Mathematical Methods in Philosophy

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Mathematics and philosophy have historically enjoyed a mutually beneficial and productive relationship, as a brief review of the work of mathematician-philosophers such as Descartes, Leibniz, Bolzano, Dedekind, Frege, Brouwer, Hilbert, Gödel, and Weyl easily confirms. In the last century, it was especially mathematical logic and research in the foundations of mathematics which, to a significant extent, have been driven by philosophical motivations and carried out by technically-minded philosophers. Mathematical logic continues to play an important role in contemporary philosophy, and mathematically trained philosophers continue to contribute to the literature in logic. For instance, modal logics were first investigated by philosophers, and now have important applications in computer science and mathematical linguistics. The theory and metatheory of formal systems was pioneered by philosophers and philosophically-minded mathematicians (Frege, Russell, Hilbert, Gödel, Tarski, among many others), and philosophers have continued to be significantly involved in the technical development of proof theory, and to a certain degree also in the development of model theory and set theory. On the other hand, philosophers use formal models to test the implications of their theories in tractable cases. Philosophical inquiry can also uncover new mathematical structures and problems, as with recent work on paradoxes about truth. Areas outside mathematical logic have also been important in recent philosophical work, e.g., probability and game theory in inductive logic, epistemology, and the philosophy of science. In fact, it seems that technical mathematical work is currently enjoying something of a renaissance in philosophy.

The workshop on "Mathematical Methods in Philosophy" brought together eminent and emerging researchers who apply mathematical methods to current issues in philosophy. These mathematical methods come mainly from the fields of mathematical logic and probability theory, and the areas of application include philosophical logic, metaphysics, epistemology, the philosophy of mathematics, and the philosophy of science.

1 Overview of the Field

1.1 Philosophical logic

Philosophical logic includes logical systems such as logics of possibility and necessity (alethic modal logic), of time (temporal logic), of knowledge and belief (epistemic and doxastic logic), of permission and obligation (deontic logic). This area is unified by its methods (e.g., relational semantics, first introduced by philosophers Saul Kripke and Jaakko Hintikka in the 1950s, algebraic methods, proof theory), but it has diverse applications in philosophy. For instance, logics of possibility and time are mainly useful in metaphysics whereas logics of knowledge and belief are of interest to epistemology. However, the methods employed in the study of these logics is very similar. Other related logics which have important applications in philosophy are many-valued logics, intuitionistic logic, paraconsistent and relevance logics.

1.2 Foundations and philosophy of mathematics and computation

One of the main advances at the intersection of mathematics and philosophy is the development of foundations of mathematics in the early 20th century (type theory and set theory). Intra-mathematical considerations of course played a very important role in motivating such developments, but philosophical concerns shaped and drove much of this development. The same is true for the subsequent development of disciplines of mathmatical logic such as proof theory and computability theory.

1.3 Formal theories of truth and paradox

The nature of truth is a central topic in metaphysics and philosophy of logic, and work on truth is closely connected to epistemology and philosophy of language. Significant advances have been achieved over the last 30 years in formal theories of truth, and there are close connections between philosophical work on truth and model theory (especially of arithmetic). One of the most important approaches to truth are the revision theories of truth first introduced by Saul Kripke and Solomon Feferman.

1.4 Formal epistemology

Formal epistemology is an emerging field of research in philosophy, encompassing formal approaches to ampliative inference (including inductive logic), game theory, decision theory, computational learning theory, and the foundations of probability theory.

1.5 Set theory and topology in metaphysics

Set theory has always had a close connection with mereology, the theory of parts and wholes, and topology has also been fruitfully applied in metaphysics.

2 Presentation Highlights

2.1 Philosophical Logic

Steve Awodey and **Kohei Kishida** (Carnegie Mellon University), *Topological semantics for first-order modal logic*

Awodey presented a new theorem, which extends Tarski's classical topological completeness result from propositional to first-order S4 modal logic.

Dave DeVidi (University of Waterloo), *Non-constructive uses of constructive logics*

While intuitionistic and other constructive logics have their first home in foundations of mathematics, their appearance in, for instance, metaphysical debates apart from mathematics is familiar, thanks to the work of philosophers such as Michael Dummett. In such cases, the reasons offered for supposing that constructive logic is correct are recognizably akin to those offered by mathematical constructivists. In recent times, though, it has become increasingly common to see versions of constructive logic advocated for philosophical purpose—as part of a solution to a paradox, for instance—when no appeal to constructivist motivations is offered and no plausible one seems possible. It is not uncommon to see such proposals rejected on the grounds of incompatibility with constructivism ("no intuitionist could consistently say *that*"). This objection is beside the point if the appeal to constructive logic has some suitable non-constructivist motivation—for then the name "intuitionistic logic" (e.g.) becomes a historical curiousity, instead of an indication of who may appeal to that logic as the correct one. This response is often claimed, but seldom defended. DeVidi described some cases of this sort, and considered the prospects for giving a non-constructivists but philosophically satisfactory defense of the claim that some-or-other constructive logic is the correct one for certain purposes.

Eric Pacuit (University of Amsterdam) and Horacio Arlo-Costa (Carnegie-Mellon University), *Quantified Classical Modal Logic and Applications*

Pacuit introduced and motivated the study of classical systems of first-order modal logic. In particular, he focussed on the study of neighborhood frames with constant domains and offer a series of new completeness results for salient classical systems of first order modal logic. He discussed general first-order neighborhood and offer a general completeness result for all classical systems of first-order modal logic. Finally, he showed how to extend this analysis to freely quantified classical modal logic.

Graham Priest (University of Melbourne and University of St. Andrews), *Many-valued modal logic*
In standard modal logics, the worlds are two-valued. There is no reason why this has to be the case, however: the worlds could be many-valued. In this talk, Priest looked at many-valued modal logics. He started with the general structure of such logics. To illustrate this, he considered modal logic based on Łukasiewicz's continuum-valued logic. Priest then considered one many-valued modal logic in more detail: modal First Degree Entailment (FDE). Tableaux for this and its special cases (K_3 and LP) were provided. Modal many-valued logics engage with a number of philosophical issues. The final part of the talk illustrated with respect to one such: the issue of future contingents.

Timothy Williamson (University of Oxford), *Adding probabilities to epistemic logic*
Williamson used a case study to illustrate the philosophical interest of adding epistemic probabilities to standard possible world models of epistemic logic. It is familiar that the non-transitivity of the accessibility relation between worlds corresponds to the failure of the KK principle—if you know, you know that you know. How far can we turn the screw with counterexamples to the KK principle? That is, how low can your epistemic probability that you know p go at a world at which you do in fact know p? Answer: As close to 0 as you like. Some of the relevant models can be instantiated in quite realistic settings. Willimason considered implications for debates about the standard of epistemic warrant required for assertion and about apparent counterexamples to otherwise plausible closure principles for knowledge.

2.2 Theories of truth and paradox

JC Beall (University of Connecticut) and **Michael Glanzberg** (University of California, Davis), *Truth and paradox*
Beall and Glanzberg aimed to give a big-picture sketch of truth and paradox – chiefly, the Liar (but also related truth-theoretic paradoxes).

Approaches to the Liar that they mentioned are all marked by the ways they navigate between completeness and consistency. Some key examples of these approaches include those which:

- Reconsider logic:
    1. Paraconsistent: the Liar teaches us that EFQ fails, that some sentences are true and false, but our language is nonetheless non-trivial (i.e., some sentences are 'just true').
    2. Paracomplete: the Liar teaches us that LEM fails, in some way that avoids a variant Liar which reinstates the paradox.
- Reconsider the semantics:
    1. Contextual: the Liar teaches us that truth is contextually sensitive, shifting the extension of 'true' from context to context.
    2. Revision Theory: the Liar teaches us that 'true' is governed by *rules of revision*.

Each of these options seeks to reject some portion of consistency or completeness, and yet present a coherent and appealing environment in which logic and semantics can coherently proceed.

Solomon Feferman (Stanford University), *A nicer formal theory of non-hierarchical truth*
A new formal theory S of truth extending PA is introduced, whose language is that of PA together with one new unary predicate symbol $T(x)$, for truth applied to Gdel numbers of suitable sentences in the extended language. Falsity of x, $F(x)$ is defined as truth of the negation of x; then the formula $D(x)$ expressing that x is a determinate meaningful sentence is defined as the disjunction of $T(x)$ and $F(x)$. The axioms of S are those of PA extended by (I) full induction, (II) strong compositionality axioms for D, and (III) the recursive defining axioms for T relative to D. By (II) is meant that a sentence satisfies D if and only if all

its parts satisfy D; this holds in a slightly modified form for conditional sentences. The main result is that S has a standard model. As an improvement over earlier systems developed by Feferman, S meets a number of leading criteria for formal theories of truth that have been proposed in the recent literature, and comes closer to realizing the informal view that the domain of the truth predicate consists exactly of the determinate meaningful sentences.

Volker Halbach (University of Oxford), *The Kripke-Feferman theory of truth*
Feferman proposed to axiomatize Kripke's theory of truth in classical logic. The resulting theory is called the Kripke-Feferman (KF) theory of truth. I argue that this theory introduces some unwanted features because it relies on classical logic, and that Kripke's theory should be axiomatised in partial logic.

It has been argued by Reinhardt that nevertheless KF may be taken as a tool for generating theorems of a theory of truth in partial logic by focusing on those sentences A that can be proved to be true in KF. Halbach argued that this justification of KF fails, as a natural axiomatisation of Kripke's theory in partial logic is proof-theoretically much weaker than the theory generated by KF.

Jeff Ketland (University of Edinburgh), *Truth and reflection*
Say that a truth theory is deflationary if, when added to any of a suitable class of base theories, the result is a conservative extension. Say that a truth theory is reflective if, when added to any of a suitable class of base theories, the reflection principles for that theory become theorems. Reflective truth theories are desirable; for, just as when we accept a statement A, we should accept "A is true," similarly, when we accept a theory T, we should accept "All axioms of T are true." Stewart Shapiro (1998) and Ketland (1999) noted that these conditions are incompatible: reflective truth theories are non-conservative, and thus non-deflationary. In particular, Tarski's compositional theory of truth is reflective (as are more sophisticated "self-applicative" truth theories, such as the Kripke-Feferman theory), and thus non-deflationary. It seems correct to conclude then that deflationism about truth is incompatible with results in mathematical logic. Several authors (Field, Azzouni, Halbach and Tennant) have presented responses to this argument against deflationism. Ketland surveyed these responses and offered some replies.

Greg Restall (University of Melbourne), *Modal models for Bradwardine's theory of truth*
Restall introduced Stephen Read's reconstruction of Bradwardine's theory of truth, and provided it with a simple model theory. This model theory can be used to provide a fixed-point construction to extend any classical theory with a Bradwardine truth predicate which diverges from Tarskian truth only on ungrounded sentences.

2.3 Foundations and philosophy of mathematics and computability

Grigori Mints (Stanford University), *Effective content of non-effective proofs*
Methods of proof theory allow to extract effective bounds from some non-effective proofs and point out possibilities of obtaining sharper bounds from mathematical proofs, sometimes depending of fewer parameters. We survey several such applications and illustrate the approach for a proof of Herbrand's theorem using compactness. A new cut elimination method (in particular a new proof of Herbrand's Theorem) is obtained here by "proof mining" (unwinding) from the familiar non-effective proof. That proof begins with extracting an infinite branch when the canonical search tree for a given formula of first order logic is not closed. Our reduction of a cut does not introduce new cuts of smaller complexity preserving instead only one of the branches.

Wilfried Sieg (Carnegie Mellon University), *Church without dogma: axioms for computability*
Church's and Turing's theses dogmatically assert that an informal notion of computability is captured by a particular mathematical concept. Sieg presented an analysis of computability that leads to precise concepts, but dispenses with theses.

To investigate computability is to analyze processes that can in principle be carried out by calculators. Drawing on this lesson we owe to Turing and recasting work of Gandy, Sieg formulated finiteness and locality conditions for two types of calculators, human computing agents and mechanical computing devices; the distinctive feature of the latter is that they can operate in parallel.

The analysis leads to axioms for discrete dynamical systems (representing human and machine computations) and allows the reduction of models of these axioms to Turing machines. Cellular automata and a variety of artificial neural nets can be shown to satisfy the axioms for machine computations.

2.4 Mathematics and logic in metaphysics

Gabriel Uzquiano (University of Oxford) and Stewart Shapiro (The Ohio State University), *Ineffability and reflection*

We know that not all concepts have extensions associated with them. In this contribution, Uzquiano and Shapiro explored the hypothesis that a concept F lacks an extension if and only if F is ineffable, by which they mean, roughly, that no concept at least as large as F is describable by logical vocabulary alone. They are interested in this hypothesis largely because it seems to them to give partial expression to the inchoate thought that the universe is ineffable. A first approximation to this thought takes the form of a second-order reflection schema on which, given a concept H at least as large as a concept F to which no extension corresponds, a sentence of pure second-order logic is true when relativized to the instances of H only if it is true when relativized to strictly fewer objects. We

One may be able to express the thought behind this reflection schema in finite compass by a sentence of a third-order language. However, once we allow ourselves the resources to do this, we find ourselves in a position to describe what is for a concept to be ineffable by the vocabulary of pure third-order logic, which betrays the very thought with which we started. This situation generalizes and, in their contribution, Uzquiano and Shapiro looked at the tension between, on the one hand, the drive to express the ineffability of the universe and, on the other, the constraint to remain faithful to it.

Harvey Friedman (The Ohio State University), *Concept calculus*

Friedman's Concept Calculus provides an unexpected exact correspondence between ordinary everyday thinking about ordinary everyday things and abstract mathematics.

As an example, Friedman identified a large range of principles involving just the two informal binary relations "better than" and "much better than," which give rise to a variety of formal systems which are mutually interpretable with a variety of standard formal systems from logic whose strengths range from weak arithmetics to various large cardinals.

It appears that an enormous range of informal concepts lend themselves to closely related investigations. For example, we have developed a kind of naive physics based on informal notions of time and space and point mass, which also corresponds, by mutual interpretation, to these same formal systems from logic.

The hope is that concept calculus can serve as a tool for organizing and analyzing metaphysical concepts that is in rough analogy with the way that the Newton/Leibniz calculus serves as a tool for organizing and analyzing physical concepts.

2.5 Philosophical issues and logic

Delia Graff Fara (Princeton University), *Relative identity and de re modality*

Fara defended the materialist thesis that material things are identical to the matter that composes them, by appealing to the semantic view that names are predicates; and by proposing and investigating a version of David Lewis's counterpart theory that appeals, rather than to Lewis's own modfied similarity relation, to relations of *relative* identity in the analysis of *de re* temporal and modal claims. This was carried out in the context of a metaphysics that's both actualist and three-dimensionalist.

Hannes Leitgeb (University of Bristol), *Applications of mathematics in philosophy: four case studies*

As we all know, mathematical methods are of crucial importance in science. Many believe that mathematics will play a similar role for philosophy once philosophical theories have reached a sufficient degree of complexity; to some degree, this has already happened. Leitgeb tried to support this thesis by stating four examples which are chosen (conveniently) from his own work:

1. Similarity, Properties, and Hypergraphs
2. Nonmonotonic Logic and Dynamical Systems
3. Belief Revision for Conditionals and Arrow's Theorem
4. Semantic Paradoxes and Non-sigma-Additive Probability Measures

Yiannis Moschovakis (UCLA), *Synonymy*

Moschovakis discussed some historical approaches to synonymy, and focussed on the mathematical problems of decidability which arise when senses are modelled rigorously in formalized fragments of language. About half of the talk was be dedicated to an exposition of the theory of referential intensions, by which (in slogan form) *the sense of a term is the natural algorithm which determines its denotation*. This modelling of meanings leads to both theorems and difficult open problems in the logic of synonymy.

Gillian Russell (Washington University, St. Louis), *One true logic?*

In their 2006 book *Logical Pluralism*, Beall and Restall argue that there is more than one correct logic. Russell examined that claim and present a different argument for a similar view.

Kai Wehmeier (University of California, Irvine), *Identity is not a relation*

Frege, Russell, and the early Wittgenstein all struggled with the notion of a binary relation that every object bears only to itself. In the *Tractatus* we even find an outright rejection of the notion, together with some gestures as to its eliminability from predicate logic. In the talk, Wehmeier sketched what seems to be the most promising argument against the existence of a binary relation of numerical identity, and discuss a few related logical issues.

Byeong-Uk Yi (University of Toronto), *Is logic axiomatizable?*

Yi defended the negative answer to the question in the title, "Is logic axiomatizable?," by considering sentences that involve plural constructions. He also compared his argument for the non-axiomatizability of logic with the usual argument for the non-axiomatizability of second-order logic and with Tarski's ω-consequence example in the beginning of his paper "On the concept of logical consequence," and how it relates to David Kaplan's proof of inexpressibility of certain sentences in elementary languages.

3 Outcomes of the Meeting

3.1 Surveys

One particular aim of the workshop was to provide the participants with a sense of the range of topics, the state of current research, the interconnections, and the important trends are in philosophical logic and related areas. To this end, the organizers invited three survey talks. These surveys provided an overview of the development of the field in the last 20–50 years, of the current state of the art, of the main open problems, and of anticipated future trends and developments:

Branden Fitelson (University of California, Berkeley), *Survey on formal epistemology: Some propaganda and an example*

Fitelson discussed various threads of "formal philosophy," as he prefers to call the field of formal epistemology. He gave a survey of the development of confirmation theory and the uses of probability theory in it, and ended with an illustrative application of confirmation theory to the problem of induction.

Markus Kracht (UCLA), *The certain past and possible future of modal logic*

The origins of modal logic are somewhere in philosophy. However, for more than fifty years there is also a more "technocratic'; approach to the field that applies mathematical methods. Over time, it has created its own terminology and, inevitably, its own problems that it likes to deal with. Other areas of application have also been found, for example computer science. While the techniques and results for propositional modal logic are by now fairly widely known even outside the circle of mathematicians, in the domain of modal predicate logic there still is some lack of knowledge transfer between philosophers and mathematicians. Kracht outlined the past developments of modal logic with special attention to modal predicate logic, where, he argued, the greatest promise for 'technocratic' modal logic is still to be found.

Stewart Shapiro (The Ohio State University), *Life on the ship of Neurath: mathematics in the philosophy of mathematics*

Shapiro gave an "idiosyncratic" survey of the use of mathematics to support or otherwise assess programs in the philosophy of mathematics. It covered the "big three" views that dominated thinking in the early decades of the twentieth century: formalism, intuitionism, and logicism, and then moveed onto contemporary descendants of these views: *ante rem* structuralism, Scottish neo-logicism, fictionalism, and various reconstructive nominalisms.

3.2 Proceedings

A proceedings volume collecting selected papers from the workshop is planned. It will apear as a special issue of the *Journal of Philosophical Logic*.

4 Acknowledgements

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