she proposed at the workshop was a shearlet multiresolution analysis, which could provide a means to resolve anisotropic structures efficiently. Along the way she developed a new type of directional and non-stationary bivariate subdivision schemes, which have the capability of adaptively changing the orientation of the data during the subdivision process.

3. Mathematics of signal processing

Signal processing is the analysis, interpretation and manipulation of signals. The signals in the real world are analog signals. To process these signals in computers, we need to convert the signals to digital form. In this workshop, Ozgur Yilmaz described his joint work with Ingrid Daubechies, Sinan Gunturk, and Yang Wang on a new robust quantization algorithm for analog-to-digital conversion. Sherman D. Riemenschneider discussed time-frequency representations of signals from pulsars. Qiyu Sun investigated sampling and reconstruction of signals with finite rate of innovation. Don Hong presented some interesting results on applications of multiscale tools, such as wavelets, to medical proteomic data analysis.

In digital signal processing and digital communications, an analog signal is converted into a digital signal by an A/D (analog-to-digital) conversion device. Some state-of-the-art A/D conversion techniques such as sigma-delta modulation and beta encoding were discussed in the interesting talk given by Ozgur Yilmaz. He introduced the "golden ratio encoder", a novel A/D conversion scheme that can be implemented robustly on analog hardware and enjoys an exponentially precise conversion performance. In particular, the new scheme exploits certain algebraic properties of the golden mean and produces robust quantized representations without requiring any ”precise multiplication” and ”precise comparison” to be performed in the analog hardware.

Over the last 12-15 years, radio astronomers have completed several surveys of the skies and have found many new pulsars. They have provided some of the best tests for Einsteins theory of relativity and are quite accurate “timing devices” for physical phenomena. In the current approach, the data is collected into a filterbank corresponding to the different bands sampled in decreasing order of frequency. The amount of data to be mined for pulsar data is enormous. The challenge for time-frequency analysts is to develop efficient algorithms to accomplish this with minimal human interaction and with an accuracy that would also allow the identification of candidates with very small periods (on the order of 1 millisecond or faster). In his talk, S. D. Riemenschneider proposed to use the Empirical Mode Decomposition to solve this important but difficult problem.

The sampling theorem is the fundamental tool allowing the processing of real signals via digital signal processors. Signals with finite rate of innovation are those signals that can be determined by finitely many samples per unit of time. In his talk, Qiyu Sun provided an algorithm to restore the signal from its noisy samples. The algorithm presented in the talk could be useful to medical signal processing, and hence possibly leads to an earlier and more complete diagnosis of cancer and heart attack (and thus a greater cure rate).

Proteomics is the study of proteins and the search for information about proteins. Comparable to the exciting development of nuclear magnetic resonance methods, mass spectrometry (MS) entered a phase of rapid growth beginning with the introduction of soft ionization methods such as matrix-assisted laser desorption ionization (MALDI). In real applications, though MALDI MS allows direct measurement of the protein “signature” of tissue, blood, or other biological samples, and holds tremendous potential for disease diagnosis and treatment, key challenges remain in the processing of MALDI MS data. The new generation of mass spectrometers produces an astonishing amount of high-quality data in a brief period of time, leading to inevitable data analysis bottlenecks. In his talk, Don Hong presented some fascinating results of the research conducted at the Vanderbilt Ingram Cancer Center (VICC) and supported by NSF IGMS. He developed a software package for proteomic data processing using wavelet-based algorithms. Applications to real MS datasets for different cancer research projects in VICC demonstrated that the algorithm was efficient and satisfactory.

4. Wavelet methods for image processing

Wavelet methods and variational methods are two important approaches to image processing. These two methods were thoroughly investigated in this workshop. In particular, Zuowei Shen and Lixing Shen underlined the wavelet approach based on redundant systems such as wavelet frames. Zuowei Shen gave a talk on image reconstructions based on tight wavelet frames, while Lixin Shen presented his recent work on restoration of chopped and nodded images by using framelets.

One important class of problems in image processing is the recovery of an image when parts of it are missing. This occurs for example when one tries to remove a scratch on a precious photo or to fill in cracks in ancient drawings. These are inpainting examples where the data are missing in the image domain. However, there are also problems where the data are missing in the transformed domain. Tomography in medical imaging and infrared imaging in astronomy are example that can be casted as inpainting in the transformed domain. Ideally, the restored image should possess shapes and patterns consistent with the given data in human vision. Therefore, it is desirable to extract information such as edges and texture from the observed data to replace the corrupted part in such a way that it would look natural for human eyes. In his talk, Zuowei Shen emphasized the important role played by tight frames as redundant systems in image reconstructions. The main idea of the tight frame based iterative algorithm perturbs the frame coefficients by thresholding, which also removes noise and sharpens edges, so that information contained in the available coefficients will permeate into the missing frame coefficients. Here, the redundancy is very important, since the available coefficients contain information of the missing coefficients only if the system is redundant. The limit of the tight frame based iterative algorithm minimizes a cost functional that balances the edge preserving and closeness to the given data together with the smoothness of the solution.

In ground-based astronomy at mid-infrared wavelengths, the weak astronomical signal is often corrupted by the overwhelming thermal background produced by the atmosphere and the telescope. To extract this celestial source, we need to eliminate the effect of the background. A common approach called *chop-and-nod* is employed. The restoration of the celestial signal from the observed chopped and nodded image is obtained through the projected Landweber method. However, the restored images with the projected Landweber method always have some kinds of artifacts. In his talk, Lixin Shen presented his joint work with J. F. Cai, R. Chan and Z. W. Shen on a tight frame based method for solving the problem. They proposed an iterative algorithm to recover the celestial signal. The convergence of the proposed algorithm was proved on the basis of convex analysis and optimization theory. Testing for simulated and real images demonstrated significant improvements of their method over the projected Landweber method.

5. PDE based numerical algorithms for image processing

Quite often natural images contain certain noises due to a variety of reasons such as the movement of a camera and the quality of its lens. Image denoising is one of the commonly used processing in many areas of sciences and industry. A popular method for image denoising in image processing is the total-variation-based Rudin-Osher-Fatemi (ROF) model. However, the partial differential equations derived from the model are highly nonlinear and singular. It is a challenging problem to find efficient numerical algorithms to solve such equations. In this workshop, Bradley Lucier, Qianshun Chang, Bin Han, and Yang Wang reported their recent results on this important problem.

Bradley Lucier gave a talk on some numerical methods for the ROF model of image smoothing. On the basis of his joint work with Stacey Levine and Antonin Chambolle, he discussed a projection technique for both anisotropic and "isotropic" upwind discrete numerical methods and proved the convergence of methods. He also employed multigrid/multiscale algorithms that speed up the computations significantly. Furthermore, he provided examples to illustrate some qualitative properties of solutions of the numerical schemes.

In his talk, Qianshun Chang presented his joint work with Wei-Cheng Wang and Jing Xu on a robust algorithm for both deblurring and denoising. He placed emphasis on the role played by the algebraic multigrid (AMG) method in numerical solutions of the nonlinear partial differential equations derived from the ROF model. He gave a convergence analysis and used various techniques to accelerate convergence. Moreover, he provided numerical experiments which demonstrated that their algorithm was efficient and robust over a wide range of parameters from images with large noise-to-signal ratios (SNR) and strong blur, to purely blurred images without noise.

In his talk, Bin Han presented his joint work with W. Dahmen and V. Pasyuga. He highlighted applications of adaptive wavelet schemes to numerical solutions of nonlinear partial differential equations in image denoising and indicated that adaptive wavelet schemes could resolve the singularities better and improve the numerical performance. He also discussed their work on the theoretical convergence rates of the adaptive wavelet scheme for such nonlinear variational problems in any dimension.

Yang Wang presented his joint work with Tony Chan and Haomin Zhou on a total variation scheme for denoising based on wavelet decomposition. Noise in natural color photos have special characteristics that are substantially different from those that have been added artificially. On the basis of good understanding of the characteristics of digital noise in natural color images, he proposed the Multiscale Wavelet TV (MWTV) method for denoising color photos and demonstrated that the MWTV method had outstanding denoising capabilities for natural color images.

6. Learning Theory

Mathematical Learning Theory has been a very active research area recently. Based on support vector machines and non-parametric estimation in Statistics, Learning Theory has broad applications to such areas as feature extraction, pattern recognition, neural networks, and data mining. Steve Smale established a solid ground for Mathematical Learning Theory based on optimal recovery in Approximation Theory coupled with Probability Estimation. In particular, for kernel machine learning, its formulation in terms of certain minimization problem leads to the application of the powerful tools from reproducing kernel Hilbert spaces. Smale was not able to participate in the workshop. But his co-worker Ding-Xuan Zhou gave a talk on learnability of Gaussians with flexible variances. Yuesheng Xu presented a talk on his recent joint work with Haizhang Zhang on refinable kernels.

The talk of Ding-Xuan Zhou was about multi-scale structures of several algorithms in manifold learning. It is in the research direction of extracting and processing information from massive data which scientists and engineers in various fields face nowadays. The inference problems associated with high-dimensional data offer fundamental challenges to modern science and technology. A promising paradigm in addressing these challenges is the observation or belief that high-dimensional data arising from physical or biological systems can be effectively modelled or analyzed as being concentrated on or near a low-dimensional manifold. This has led to the hot topic of manifold learning aiming at many applications: dimensionality reduction, clustering, sparse representation, feature selection, and gene expression analysis. A variety of efficient learning algorithms have been introduced such as graph Laplacian, Hessian eigenmap, and diffusion maps, most of which involve Gaussian kernels because of the isotropic nature of their radial basis function form. In his talk, Zhou showed that the union of the unit balls of reproducing kernel Hilbert spaces generated by Gaussian kernels with flexible variances is a uniform Glivenko-Cantelli class. This verifies the uniform convergence of many learning algorithms involving Gaussians with changing variances, and reveals how Gaussian kernels are used to extract information with different frequency levels when the variances change.

In his talk on refinable kernels, Yuesheng Xu introduced the notion of refinable kernels and gave various characterizations of refinable kernels. The study of refinable kernels is motivated by efficiently updating kernels in mathematical learning from training data. The concept of refinable kernels leads to the introduction of wavelet-like reproducing kernels. In particular, he presented characterizations of translation invariant refinable kernels, and refinable kernels defined by refinable functions. This study leads to multiresolution analysis of reproducing kernel Hilbert spaces. Xu's talk bridged mathematical learning theory with applied harmonic analysis. His talk showed that the well-established concepts in the area of applied harmonic analysis can be used in mathematical learning to make the learning algorithms more efficient. Meantime, research in mathematical learning theory raises new challenging problems related to applied harmonic analysis. Interaction between these two areas is crucial for further developments of both area. This workshop provided a platform for people from these two areas to communicate with each other. It is expected that such communication will lead to fruitful and positive interaction between these two areas.

7. Mathematics of Imaging

This workshop was well represented by active researchers interested in the theory and applications of the rapidly developing discipline of Imaging Mathematics. As already described in Sections 4 and 5 above, image noise removal (also called digital image de-noising) is currently an active research area in Applied Mathematics. Although it has been a classical problem for several decades, yet due to the emerging sensor technology and rapid advancement of digital image devices, innovation of effective and efficient digital image de-noising methods has recently become a pressing issue, particularly to the sensor manufactures, both for government sector and the huge consumer market. For instance, consumers continue to demand higher picture resolutions but smaller and lighter devices, such as still pocket cameras, cellular cams, and even tiny digital video cams. Just like Moores law for the chip manufacturers, the trend of such demand does not seem to end soon. However, higher resolutions and smaller devices require more pixel elements crammed in a smaller image sensor matrix, resulting in creating higher thermal noise (called black current) due to increasing contamination of photo-electrons free from neighboring light sensors when the image sensor is stricken by light photons, particularly in lower light environment.

The classical mathematical approach of image smoothing in terms of Gaussian convolution, known to be equivalent to the isotropic heat diffusion process, has been extended over a decade ago to anisotropic diffusion PDE models to preserve and even enhance image features, particularly image edges, while reducing image noise. The most well-known models are the Perona-Malik model with backward diffusion capability and the total variational (or TV) model introduced by Rudin, Osher, and Fatemi to eliminate diffusion in the gradient (or normal) directions of image edges.

A more recent exciting development initiated by Tomasi and Manduchi in 1998, is the extension of the Gaussian filter (called spatial filter) by attaching to it a multiplicative radiometric filter component, in terms of differencing with neighboring pixel values. Though ad hoc, yet just as effective and certainly more efficient as compared with anisotropic diffusion, this so-called bilateral filtering approach has created a tremendous significant impact, not only to the digital imaging community, but currently to the community of computer graphics as well.

Over the past few years, there has been a lot of speculation that the two above-mentioned developments, anisotropic diffusion and bilateral filtering, are very likely to be intimately related, and the evidence was quite convincing, particularly through adaptive filtering as well as the currently popular research area, called diffusion maps and geometry, introduced by Coifman and Lafon. In the lecture delivered by Charles Chui, the on-going joint research program with Jianzhong Wang on the development of a unified theory, based on the concepts of anisotropic diffusion, diffusion maps, and nonlinear filtering, was briefly discussed. In particular, the (advection) diffusion partial differential equations derived from the bilateral filters are formulated, with solutions of the PDE given by the (stationary) diffusion mapping process (of Coifman and Lafon) by using the bilateral filters as diffusion kernels. In addition, the corresponding discrete-time PDE formulations are established, with solutions given by the (non-stationary) diffusion mapping process, meaning that the diffusion kernels are being up-dated, with outputs at each iterative step as inputs for the next iterations.

The presentation by Jianzhong Wang was continuation of Chuis talk, showing the central ideas for formulating the PDEs as well as key steps of the proofs. In particular, it is shown that the iterative bilateral filtered outputs provide the solutions of the diffusion PDE's in the sense of infinitesimal, with fairly sharp orders of estimation.

As mentioned above, the TV model of anisotropic diffusion has created a huge impact to the study of digital image de-nosing. In the lecture by Xavier Bresson, the topic of fast minimization of the vectorial TV norm was discussed. In this talk, a regularization algorithm for color/vectorial images is proposed. The proposed algorithm is fast, easy to implement, and mathematically well-posed. Based on a dual formulation, the regularization model may be regarded as a vectorial extension of the dual approach considered by Chambolle for gray-scale/scalar images. The proposed model has been applied to various color image processing tasks, including image decomposition, image inpainting, image deblurring, and image denoising defined on manifolds.

A higher-order diffusion PDE model for digital image de-noising introduced by You and Kaveh incorporates the feature of image segmentation and addresses such drawbacks as stair-casing of the TV model and the anisotropic diffusion model of Perona-Malik. Unfortunately, this fourth order PDE is not well-posed. In her talk, Catherine Dupuis discussed the discrete version, illustrated with numerical simulations the sensitive dependence of its solutions on the initial data, and for the one-dimensional spatial setting, proved a weak upper bound on the coarsening rate of the discrete-in-space version of this equation, by following a recent technique of Kohn and Otto.

Yet another diffusion PDE model for digital image de-noising is the diffusion-reaction equation, derived by using the steepest decent approach to solving the Euler-Lagrange equation of the minimum-energy functional with balanced internal energy component governed by the noisy image data. In her talk, Gerlind Plonka discussed certain TV-based diffusion-reaction numerical solution considered as a digital variant of a diffusion-reaction type equation as suggested by Nordström. The discrete formulation leads to certain nonlinear algebraic equations, eliminating the need of any knowledge of numerical PDE solver for the analysis and application of the PDE model. Numerical examples are used to compare the de-noising properties of this approach against various image de-noising methods in the literature.

Perhaps the most exciting extension of the problem of digital image de-noising is digital image inpainting, for which certain missing data are to be recovered from the remaining portion of the digital image which is contaminated with noise. In addition to the wavelet-frame approach presented by Zuowei Shen, as described in Section 4 above, there are two talks in this workshop on image inpainting based on the TV minimum-energy model, which naturally leads to solution of certain PDEs via the steepest decent approach to solving the Euler-Lagrange equations.

In his talk, Xue-Cheng Tai discussed his joint work with Osher, Holm, and Rahman on certain TV-Stokes PDE model, based on geometric consideration as opposed to the standard fluid mechanics formulation. By using the TV energy minimization model of isophote curves (or level sets) of the input noisy image with missing data, the steepest decent solution is used to formulate the minimum-energy functional of the image gradient field to be restored, as the orthogonal vector field of the isophote curves. Analysis and numerical experiments show that better image properties, both in de-noising and inpainting, can be achieved by using the corresponding modified higher-order diffusion PDE.

The other digital image inpainting talk delivered by Haomin Zhou is a joint work with Tony Chan and Jackie Shen. In this talk, the image to be inpainted and de-noised is represented in the wavelet domain, and the goal is to clean up and restore the missing wavelet coefficients. The TV model is used as the external energy functional, and the internal energy functional constitutes the known wavelet coefficient data. As usual, the Euler-Lagrange PDE is formulated, but in terms of a discrete version of the steepest decent approach. Solution of the proposed models (one with and the other without noise contamination) could allow effective and automatic control over certain geometric features of the inpainted images including sharp edges, even in the presence of substantial loss of wavelet coefficients, including in the LL (thumb-nail) frequency band.

8. Representation and Analysis of Complex Data

One of the biggest challenges to the applied and computational mathematicians is to develop methods and algorithms for manipulation, representation, visualization, and understanding of large complex data sets, particularly in the high dimensional setting. Problems that range from data-mining to point-cloud visualization have a tremendous number of applications in virtually all disciplines, including the medical imaging field, various manufacturing industries, the entertainment sector, and the consumer-electronic market.

In particular, analysis of (and on) data sets is of great importance in a wide variety of applications, where one is confronted with large amounts of data in high dimensions, and where such tasks as finding features, compressing data, de-noising, and building predictors are of primary concern. In his talk, Mauro Maggioni discussed his recent and current development of harmonic analysis on data sets. It is noted that in many disparate contexts, data sets have a certain geometry that can be analyzed quantitatively, and can be exploited in order to achieve superior performance in the tasks mentioned above. This data geometry also instrumental for building useful dictionaries of functions on the data, and thereby, analogous to generalized Fourier and wavelet based approaches, allowing to lift signal processing tools to the analysis and manipulations of functions on data sets.

In a more theoretical development in the same direction, Hrushikesh Mhaskar presented his joint work with Mauro Maggioni on diffusion polynomial frames on metric measure spaces. In this talk, he showed the method of construction of a multiscale tight frame based on an arbitrary orthonormal basis for the L^2 space of an arbitrary sigma finite measure space. He also discussed the approximation properties of the resulting multiscale in the context of Besov approximation spaces, which are characterized both in terms of suitable K–functionals and the frame transforms, with the only major condition required being uniform boundedness of certain summabilility operator. Sufficient conditions were mentioned for this to hold in the context of a general class of metric measure spaces, where the assumption of finite speed of wave propagation might not hold. The theory is also illustrated by using the approximation of the characteristic functions of caps on a dumbell manifold, and applied to the problem of recognition of hand-written digits. It was claimed that the methods outperforms the comparable methods for semi–supervised learning.

The study of harmonic analysis of and on data sets as discussed in Mauro Maggionis presentation is an essential contribution to the advancement of the research direction of diffusion maps and diffusion geometry, as mentioned above in the description of the talk by Charles Chui. One of the important applications is data-set segmentation and clustering, and it happens that this new mathematical development has significant impact to the existing tools used by the computer science community, such as normalized cuts and graph partitioning. In the talk by Jerome Darbon, some deep theoretical links between certain maximum-flow based approach, known as ”graph-cuts”, and the total variation (TV) minimization problem was discussed.

On the other hand, explicit representation of complex data in requires the use of local coordinates. In the discussion of mean value representations and curvatures of compact convex sets, S. L. Lee considered representation of points and functionals on a set by the extreme points or boundary of the set, and thus extending the notions Barycentric coordinates of simplexes, and in some sense, the theorems of Krein-Millman and of Choquet. In conjunction with the construction of one-one transformation and parametrization of meshes in $\mathbf{R}^3$, it was mentioned that Floater has constructed new coordinates, called the *mean value coordinates*, for the representation of points in the kernel of star-shaped polygons in $\mathbb{R}^2$ and polyhedrons in $\mathbf{R}^3$ in terms of extreme points of the star-shaped regions. Floater's construction was motivated by the mean value property of harmonic functions, but it was shown recently that the mean value coordinates can be derived from a *mean value representation,* which is more naturally associated with conservative vector fields in $\mathbb{R}^2$ and divergence free vector fields in $\mathbf{R}^3$. Interestingly, in this divergence free framework, the mean value representation is intimately connected with the Minkowski problem that relates positive functions on the unit $(n-1)$-dimensional sphere in $\mathbb{R}^n$ that are orthogonal to the first harmonics, to the curvature of compact strictly convex hyper-surfaces through its Gauss map.

The vector fields are also intimately related to homogeneous functions, which provide a general method for the construction of a large class of coordinates based on the curvature of compact strictly convex hyper-surfaces, which extend the barycentric coordinates. In the divergence free framework, the vector fields, $\mathbf{F}(\mathbf{r}) = \mathbf{r}/\|\mathbf{r}\|^{n+1}$, $\mathbf{r} \in \mathbf{R}^n$, that produce Floater's mean value coordinates for $n = 2, 3$ are associated the curvature of the unit $(n-1)$-dimensional sphere. This shows that the mean value coordinates are indeed a simple case of mean value representation based on the geometry of compact strictly convex hyper-surfaces.

In his talk, S.L. Lee gave the relationship between homogeneous functions and the corresponding vector fields for mean value representation and use them, in conjunction with Minkowski problem, to construct mean value representation based on the curvatures of compact strictly convex hypersurfaces in $\mathbf{R}^n$.

Another approach for representing scattered data in the three-dimensional Euclidean space is the more classical method called surface subdivisions. This approach is quite effective if the subdivision schemes are interpolating. In addition, since subdivision is based on refinability, which is, in some sense, equivalent to the mathematical structure of multiresolution approximation (or MRA), the notion of wavelets can be adopted to add features to the subdivision process. In her talk, Maria Charina presented her joint work with Joachim Stöckler on sibling frames for interpolatory subdivisions, and discussed a certain local matrix factorization technique, inspired by an earlier work of Chui-He-Stöckler, for the construction of non-stationary sibling frames with at least one vanishing moment. The construction method is demonstrated by examples of certain 4–point and the butterfly interpolating surface subdivision schemes.

There are various classical approaches to representing complex data, particularly those commonly used in the statistics community. In the talk by Ming-Jun Lai, bivariate splines are used to realize the functional regression models. More precisely, a functional linear regression model was considered, where the explanatory variable is a random surface and the response is a real random variable with noise. Bivariate splines are then used to represent the random surfaces. It was shown that under the boundedness assumptions on the regressors in the sample and that the regressors span a sufficiently large function space, then the bivariate splines approximation properties achieve the consistency of the estimators. It was claimed that simulation results illustrate the high quality of the asymptotic properties for various situations.

9. Polynomial and Wavelet Frames

Another research area in Applied Harmonic Analysis that has been under rapid development in recent years is the theory and analysis of frames. Frames add redundancy in comparison with bases, and therefore are more suitable for certain applications that require representors with more desirable algebraic and geometric properties as well as more flexibility to adapt to changes. Algebraic and trigonometric polynomials, splines, and wavelets are among the most commonly used functions for building frames with prescribed properties.

In the talk by Jeff Geronimo, the topic of stable polynomials was discussed, with emphasis on their Christoffel-Darboux formulas. For digital filtering, a stable polynomial in one variable is one that has all its zeros exterior to the closed unit disk. Such polynomials also play an important role in the theory of polynomials orthogonal on the unit circle, the construction of Szego-Bernstein measures, and auto-regressive models. It is well-known that one-variable stable polynomials satisfy a Christoffel-Darboux formula. In two variables, it is shown in this talk that there is a similar link between stable polynomials and a Christofell-Darboux formula which leads to a surprising number of interesting applications.

Polynomials can be used as building blocks to construct localized bases and frames. Jürgen Prestin presented his joint work with F. Filbir and H. N. Mhaskar on exponentially localized polynomial bases and frames on the interval $[-1, 1]$. As an example, for any given function f defined on $[-1, 1]$, a sequence of algebraic polynomials is constructed to have the properties of fastest uniform convergence rate (i.e. the order of best uniform approximation) to f on $[-1, 1]$ as well as the geometrical convergence rate at each point on $[-1, 1]$ where f is analytic. The method of construction allows the construction of exponentially localized kernels based only on certain summability estimates. In turn, the localization property also enables derivation of the characterization of local Besov spaces on the interval. The method is extended to the unit sphere, and an example of an exponentially localized basis of trigonometric polynomials was also given for periodic functions.

For spline functions and wavelets, the topic of prediction schemes for adaptive approximation was presented by Kai Bittner. In this joint work with Karsten Urban, the adaptive wavelet approach is used as the core ingredient for adaptive evaluation of nonlinear functions. In this regard, an efficient adaptive method for approximate evaluation of nonlinear functions of wavelet expansions is developed in terms of semi-orthogonal spline wavelets. The desired accuracy is achieved by simultaneous approximation in terms of wavelets and approximate computation of the corresponding wavelet coefficients. The computational complexity of this proposed method has the same asymptotic order as the best n-term tree approximation.

Almost all wavelet bases and frames have been constructed by using the architecture of multiresolution approximation (MRA), at least in the weak sense. Families of bivariate refinable scaling functions were discussed in the talk by Jian-ao Lian. In this talk, by reformulating the conditions for polynomial reproduction order with arbitrary dilation matrix, families of bivariate refinable scaling functions with any desirable symmentry and polynomial reproduction order are constructed.

The notion of tight wavelet frames is a natural generalization of orthonormal wavelet bases, in the sense that the Parseval identity is preserved. Wenjie He presented his joint work with Charles Chui on derivation of useful conditions for the existence of vector-valued tight frames of multi-variate wavelets. In particular, the unitary extension principle (UEP) is adopted for the existence proof. A main result is that for any compactly supported refinable function-vector $\Phi = [\phi_1, \ldots, \phi_r]^T \in (Ł_2(\mathbb{R}^s))^r$ with dilation matrix A and two-scale symbol $P(\mathbf{z})$, certain positivity condition imposed on $P(\mathbf{z})$ guarantees the existence of tight frames of compactly supported multi-variate wavelets based on UEP. This is a significant generalization of the uni-variate setting, for which the problem can be solved by applying a matrix-valued version of the scalar-valued Riesz Lemma.

Another contribution to the advancement of tight-frame wavelets was discussed by Joachim Stöckler. Most of the results presented in this talk were obtained in his student Kyoung-Yong Lee for his doctoral thesis. The idea is to apply the Hilbert transform to an MRA tight wavelet frame to generate another MRA tight wavelet frame. Specifically, generators of tight wavelet frames as linear combinations of B-splines of order $m \geq 2$, obtained by applying the work of Chui-He-Stöckler are considered, and explicit formulations of their approximate Hilbert transforms in terms of certain linear combinations of B-splines of order $m + 1$ are derived. An application of the approximate Hilbert transform to filtered back-projection was also outlined in this presentation.