Workshop on Nonholonomic Dynamics and Integrability
January 28 – February 2, 2007

Monday

Speaker: **L. Bates** (University of Calgary)
Title: *What happened to the Hamilton-Jacobi equation*
Abstract: By looking at some examples we attempt to explain why there is no Hamilton-Jacobi equation in nonholonomic dynamics and the implications this has for the integration of completely integrable nonholonomic systems.

Speaker: **J. Sniatycki** (University of Calgary)
Title: *Conservation laws, symmetry and reduction*
Abstract: For a non-holonomically constrained mechanical system, I shall describe the distributional Hamiltonian formulation of its dynamics, formulate a non-holonomic analogue of Noether's theorem, and discuss the notion of symmetry of the system. I shall discuss various types of constants of motion and singular reduction of symmetries.

Speaker: **Yu. Baryshnikov** (Bell Labs)
Title: *Spherical billiards with many 3-periodic orbits*
Abstract: It is known that a planar Birkhoff billiard cannot have a 2-parameter family of 3-periodic orbits, while spherical billiard domains can (for example, the spherical geodesic triangle with right angles). I will explain why, and describe spherical billiards having this property.

Speaker: **V. Zharnitsky** (University of Illinois)
Title: *Periodic orbits in outer billiards*
Abstract: The study of large sets (having positive measure) of periodic orbits in Birkhoff billiards is important for the spectral asymptotics in the corresponding wave equations.

We will describe a recently developed new approach to this problem relying on exterior differential systems and we will show that the set of 4-period orbits in the outer billiard has an empty interior. This is joint work with Alexander Tumanov (Math UIUC).

Tuesday

Speaker: **A. Agrachev** (SISSA)
Title: *Rolling balls and octonions*
Abstract: I am going to discuss hidden symmetries of the classical nonholonomic kinematic system (a ball rolling over another ball without slipping or twisting) and explain the geometric meaning of basic invariants of vector distributions.

Speaker: **R. Montgomery** (UC Santa Cruz)
Title: *G_2 and the Rolling Distribution.* (Joint with Gil Bor)
Abstract: The act of rolling one surface along another surface without slipping or spinning defines a rank 2 distribution on 5-manifold, the 5-manifold being a circle bundle over the product of the two surfaces. This distribution is manifestly invariant under the product K of the isometry groups of the two surfaces. When the two surfaces are spheres then K is thus the product of two rotation groups, one for each sphere. However, something miraculous happens when the ratio of radii of the spheres is 1:3: the local symmetry group of the rolling distribution becomes much larger. This local automorphism group becomes first exceptional Lie group, namely, the split real form of the Lie group G_2. We will sketch the proof of this fact

using two explicit constructions, and relying heavily on the theory of roots and weights for the Lie algebra of G_2. Paper available at: math.DG/0612469

Speaker: **A. Bloch** (University of Michigan)
Title: *Connections between nonholonomic mechanics and control*
Abstract: In this talk I will discuss the relationship between nonholonomic mechanics and nonlinear control theory. Systems subject to nonholonomic constraints have natural links to nonlinear control systems as the constraints often induce good controllability properties. I will discuss the important distinction between kinematic and dynamic nonholonomic systems and will describe the different optimal control problems that arise for these two classes of systems. I will discuss integrable systems that arise naturally in optimal control of nonholonomic systems and in nonholonomic systems themselves. I will also discuss aspects of stability and stabilization of nonholonomic systems and discuss how one can get asymptotic stability in certain classes of nonholonomic systems even in the absence of external dissipation.

Speaker: **M. Levi** (Penn State)
Title: *A simple example of Arnold diffusion*
Abstract: In this joint work with Vadim Kaloshin we give a simple geometrical explanation of Arnold diffusion. The idea will be illustrated for the case of a particle in a periodic potential in $\mathbf{R}^3$, and, in a slightly different setting, for the geodesic flow with time-periodic metric.

Wednesday

Speaker: **D. Zenkov** (North Carolina State)
Title: *Momentum conservation, integrability, and applications to control*
Abstract: Numerous examples show that momentum dynamics of nonholonomic systems is remarkably different from that of holonomic/Hamiltonian systems. For example, symmetries do not always lead to spatial momentum conservation as in the classical Noether theorem. We will discuss nonholonomic momentum conservation relative to the body frame and its role in the theory of integrable nonholonomic systems. A new integrable nonholonomic system will be introduced. We then will discuss applications of momentum dynamics to control of nonholonomic systems.

Speaker: **Yu. Fedorov** (Universitat Politechnica de Catalunya)
Title: *Discretization of integrable nonholonomic systems on Lie groups*
Abstract: Recently the formalism of variational integrators (discrete Lagrangian systems) was extended to systems with nonholonomic constraints. We briefy describe this formalism and apply it to the case when the configuration space is a Lie group G and the discrete Lagrangian is left-invariant, while discrete constraints are left- or right-invariant with respect to the action of G. As examples, we construct discretizations of several classical integrable nonholonomic systems with an invariant measure, in particular, of the celebrated Chaplygin nonholonomic sphere problem. It appears that the resulting discrete dynamics is similar to that of the continuous models. We then propose a method of choosing left-invariant discrete nonholonomic constraints that ensures preservation of the energy integral in the discretizations. The conservation of an invariant measure in the discrete systems will also be discussed.

Speaker: **V. Jurdjevic** (University of Toronto)
Title: *Rolling sphere problems on spaces of constant curvature.* (Joint with J. Zimmerman)
Abstract: The rolling sphere problem on Euclidean space $\mathbb{E}^n$ for $n \geq 2$ consists of determining the path of minimal length traced by the point of contact of the unit sphere $\mathbb{S}^n$ on $\mathbb{E}^n$ as it rolls without slipping between two specified points of $\mathbb{E}^n$ and from a given initial rotational configuration to a prescribed terminal rotational configuration.

In this lecture I will present the results, in which the rolling sphere problem is extended to situations in which a sphere $\mathbb{S}^n_\rho$ of radius ρ rolls on a stationary sphere $\mathbb{S}^n_\sigma$ of radius σ, and to the hyperbolic analogue in which the spheres $\mathbb{S}^n_\rho$, $\mathbb{S}^n_\sigma$ are replaced by the hyperboloids $\mathbb{H}^n_\rho$, $\mathbb{H}^n_\sigma$ having hyperbolic radii ρ, σ with $\sigma \neq \rho$. The notion of rolling is taken in an isometric sense; the length of the path of the point of contact is measured by the metric of the stationary manifold and the orientations of the rolling manifold are expressed by the elements of its isometry group. This larger geometric perspective, that encompasses both the Euclidean and the hyperbolic geometries, also includes the unit hyperboloid $\mathbb{H}^n$ rolling isometrically on $\mathbb{E}^n$.

Speaker: **T. Tokieda** (Cambridge)
Title: *Slipping and rolling toys and their integrability*
Abstract: I will discuss, both on the board and through toy demonstrations, a number of nonholonomic problems which look integrable—conserved quantities, quasi-periodicity, etc.—but seem awkward to fit into the current models of integrability. Other signatures they exhibit are chirality and finite-time singularity, and I argue that these ought to be a generic part of physically realistic models of nonholonomic integrability.

Thursday

Speaker: **M. de Leon** (CSIC Real Academia de Ciencias)
Title: *Hamilton-Jacobi theory for nonholonomic mechanical systems*
Abstract: We develop a Hamilton-Jacobi theory for nonholonomic mechanical systems. The results are applied to a large class of nonholonomic mechanical systems, called Chaplygin systems.

Speaker: **W. Respondek** (INSA de Rouen)
Title: *Integrability and non-integrability of sub-Riemannian problems*
Abstract: We study the problem of optimal laser-induced population transfer in n-level quantum systems. This problem can be represented as a sub-Riemanniann problem on $SO(n)$ and it is known that for $n = 3$ the Hamiltonian system associated with PMP (Pontryagin Maximum Principle) is integrable. In the first part of the talk, we will show that this changes completely for n larger than 3. Namely, the adjoint equation of PMP does not possess any first integral independent of the Hamiltonian on the leaves of the symplectic foliation. In proving non-integrability we use the Morales-Ramis theory.

In the second part we will show that the above-mentioned integrability of the adjoint equation for $SO(3)$ is a particular case of a more general result. Namely, we prove that the adjoint geodesic equation for a 3-dimensional homogeneous sub-Riemannian space possesses an additional quadratic first integral if and only if the space is symmetric.

Our talk is based on joint results with Andrzej Maciejewski (University of Zielona Gora).

Speaker: **Yu. Sachkov** (University of Pereyaslavl))
Title: *Maxwell strata and conjugate points in Euler's elastic problem*
Abstract: In 1744 Leonard Euler considered the following problem on stationary configurations of elastic rod. Given a planar elastic rod with fixed endpoints and tangents at the endpoints, it is required to find possible profiles of the rod with the given boundary conditions. Euler derived differential equations for stationary configurations of the rod, reduced them to quadratures, and described their possible qualitative types. Such configurations are called Euler elastic.

The question on stability of Euler elastic was solved only in some partial cases. In the talk we describe the full solution to the problem of stability of Euler elastic. In addition to this local problem, the corresponding global optimal control problem is also considered. Stability of Euler elastic corresponds to local optimality of extremals of acertain optimal control problem. It is known that extremals cannot be optimal after Maxwell points (where distinct extremal curves with the same length and cost functional meet one another) or after conjugate points (at the envelope to the family of extremal trajectories).

The group of discrete symmetries of the system of extremals is generated by the group of discrete symmetries of the equation of a pendulum. Maxwell points are described via the study of fixed points of the action of this symmetry group. Maxwell points for all types of elastic are found.

Speaker: **A. Ruina** (Cornell)
Title: *Some mechanics perspectives on non-holonomic constraints*
I will briefly discuss some of these simple observations about non-holonomic systems. 1) Despite common mythology, equations of motion can sometimes be found by simple means (with video). 2) The most common non-holonomic systems, cannot by virtue of their symmetry, have the most interesting of non-holonomic features, asymptotic stability. 3) The word "non-holonomic" might sensibly be replaced with "skates and wheels". 4) Despite a 109 year history there are no established equations of motion for a reasonably-general non-holonomic bicycle (with video). 5) There seems to be an intimate connection between zero-energy controllers and steerable non-holonomic constraints. 6) A mechanical implementation of a non-holonomically constrained harmonic oscillator (w/ Larry Bates).

Friday

Speaker: **L. Garcia-Naranjo** (University of Arizona)
Title: *Almost Poisson bracket for nonholonomic systems on Lie groups*
Abstract: Nonholonomic mechanical systems are not Hamiltonian. One can however describe their dynamics in term of a bracket of functions that fails to satisfy the Jacobi identity. Now one speaks of an almost Poisson bracket. This approach avoids dealing with Lagrange multipliers, but, in practice, is difficult to implement because it involves heavy computations in coordinates.

I will consider the so-called LL and LR systems where the configuration space is a Lie group and both the Hamiltonian and the constraints have invariance properties. These invariance properties will allow us to give a geometric construction of a bracket for the description of the system on a reduced space. This construction avoids computations in coordinates and provides relatively simple formulas for the bracket. The idea involved in the construction generalizes the theories of Lie-Poisson and semidirect product reduction to the nonholonomic setting. The constraint functions of the resulting bracket are Casimirs, so the constraints are satisfied automatically.

Speaker: **P. Lee** (University of Toronto)
Title: *Infinite-dimensional geometry of optimal mass transport*
Abstract: We consider the following nonholonomic version of the classical Moser theorem: given a bracket generating distribution on a manifold, two volume forms of equal total volume can be isotoped by the flow of a vector field tangent to this distribution. We discuss these results from the point of view of an infinite-dimensional non-holonomic distribution on the diffeomorphism groups. Furthermore, in the 60's Arnold showed that the Euler equation can be thought of as the geodesic flow on the group of volume-preserving diffeomorphisms. In a similar fashion, Otto showed that the mass transport problem can be consider as the geodesic problem on the space W of all volume forms with the same total volume. In particular, the space W can be regarded as the quotient of the group of all diffeomophisms by the subgroup of volume preserving ones, while the geodesic flow on the diffeomorphism group, given by the Burgers equation, is closely related to that on the space W. It turns out that this relation between diffeomorphism group and the space W can be understood via Hamiltonian reduction.