L-functions, ranks of elliptic curves, and random matrix theory

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July 8 – July 13, 2007

1 Overview of the Field

The group of rational points on an elliptic curve is one of the more fascinating number theoretic objects studied in recent times. The description of this group in terms of the special value of the L-function, or a derivative of some order, at the center of the critical strip, as enunciated by Birch and Swinnerton-Dyer is surely one of the most beautiful relationships in all of mathematics; also it's understanding carries a \$1 million dollar reward!

Random Matrix Theory (RMT) has recently been revealed to be an exceptionally powerful tool for expressing the finer structure of the value-distribution of L-functions. Initially developed in great detail by physicists interested in the statistical properties of energy levels of excited nuclei, RMT has proven to be capable of describing many complex phenomena, including average behavior of L-functions.

The most important invariant of an elliptic curve is the rank of its (Mordell-Weil) group of rational points; it is a non-negative integer, believed to be 0 or 1 for almost all elliptic curves. The beginnings of the subject is a conjecture (see [1]) about how often the rank is greater than or equal to 2 for the family of quadratic twists of a given elliptic curve. Each elliptic curve has an L-function associated with it; this is an entire function which satisfies a functional equation. The Birch and Swinnerton-Dyer conjecture asserts, among other things, that the order of vanishing at the central point of the L-function associated with an elliptic curve is equal to the rank. It is generally conjectured that almost all elliptic curves have rank zero or one according to whether the sign of the functional equation of the related L-function is $+1$ or -1. Rank two curves should occur with L-functions that have a $+1$ sign of their functional equation but vanish nevertheless at the central point. These are expected to be rare; the question of how rare is the subject here.

If the elliptic curve is given by $E : y^2 = x^3 + Ax + B$, and if d is a fundamental discriminant, then the quadratic twist of E by d is the elliptic curve $E_d := dy^2 = x^3 + Ax + B$. The conjecture, derived from RMT and number theory, is that E_d will have rank 2, or greater, for asymptotically $c_E x^{3/4}(\log x)^{b_E+\frac{3}{8}}$ values of d with $|d| \le x$; here b_E is one of four values (see [7]):

- $b_E = 1$ when E has full rational 2-torsion,
- $b_E = \frac{\sqrt{2}}{2}$ when E has one rational 2-torsion point,
- $b_E = \frac{1}{3}$ when E has no rational 2-torsion and the discriminant is a perfect square,
- $b_E = \frac{\sqrt{2}}{2} - \frac{1}{3}$ when E has no rational 2-torsion and the discriminant is not a square.

The constant c_E is yet to be determined, but depends on a mix of RMT, number theory, and probabilistic group theory; see [10].

This conjecture, while interesting, is not as compelling as it might be because of our ignorance of c_E. However, an absolutely convincing case for RMT can be given by considering curves of rank 2 or higher, as above, but divided into *arithmetic progressions* of d modulo some prime p.

Using RMT arguments combined with a number theoretic discretization of the problem, one is led to predict that if a is a quadratic residue mod p and b is a quadratic non-residue then the ratio of rank 2 or higher twists among $d \equiv a \bmod p$ to $d \equiv b \bmod p$ is, in the limit,

$$R_p := \sqrt{\frac{p+1-a_p}{p+1+a_p}}$$

where $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is the L-function associated with E. Those familiar with the conjecture of Birch and Swinnerton-Dyer might not be surprised to see the ratio

$$\frac{p+1-a_p}{p+1+a_p}$$

show up; however, it is the square-root, contributed by RMT, that is the surprise. Here is some data on quadratic twists of the elliptic curve of conductor 11 where the absolute value of the discriminant is at most $T = 333605031$ (here $R_p(T)$ is the computed ratio of positive even ranks amongst twists by discriminants in square residue classes modulo p to those in non-square residue classes modulo p, i.e. the empirical approximation to R_p):

p	a_p	R_p	$R_p(T)$
3	-1	1.2909944	1.2774873
5	1	0.8451542	0.84938811
7	-2	1.2909944	1.288618
13	4	0.74535599	0.73266305
17	2	1.118034	1.1282072
19	0	1	1.000864
23	-1	1.0425721	1.0470095
29	0	1	0.99769402
31	7	0.80064077	0.78332934
37	3	0.92393644	0.91867671
41	-8	1.2126781	1.2400086
43	-6	1.1470787	1.1642671
47	8	0.84515425	0.82819492

The basic calculation to obtain this result involves a ratio of conjectures for

$$\sum_{\substack{d \equiv a \bmod p \\ d \leq x}} L_{E_d}(1/2)^{-1/2};$$

the reason that one takes the -1/2 power here is due to the fact that the s'th moment of characteristic polynomials of even orthogonal matrices has its rightmost pole at $s = -1/2$. The description of this calculation and the compelling numerical evidence is in the paper [1]. The calculation is taken a step further in the paper of Conrey, Pokharel, Rubinstein, and Watkins [3] where lower order terms for the moments are incorporated and lead to an even more precise evaluation of these ratios.

The conjectures about quadratic twists can be generalized to cubic twists in two different ways. One involves the frequency of rank 2, or greater, elliptic curves within the classical family $E_m := x^3 + y^3 = m$. Here we restrict attention to m for which the sign of the functional equation of the elliptic curve associated with E_m is $+1$. One expects that amongst such m for primes $p \equiv 2 \bmod 3$ the m for which E_m has at least rank 2 will be evenly distributed among the residue classes modulo p. For primes $p \equiv 1 \bmod 3$ the conjecture is more interesting. Here we define $a_c(p)$ to be one of the three solutions of $a \equiv 2 \bmod 3$ with $4p = a^2 + 3b^2$. For example, we have $a_c(7) = 5$ for $c = 3, 4$; $a_c(7) = -1$ for $c = 1, 6$; and $a_c(7) = -4$ for $c = 2, 5$. Then, the ratio of the frequency of curves of at least rank 2 amongst $m \equiv c_1 \bmod p$ compared with $m \equiv c_2 \bmod p$ is conjecturally

$$\sqrt{\frac{p+1-a_{c_1}(p)}{p+1-a_{c_2}(p)}}.$$

Notice that

$$\sqrt{\frac{7+1-a_{2,5}(7)}{7+1-a_{3,4}(7)}} = 2 \qquad \sqrt{\frac{7+1-a_{1,6}(7)}{7+1-a_{3,4}(7)}} = \sqrt{3} \qquad \sqrt{\frac{7+1-a_{2,5}(7)}{7+1-a_{1,6}(7)}} = \frac{2\sqrt{3}}{3}$$

so that, conjecturally, curves of rank two or higher come up twice as often for $m \equiv 2 \bmod 7$ as for $m \equiv 3 \bmod 7$. Here is some data for the number of curves with rank r greater than zero for each residue class $c(\mathrm{mod} 7)$:

c	$\#r > 0$	# curves	ratio
1	109569	595982	0.184
2	125728	595952	0.211
3	59440	595912	0.100
4	58759	595903	0.099
5	125714	595963	0.211
6	110125	595937	0.185

The other way to do a cubic twist is to take a fixed elliptic curve E and a Dirichlet character χ of order 3 and consider the twisted L-function, $L_E(s, \chi) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$. David, Fearnley, and Kisilevsky [5] have shown, very surprisingly, that such twists vanish for about $x^{1/2}$ cubic twists of modulus $\leq x$, and have given precise conjectures, based on RMT, for the asymptotic frequency of this event. They also consider quintic twists and conclude that there are (barely!) infinitely many order five characters for which the twisted L-function vanishes at the central point. These predictions are based on considerations with groups of unitary matrices, whereas the previously mentioned conjectures arise from calculations on groups of orthogonal matrices.

It is interesting to begin with a weight 4 modular newform f, with integer Fourier coefficients, and similarly ask about vanishing of, say, quadratic twists of the associated L-function. In this case it is expected that there will be asymptotically $c_f x^{1/4} (\log x)^{b_f}$ vanishings at the central point of the quadratically twisted L-functions . The possible values of b_f have not been worked out here; however, if one restricts to prime discriminants, then the power on the log is expected to be $-5/8$ in both this case and the case of twists of elliptic curve L-functions. If one considers weight 6 or higher, it is expected that there will only be finitely many vanishings of quadratic twists of the associated L-functions. It is not clear what happens if one considers all such weight 6 forms and all of their twists if one then accumulates infinitely many vanishings. There is an arithmetic significance to the vanishings of the twists of the weight 4 modular forms: it is related to the rank of an associated Chow group, about which we hope to say more at a later time.

In the twists mentioned here, of course we only consider the twists for which there is a plus sign in the functional equation.

The evidence for many of the above conjectures has been accumulated by a combination of forces: Tornaria, Rodriguez-Villegas, Rosson, Mao, and Rubinstein. Much of it is based on an algorithm of Gross for finding the half-integral weight form, as a theta series involving ternary quadratic forms, whose Fourier coefficients yield the values of the twisted L-series at the central point. Prior to a workshop at the Newton Institute in February 2004, only a handful of such theta series were known. During that workshop, the first four people above worked out the obstacles to further progress and provided literally thousands of examples to the last named person who computed hundreds of millions of values for each; this provides a nice data bank for testing conjectures.

All of the above discussion has been focused on curves of even rank two or higher. The question of modeling rank 3 or higher members of a family is much more difficult; in fact it is not at all satisfactorily addressed. In the case of trying to determine rank 2 quadratic twists the Random Matrix model is based on a discretization arising from the beautiful formula, due in this form to Kohnen and Zagier [14]:

$$L_{E_d}(1/2) = \kappa_E \frac{c_E(|d|)^2}{\sqrt{|d|}}$$

where $c_E(|d|)$ is an integer and $\kappa_E > 0$. In the case of trying to determine the frequency of odd rank > 1 amongst quadratic twists, we consider the conjectural formula of Birch and Swinnerton-Dyer for the value of the derivative of an odd $L_{E_d}(s)$:

$$L'_{E_d}(1/2) = \kappa_E \frac{h_{E_d}|\mathrm{Sha}_{E_d}|}{\sqrt{d}}$$

where h_{E_d} is the height of a generating point and $|\mathrm{Sha}_{E_d}|$ is the order of the Tate-Shafarevich group. Now we don't know what kind of discretization to give h_{E_d}. It could conceivably be as small as $\log|d|$ but statistically this does not seem to be the correct model. By the work of Snaith [27], the right-most pole of the derivative of the sth moment of characteristic polynomials of odd orthogonal matrices occurs at $s = -3/2$. This might suggest, if one uses the discretization $(\log|d|)/\sqrt{|d|}$, that there are only about $x^{1/4}$ rank 3 curves among the family of twists with conductor smaller than x. However, Rubin and Silverberg [25] give examples of E which have many more rank 3 quadratic twists. In examining the limited data we have about rank 3's, an interesting phenomenon seems to appear: it looks as though $L'_{E_d}(1/2)$ cannot be as small as $(\log|d|)/\sqrt{|d|}$. Is it possible that when Sha is small then the height of a generating point is big and vice-versa? This linkage does not seem unnatural if one compares for example to the situation of the class number of a real quadratic field. There one finds that the product of the regulator times the size of the class group is always about the size of the square root of the discriminant. However, this analogy may not be correct, since this involves L-functions at the edge of the critical strip whereas we are discussing values at the center. Much more data is needed to make an informed conclusion.

2 New Directions

The initial goals for this workshop were to:

- Give a good statistical model for rank three quadratic twists.
- Find a fast algorithm to compute which curves from a quadratic twist family have rank 3.
- Understand why the size of the regulator times the size of Sha is so big for rank one curves.
- Produce theta series for thousands of weight 4 and 6 forms.
- Find an algorithm to give the theta series for odd weight form.
- Refine our understanding of the value of c_E.
- Investigate the constants $b_3(E)$ and $c_3(E)$ which arise in the case of cubic twists.

But as the workshop developed it transformed into being much more about RMT and statistics of higher rank L functions. And so a big motivation for the workshop was to see if arithmetical applications of Random Matrix Theory could be found for more exotic or higher rank L-functions. For example, there are conjectures due to Böcherer and Schulze-Pillot about the special values at the central point of L-functions associated with Siegel modular forms. One wonders whether Random Matrix models can be used to predict the frequency with which these values vanish and in general what their value distribution is. The discussion of these topics has led to a desire to be able to do explicit calculations with these L-functions. Almost nothing has been done in this direction, but a specific outcome of the workshop is the resolution to compute these.

Another example of the investigation higher rank is the work of Martin and Watkins [18] on central values of symmetric powers of L-functions of elliptic curves. For example, they find many suspected double zeros of the symmetric cubes of these. RMT should be able to use an appropriate discretization based on conjectural formulas for the special values to predict the frequency with which this happens.

Watkins [32] also reported on the implementation of an algorithm using Heegner points to check for rank 3 curves elliptic curves in families of quadratic twists in essentially the same amount of time that it now takes to do rank 2 curves. He counts rank 3 curves from a family of quadratic twists and gives evidence that they are distributed as

$$\left(\frac{p+1-a_p}{p+1+a_p}\right)^{-3/2}.$$

Watkins also has conjectures based on RMT about how many elliptic curves of conductor up to X will have rank 2, namely $cX^{\frac{19}{24}}(\log X)^{\frac{3}{8}}$.

The families mentioned above are all families of quadratic twists of a given curve. David, Fearnley and Kisilevsky have been investigating families of odd order twists, especially twists by Dirichlet characters of orders 3, 5, and 7. They predict $c_3(E)x^{1/2}(\log x)^{b_3(E)}$ rank two curves among cubic twists, while about $(\log x)^c$ rank two curves among quintic twists.

Another area of interest concerns the generalization wherein the modular form of weight 2 associated with the elliptic curve is replaced by a modular form of weight 4. In this case the arithmetic question is about the rank of a Chow group. Some of these forms are associated with Calabi-Yau threefolds (see for example the work of Helena Verrill) and so are of interest to a wide community of scientists. A small amount of data has been gathered here; more would be extremely valuable. For weight 6 newforms, only a finite number of twists are expected to vanish. Investigating how this finite number varies with the level is of interest.

Considering vanishing for twists of odd weight rational new forms would also be very interesting. No one knows any theta series to assist with numerical experiments here.

Another application for these ideas is in the context of Siegel modular forms which also have formulas for the special values of their twists.

3 Presentations

The workshop brought together people from the Random Matrix side of things with number theorists. Therefore, it was decided to feature some introductory lectures from both fields. The lecture of Rubinstein and two lectures by Keating provided this introduction on the Random Matrix Theory side of things, while Kohnen gave a series of three lectures introducing Siegel modular forms. In addition, there was a lengthy discussion session on Friday morning; a very loose transcript of that is given below.

- Monday
  - 9:00 Rubinstein: Probability models for elliptic curves Corbett Hall
  - 10:30 Darmon: Survey of special values of L-functions, BSD, and Heegner points
  - 14:30 Keating: Random matrix theory I
  - 16:00 Kohnen: Siegel modular forms I
- Tuesday
  - 9:30 Mao: Shimura correspondence and computation of L-values
  - 11:00 Darmon: Shintani lifts, p-adic families, and derivatives of quadratic twists
  - 14:00 Kohnen: Siegel modular forms II
  - 15:30 Keating: Random matrix theory II
- Wednesday
  - 9:00 Watkins: Non-trivial vanishings of odd quadratic twists
  - 10:30 Delaunay: Odd rank quadratic twists of elliptic curves
- Thursday
  - 9:30 Kisilevsky: Ranks of Elliptic Curves in Families of Cubic Extensions

- 11:00 Rubin: Ranks of elliptic curves in families of quadratic twists
- 14:00 Kohnen: Values of spinor zeta functions at the central point
- 15:30 Miller, Duenez, and Huynh: Finite conductor models for zeros of elliptic curves

- Friday
  - 9:00 - 12:00 Discussion and wrap-up

4 Discussion Topics

4.1 Statistics of N consecutive eigenvalues

Mike Rubinstein: Instead of looking at $U(N)$ for finite N, let M tend to infinity and look at N consecutive normalized eigenvalues of $U(M)$.

David Farmer: There's no way that this will be what we want. In the large M limit we have a characteristic function of whether or not overlap. Nina and Jon's picture: hasn't converged to the Gaussian, won't appear. Will throw away all the lower order terms.

Mike R: Is that a theorem?

David F: Take a huge matrix, segment of its characteristic polynomial will be exactly Gaussian and not skewed Gaussian.

Mike R: So then David thinks we can easily check wi' a single statistic, numerical statistic.

David F: yes.

Mike R: Eigenvalues will still repel. Jon?

Jon Keating: You said 2×2 matrices. There are some statistics where answer is extremely close to infinite. Local statistics largely independent of size of matrix block. Long range statistics.

David F: leading order close, not lower order. Lower order will fall apart instantly if M is not the right size.

Mike R: this is the better model to fit what we're doing with L-functions. (The mixed Hadamard and primes model).

4.2 Weyl Measure: Is it RMT or the underlying distributions that are significant for L-functions

David Farmer: In Keating's talk he showed calculations involving Weyl integration formula. Everything done in subject only uses that measure. Now, if had been told that measure but not where it came from, would you think RMT is the right thing as to where it came from, or something else? Only the *measure* on the eigenvalues we use; might be missing something (only using measure).

Mike R: What about function field analogue, different families (unitary, symplectic, orthogonal). How are the L-functions choosing?

David F: sounds natural, but not necessarily right.

Jon Keating: famous: arrival times of bus in Mexican town, spacing b/w parked cars in London. Same measure, no matrix. Comes from an entropy formula. Appears elsewhere.

David F: take N points at the origin, Brownian motion to the unit circle without allowing crossing. Unitary statistics. What if someone offered this as the hypothesis as the behavior of the L-functions.

4.3 Packaging twists by cubic Dirichlet characters

Hershy Kisilevsky: L-values from cubic twists. Envious of Rubinstein who can use half-integral weights to get millions of computations. Find some way to package things.

Henri Darmon: Might be a paper in some special situations.

Mike Rubinstein: Maybe fast Fourier technique for the ternary calculations. Maybe D^{ϵ} for a single computation on average.

Brian Conrey: to calculate one value is $\sqrt{T}$ steps, but to calculate T of them it takes about $T^{1+\epsilon}$ (O-S). Maybe if we organize it to recognize things being repeated we can save a $\sqrt{D}$.

4.4 Many (10^{12}) quadratic twists

Mike R: This would be another project, in the back of my mind. Today might be possible to do 10^{10} to maybe 10^{12} (pushing it).

Jack Fearnley: we're at about 10^6.

Mike R: It's on a logarithm scale. There are some secondary terms which we will see a bit of an improvement on.

Brian C: certain asymptotics that we just won't be able to see. Many problems trying to modify $X^A \log^B X$, tough to get at our level of data.

4.5 Interpolating integer moments

Brian C: Paul has a formula for

$$\int_{U(N)} |\Lambda(1)|^{2k-r} |\Lambda'(1)|^{2r} d\mu \tag{1}$$

for integers r and k: analytic continuation? (Note: may have the wrong exponents for the Λ's.)

Paul: If you fix k, can only evaluate for a finite number of r's. Integer points under a line in the (r, k)-plane. Know it is not the correct formula if continue. Use finitely many points, analytically continue above the line, but can get to that point by analytically continuing horizontally (which know is correct), but get two different answers.

David F: there is at least one example where replacing factorials with Gamma functions doesn't give the right thing (when going from integers to non-integers). Always say there are formulas that interpolate. There are so many of them that surely some are wrong. Here is an example. I would like to understand when it is okay to interpolate,

Mike R: related to the question: can we develop an analytic theory for the full moments. Leading term is just $a_k g_k$. Can we develop heuristics (proofs are beyond us) for the analytic theory of the full moment:

$$\int_0^T |\zeta(1/2+it)|^{2k} \, dt \; = \; \int_0^T P_k \left(\log \frac{t}{2\pi}\right) dt + O\left(T^{1/2+\varepsilon}\right). \tag{2}$$

Here we can do it for k an integer, want to interpolate for all k. We need a difference approach to the moments that will allow us to work it out. Maybe a refined version of the hybrid formula and the independence hypothesis.

4.6 Independence in the hybrid formulas

Mike Rubinstein: Hybrid formulas for L-functions, use these formulas to do the moments, assume the truncated Euler product and the truncated Hadamard product are independent in the sense that when you do the moments you can do each piece separately. Only gives the leading term but not the lower terms. How independent are they (statistical tests)? The fact that it gives the moments is a confirmation. Can we attach a statistical value to how independent on the two factors.

Jon Keating: independence is proved for the first few moments. Can prove it splits because we know what the moments are.

Mike R: doesn't mean it is independent, just an identity consistent with being independent. Maybe try and correct for the fact that they are not completely independent, maybe would help with the previous question.

Brian C: use the hybrid model to find lower order terms.

4.7 Constructing certain Siegel modular forms

Brian Conrey: Construct Siegel modular forms of degree 2 and weight 2 or 3 whose spinor L-function is primitive (means doesn't split into a product of lower L-functions).

Audience and Conrey: Upper bound for dimension computed for these spaces up to 23 and we have enough examples to exhaust (???). In principle can produce all the examples and test. As soon as given a bound can try to meet it. David F: for which levels do we know the space of cusp forms? Answer: don't have exact formula, have upper bound, generate lifts (more or less experimentally), as soon as have as many linearly independent lifts as allowed we are done. In some cases there are known formulas ($\Gamma_0(p)$ and higher weights, starting at weight 5). Hope is for p large. Conrey: what if someone told you one exists for $p = 101$: can you find it? Answer: don't know, think would be hard. David F: If it factors, as it's degree 4 it factors into either two of degree 2 or one of degree 1 and one of degree 3. Know bounds of conductors: test whether or not it factors into degree 1 or degree 2 (know candidates, check by brute force).

4.8 Special values of quadratic twists D of Rankin-Selberg on $\mathrm{GL}_2 \times \mathrm{GL}_2$

Brian Conrey: This is another degree 4 thing. This could possibly produce a bunch of degree 4 things that vanish.

Mark Watkins: have with symmetric powers. Doesn't seem to match with RMT. It seems that the ones for which RMT produces more twists that vanish actually have less twists that vanish.

4.9 Abelian surfaces

Henri Darmon (and audience and Brian Conrey): Look at twists by Dirihclet characters of order ℓ (a prime) of the L-function of an abelian surface. DFK (David, Fearnley and Kisilevsky) conjectures for elliptic curves that $\ell \geq 7$ implies only finitely many twists vanish. What about abelian surfaces?

Brian Conrey (and audience): need to know size of L-function, size of D in special value. Degree 4 thing. Are these standard L-functions? Are they primitive? Can compute the value of the L-function, if small enough declare it to be zero.

Henri Darmon: when twist pick up a D^2?

Brian Conrey: weight k get $1/|D|^{(k-1)/2}$. This goes to $1/|D|^{(k-1)/4}$ when take the square root. If $\frac{k-1}{4} \leq 1$ expect infinitely many non-trivial vanishings, and if $\frac{k-1}{4} > 1$ then only finitely many. If can answer the question as to what the power of D, would lead to the conjecture.

4.10 Miscellaneous questions

- Can one predict how often one will get vanishing of a triple product L-function?
- If one looks at an elliptic curve over say a real quadratic number field, can one effectively do a discretization and give an RMT model which predicts the frequency of vanishing? Or does the fact that there are very small integers prevent this?
- Can one predict the frequency with which various finite groups appear when studying the group of points modulo a prime p in the family of quadratic twists of a fixed elliptic curve?
- For a fixed cusp form of weight 6, we expect that for only finitely many quadratic twists the L-function will vanish. What if one varies the cusp form? How large a finite number can one get?

5 Results at the workshop

Interestingly there were a few real-time developments that occurred at the workshop.

One is that we carried out an experiment to help us understand the asymptotic number of elliptic curves with even positive rank in the family if quadratic twists of a given elliptic curve. We had previously understood that Delaunay's heuristics for Tate-Shafarevich groups are relevant for this problem, yet some subtle behaviour seems to complicate the answer. This depends on certain exceptional primes, and, at the workshop, we formulated and numerically tested a hypothesis that these exceptional primes are governed by classical Cohen-Lenstra heuristics for class groups. This gave an important step towards fully understanding the asymptotics.

Another is that some of the computational experts at the workshop (Tornaria, Watkins, and Rubinstein) figured out how to do some computations of interest to David, Fearnley, and Kisilevsky and generated lots of data for their experiments that at first seemed to be beyond reach.

6 Conclusion

The main achievement of the workshop was to bring to the forefront many new opportunities for interactions between RMT and number theory. There is now a motivation to develop algorithms for counting and computing L-functions from higher rank groups, especially for the L-functions for Siegel modular forms. Certainly many more questions were raised than were solved. This bodes well for the future of the subject.

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