Modern Approaches in Asymptotics of Polynomials

Sunday, November 11 to Friday, November 16, 2007

Peter Borwein (Simon Fraser University), Doron Lubinsky (Georgia Institute of Technology), Ed Saff (Vanderbilt University).

1. Objectives

The focus of the conference is sequences of polynomials, their zeros, and asymptotic behavior – as well as related potential theoretic issues, such as distribution of points on a sphere. The aim is to bring together experts who have different approaches to these questions – for example those using potential theory, those mixing approximation and number theoretic techniques on integer polynomials, and those using Riemann-Hilbert techniques for asymptotics of sequences of (mainly orthogonal) polynomials. There has not been any meeting focusing on this cross-section of researchers in the past few years. We expect the communication of ideas and methods from these different approaches will encourage new techniques and research across several topics.

We also expect that the young researchers present will benefit from exposure to the leading different approaches.

2. Overview

RELEVANCE, IMPORTANCE AND TIMELINESS

In recent years, asymptotics of orthogonal polynomials have been used to study random matrices, combinatorial questions such as the longest increasing subsequence in a given sequence, Toda lattices, and weighted approximation. The potential theory that underlies some of these asymptotics has been used in distributing points on spheres and manifolds and in studying the distribution of zeros of sequences of polynomials. Zeros of integer polynomials, and the behavior of integer polynomials has been explored with a view to applications in number theory. The problems within the focus of the conference are widely applied, highly regarded, and very active areas of research. The conference is timely, and has a different focus from any other that we know of. At least 5 of the participants are young researchers (including graduate students and postdocs).

3. Scientific Program

The program consisted of nine one hour lectures, and 29 half hour lectures. There was ample time in between for questions, and discussion, with group discussions in the evenings, and on the free Wednesday afternoon.

4. A Review of the Meeting and Some Outcomes

There was a lively exchange between researchers in several different topics, including:
(I) Riemann-Hilbert methods for asymptotics of univariate orthogonal polynomials, with applications to random matrices;
(II) Classical operator theory methods for asymptotics of univariate orthogonal and related polynomials;
(III) Multivariate polynomials, with applications to multivariate orthogonal polynomials, distribution of points on manifolds; and multivariate approximation;
(IV) Extremal and integer polynomials with a number theoretic flavor, including applications to Diophantine approximation, Mahler measure, and nearly unimodular polynomials;
(V) Potential theoretic methods, including old problems like the capacity of the Cantor set, and newer ones, such as Saff's weighted approximation problem involving external fields, and the application of polynomial pullbacks to a host of problems;
(VI) Classical analytic techniques with applications to universality limits;
(VII) Approximation by rational and meromorphic functions in one variable, involving Hankel operators, Cauchy Transforms, and applications to numerical analysis.

Of course, many of these topics intersect, and this was reflected in the talks. Thus, for example, Arno Kuijlaars' hour long talk on a coupled random matrix model involved (bi)orthogonal polynomials, random matrices, Riemann-Hilbert asymptotic techniques, and Riemann surfaces. This depth was also evident in Percy Deift's introduction to Riemann-Hilbert techniques, and in Andre Martinez-Finkelshtein's treatment of Bessel processes and multiple orthogonal polynomials. Jinho Baik presented asymptotic expansions of Tracy-Widom distributions that arise in random matrix theory, combinatorics, and probability. Thomas Kriecherbauer showed how Riemann-Hilbert methods can be used to give new, and sharper insights into the old problem of the distribution of zeros of partial sums of exp(z). Ken McLaughlin showed how Riemann-Hilbert methods may be extended beyond the case of analytic fields, using the delta bar approximation method: only two derivatives are required of the external field. It was evident that the range of applicability, and the depth and power of Riemann-Hilbert methods is constantly expanding.

Barry Simon displayed the power of operator theoretic and classical analytic methods in obtaining asymptotics in very general situations for orthogonal polynomials on the real line and unit circle. Applications to universality limits were also part of Simon's focus. In a related vein, Irina Nenciu delivered a deep talk on the Ablowitz-Ladik equation, its relation to orthogonal polynomials on the unit circle, Toda flows, Poisson brackets, and Hamiltonian structures. One outcome is that the classical relationship between orthogonal polynomials on the unit circle and real line, generates a similar relationship between Ablowitz-Ladik and Toda flows.

Vili Totik explored the power of polynomial pullbacks and potential theory. Polynomial Pullbacks were introduced by Geronimo and Van Assche as a means of studying extremal and orthogonal polynomials on several intervals, and self similar sets. As developed by Vili Totik and others, they have been used to extend Markov-Bernstein inequalities, asymptotics for Christoffel functions, universality limits, asymptotics of best approximation to general compact sets. Franz Peherstorfer provided asymptotics for sup-norm Christoffel functions on one (and several) intervals, involving elementary functions and Zolotarev type polynomials. New methods for establishing universality limits for orthogonal polynomials were outlined by Doron Lubinsky.

An old problem in potential theory is the logarithmic capacity of the Cantor set. Thomas Ransford, author of a classic text on potential theory, presented a method that provides rigorous upper and lower bounds for this capacity, together with an extrapolation to the limit, which is approximately 0.220949102189507. This talk led to several discussions on how to extend and accelerate the current numerical procedures. Potential theory, but with external fields, was the focus of Peter Dragnev's talk. He showed how the iterated balayage algorithm can be used to deduce new information about the support of the equilibrium measure for Riesz energy problems on the sphere, as well as non-standard external fields involving signed equilibrium measures. David Benko discussed Saff's weighted approximation problem, which involves weighted polynomials and delicate potential theory with external fields.

The interplay between number theory and polynomial asymptotics was laid out in a variety of talks. Hugh Montgomery presented several such connections, including: polynomials with small Mahler measure, polynomials with integral coefficients that are small on the entire unit circle, and polynomials that arise in Turan's power sum method. Igor Pritsker showed just how close is the connection between classical complex analysis, potential theory, and number theory, through his the solution (joint with A. Baernstein) to an extremal problem involving moments of equilibrium measures. This has important applications to norms of factors of polynomials, Mahler measure, and minimal norm integer polynomials.

Also in a polynomial number theoretic vein, Peter Borwein outlined the recent solution to a very old problem of Littlewood on the number of zeros of cosine polynomials with coefficients 0 and 1. Littlewood conjectured that such a cosine polynomial with N terms cannot have much less than N zeros, but the conjecture turned out be false, there can be a lot less than N zeros. His colleague Stephen Choi continued the number theoretic theme with a penetrating study of Mahler measure and Lp norms of polynomials with coefficients -1 and 1 or -1, 0, and 1. Choi and his coworkers determined the asymptotic behavior of the mean value of the Mahler measure and Lp norm (averaged over the finitely many such polynomials of a given degree), as the degree grows to infinity. Karl Dilcher's most interesting talk dealt with the Stern sequence, and the polynomials with 0 and 1 coefficients that it generates. Tamas Erdelyi presented his most recent results on ultraflat polynomials, establishing necessary conditions that preclude properties such as conjugacy or skew-reciprocality. Ping Zhou showed an interesting technique to establish irrationality of certain multivariate series.

Multivariate problems and polynomials were the subject of talks by Zeev Ditzian, Jeff Geronimo, Doug Hardin, Sergiy Borodachov, and Andras Kroo. Doug Hardin dealt with the fundamental problem of distributing points "evenly" on a manifold. This of course has a close connection with Steven Smale's problem of distributing points on a sphere. Depending on the parameter in the Riesz potential, the minimal discrete energy used for finding the points, varies between a logarithmic energy problem, and a best packing one. Sergiy Borodachov continued this theme, with a careful analysis of the asymptotic behavior of minimal Riesz energy for arbitrary compact sets, not just rectifiable ones. While there need not be a limit of the scaled discrete energy, the lim sup may be expressed in terms of a certain outer measure that is independent of the specific set.

Jeff Geronimo presented recent results on bivariate analogues of the classical univariate Bernstein-Szego polynomials. These included recurrence relations, positive polynomials, and the factorization of the latter. Andras Kroo took a more approximation theoretic line, examining the density of homogeneous polynomials in spaces of continuous functions on convex and star-like surfaces. Kroo displayed the role of the geometry of the surface in determining density or its failure. Zeev Ditzian presented a unified approach to sharp Jackson inequalities that works for spherical harmonic polynomials on the unit sphere, but also in a variety of univariate contexts.

The classical and difficult topic of rational approximation was discussed by Laurent Baratchart, Hans-Peter Blatt, Vasiliy Prokhorov, Herbert Stahl and Maxim Yattselev. Laurent Baratchart focused on asymptotic uniqueness of best rational approximants in the L2 norm to Cauchy transforms of complex measures. Maxim Yattselev developed this topic further, providing asymptotics for the errors of best rational approximation. Vasiliy Prokhorov displayed the utility of Hankel operator techniques in rational approximation, with additional applications to meromorphic approximants, and the asymptotic distribution of poles of subsequences of Pade approximants. Herbert Stahl analysed best rational approximants on the negative real axis to functions of the form r(z)+s(z)exp(z), where r and s are fixed rational functions. This problem arises in numerical analysis, and leads to a generalization of the famous (1/9)th problem, which was solved by Goncar-Rakhmanov using techniques developed by Herbert Stahl. Hans-Peter Blatt explored the phenomenon of divergence of best rational approximants to a function on [-1,1], outside this domain. The classic example is the function |x|, whose diagonal best rational approximants do converge in the left and right-half planes, but whose off-diagonal rational approximants diverge.

A range of generalized orthogonal polynomials were considered by Erwin Mina-Diaz, Guillermo Lopez Lagomasino, Francisco Marcellan, and Avram Sidi. Erwin Mina-Diaz provided an exact series representation of orthogonal polynomials over an analytic Jordan curve, and Carleman orthogonal polynomials over the interior of that curve. This leads to strong asymptotics for the polynomials, as well as asymptotic information inside the curve, under additional conditions. Guillermo Lopez showed that it is possible to construct an analogue of the Stieltjes polynomials on the unit circle, leading to unit circle analogues of Gauss-Kronrod quadrature. Francisco Marcellan surveyed old and new results on asymptotics of Sobolev orthogonal polynomials. Avram Sidi described asymptotics of polynomials that are biorthogonal to systems of exponentials. The latter arise in convergence acceleration techniques, and in numerical integration of singular integrands.

Xin Li and Hrushikesh Mhaskar examined applications related to the general theory of orthogonal polynomials. Hrushikesh Mhaskar showed how the analyticity of functions can be checked at a point using polynomial frames. The latter use Cesaro means of orthonormal expansions. Xin Li discussed an extremal problem for functions in a weighted L2 space whose first N moments form a geometric progression. This leads to a new criterion for determinacy of the classical moment problem.

Many research problems were raised at the conference. These include:
(i) Methods for establishing universality limits in the bulk and edge of the spectrum, for the most general settings;
(ii) The convergence behavior of subsequences of diagonal Pade approximants that might replace the disproven Baker-Gammel-Wills conjecture, perhaps via Hankel theory methods;
(iii) New directions in asymptotics of Sobolev orthogonal polynomials;
(iv) Convergence acceleration for the sequences of upper and lower bounds for the capacity of the Cantor set, generated by the algorithm of Ransford and his coworkers;
(v) The zero distribution of sequences of polynomials that are biorthogonal to powers of a fixed function;
(vi) Zero distribution of Littlewood and similar polynomials, as well as mean statistics of various quantities for such classes;
(vii) Extensions of unit circle phenomena to the real line, or vice-versa, including flows and various quadratures;
(viii) Riemann-Hilbert analyses of partial sums of entire functions, and Riemann-Hilbert analyses of multi-orthogonal, and bi-orthogonal polynomials;
(ix) The support and behavior of equilibrium measures and related discrete and continuous energies on manifolds.