The Cop, the Robber and the dark side of a graph

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The classical Cop-Robber game

- 2 player: the cop and the robber
- They move alternately beginning with the cop
- On their first move, each player choose a vertex of the graph and move on it
- Then, at its turn, each player can choose either to stay on its current position or move to an adjacent
- The goal is for the cop to catch the robber (i.e., to be on the same position as the robber. For the robber the goal is to prevent it to happen.

Definitions

- a *cop-win* graph is a graph for which the cop has a winning strategy
- A *cordal* graph is a graph whose only induced cycles are triangles
- A *bridge* graph is a graph whose only isometric cycles are triangles

Definitions (2)

- A simplicial vertex de G is a vertex whose closed neighborhood is a clique

- A graph G is *simplicially dismountable* if there exists a well order $\angle_s$ on $V(G)$ s.t. $\forall\, x \in V(G)$ (except (possibly) the greatest element):
  x is simplicial in $G - \{y \in V(G) \mid y \angle_s x\}$

- A graph G is *simplicially constructible* if there exists a well order $\angle_s$ on $V(G)$ s.t. $\forall\, x \in V(G)$ (except the smallest element):
  x is simplicial in $G - \{y \in V(G) \mid x \angle_s y\}$

Definitions (3)

- A dominated vertex de G is a vertex whose close neighborhood is included in the close neighborhood of another vertex

- A graph G is *dismountable* if there exists a well order $\angle_s$ on $V(G)$ s.t. $\forall\, x \in V(G)$ (except (possibly) the greatest element): x is simplicial in G-$\{y \in V(G) \mid y \angle_s x\}$

- A graph G is *constructible* if there exists a well order $\angle_s$ on $V(G)$ s.t. $\forall\, x \in V(G)$ (except the smallest element): x is simplicial in G-$\{y \in V(G) \mid x \angle_s y\}$

If G is finite

[Nowakowski,Winkler+Quilliot]

G is cordal
$\Leftrightarrow$
G is simplicially dismountable
$\Leftrightarrow$
G is simplicially constructible

G is cop-win
$\Leftrightarrow$
G is dismountable
$\Leftrightarrow$
G is constructible

If G is infinite !!!!

And for the bridge graphs ?

- **Theorem [Anstee, Farber]** *Let G be a finite connected graph. Then*
  G is bridged iff it is cop-win and contains no induced cycles of size 4 or 5.

- **Theorem [Chastand, Polat, L]** *Let G be a graph (finite or infinite). Then*
  G is bridged iff each breath first search (BFS) of the vertices of G induces a construction order $\angle_d$ *s.t.* $\forall x \in V(G)$

  x is dominated by its BFS-father in $G - \{y \in V(G) \mid x \angle_s y\}$.

Two questions that lead me in Cop-win problems

- **Question 1 [Farber]** *Does a bridge graph of finite diameter always cop-win?*

- **Question 2 [Hahn, Sauer, Woodrow]** Is it always possible to extend a finite subgraph of a bridged graph *G* to a ***finite bridged subgraph*** *of G?*

Related to Question 1

- In a finite bridged graph, supposing an intelligent robber, how can we determine the **minimal number of rounds** the cop will need to catch the robber?

- **Definition** Let G be a (finite) graph, then
  $G^{(0)} := G$
  $G^{(n+1)} := (G^{(n)})'$
  where $H' := H - \{x \in V(H) \mid x \text{ is dominated in } H\}$

- **Théorème [Polat, L]** *Let G be a finite graph. Then*
  G is cop-win iff $G^{(n)}$ is a clique for some integer n.

  Moreover, le minimal number of rounds before the cop win is either n or $n+1$.

Other results

- **Theorem [Hahn, L, Sauer, Woodrow]** *For every integer $n>0$, there exists a finite cordal graph of diameter 2 s.t. the cop needs at least n rounds to catch the robber.*

- **Corollary** *There exists infinite cordal graphs of diameter 2 that are robber-win.*

Thus, in the infinite case, we have to change the rules

- **Definition**
  A graph is weakly *cop-win* if
  - it is cop-win or
  - The cop has a strategy that, after a finite number of rounds, forces the robber to "run straight ahead" for ever

- The Cop will not necessarily catch the robber but at each round (except for a finite number) will secure an increasing part of the graph, and, at the limit, he will have totally secured the graph.

With those new rules:

- **Proposition [Chastand, L, Polat]**
  In the finite as in the infinite case:
  - *Trees are weakly cop-win;*
  - *Chordal graphs are weakly cop-win;*
  - *Bridged graphs are weakly cop-win;*
  - *Helly graphs are weakly cop-win;*
  - *«Constructible» graphs are weakly cop-win.*

Let us come back to Question 2

- **Question**
  Is it always possible to extend a finite subgraph of a bridged graph G to a ***finite bridged subgraph*** *of G?*

- **Theorem [Hahn, L, Sauer, Woodrow]**
  if the diameter is 2: yes.
  Moreover, the obtained finite bridged subgraph will be isometric in G.

- **Théorème [Chastand, L, Polat]** *yes.*

Let us change the rules again !!

- **Definition**
  In the game *Neighbor-Cop-win* ($\mathcal{N}$-*cop-win*), the robber will be allowed to move as far as he want provided he never move to a vertex adjacent to the cop.

  Hence, we have a very fast robber and a a slow cop only efficient in some "surveillance area" (here: the close neighborhood of its current position)

- *Note this new game is more similar to the one that defines the tree-width*

Let us change the rules again !!

- **Definition**
  In the game *Neighbor-Cop-win* (*$\mathcal{N}$-cop-win*), the robber will be allowed to move as far as he want provided he never move to a vertex adjacent to the cop.

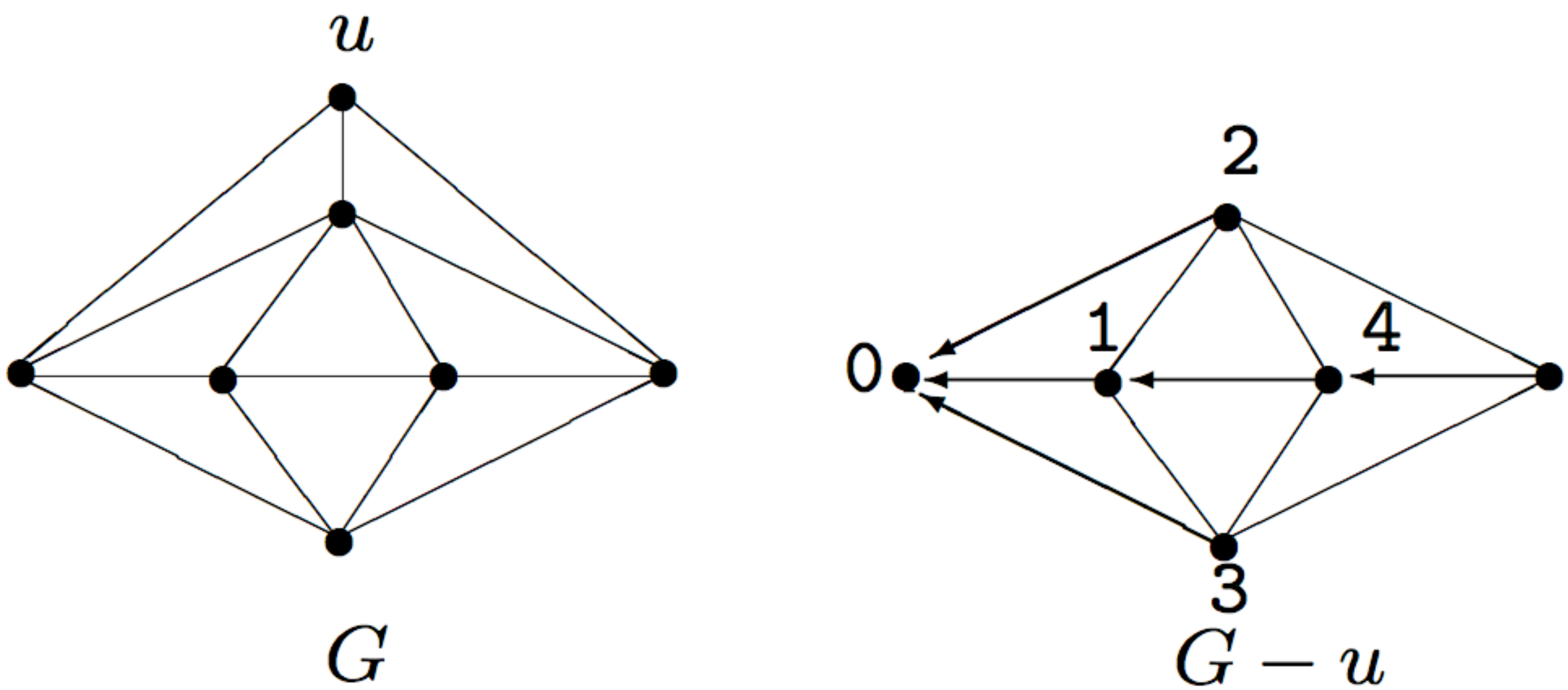


Fig.1

Neighbor-cop-win graphs

- **Proposition [L, Polat]** *Let G be a graph and x a simplicial vertex of G. Then*
  G is $\mathcal{N}$-Cop-win iff G-x also is.

- **Corollary**
  Finite chordal graphs are $\mathcal{N}$-cop-win.

- *But they are more!*

Can we say more about them ?

- **Definition :** a vertex x of G is hyper-dominated if $\mathcal{N}^2[x] \subseteq \mathcal{N}[y]$ for some vertex y.

- **Proposition [L, Polat]**
  G is hyper-dismountable $\Rightarrow$ G is $\mathcal{N}$-cop-win

  $$\not\Leftarrow$$

- It seems that we cannot say better than that !?!

- *It seems not possible to characterize $\mathcal{N}$-cop-win by a dismantling argument :-(*

Let us change the definition of graphs

- **Definitions**
  - An *undergrounded graph* G (or *u-graph*) is a graph whose set of vertices $V(G)$ has been 2-colored (light and dark). $V^l(G)$ is the set of light vertices and $V^d(G)$ the set of the dark ones.
  - $G\blacklozenge X$ is the u-graph obtained from G by transforming every vertices of $X\cap V^l(G)$ in dark vertices
  - The *light-neighborhood* of a vertex y is

    $\mathcal{N}^l_G(y){:=}\{z \in V^l(G)\text{-}\,\mathcal{N}_G[y] \mid \exists\ yz\text{-path } P \text{ t.q. } V(P\text{-}\{y,z\}) \cap V^l(G) = \emptyset\}$
  - *The freedom space of a vertex y* w.r.t a vertex x is

$F_x^G(y){:=}\{z \in V(G)\text{-}\,\mathcal{N}_G[x] \mid \exists\ yz\text{-path } P \text{ t.q. } V(P\text{-}y) \cap \mathcal{N}_G[x] = \emptyset\}$

l-simplicial and l-dominated vertices

- Definitions
  - A light vertex y of a u-graph G is *l-simplicial* if there exists a clique K in G such that
    - $y \in V(K)$ and
    - the light neighborhood of x in G-(K-y) is empty.
  - A light vertex y of a u-graphe G is *l-dominated* by a light vertex x if
    - x dominate y in G^l and
    - the freedom space $F_x^G(y)$ contains dark vertices only.

On u-graphs, the $\mathcal{N}$ **-cop-win** game becomes:

- **Definition** The cop try to catch the robber, and they both alternately playing under the following rules:
  - The cop chooses a starting position that is a **light** vertex;
  - Then, the robber also chooses a starting position that is a **light** vertex;
  - On its turn, the cop can stay on its current position or move to an adjacent **light** vertex;
  - On its turn, supposing the robber position is x, it can move to any **light** vertex y for which there exists an xy-path P such that no vertices of P-$\{x\}$ (light or dark) belong to the closed neighborhood of the cop current position.

$\mathcal{N}$-cop-win (finite) u-graphs have a dismountable type characterization

- **Definition**
  A u-graphe G is *l-dismountable* if there exists a well ordering $\angle_s$ on $V(G)$ such that $\forall\, x \in V(G)$ (except possibly the greatest element):

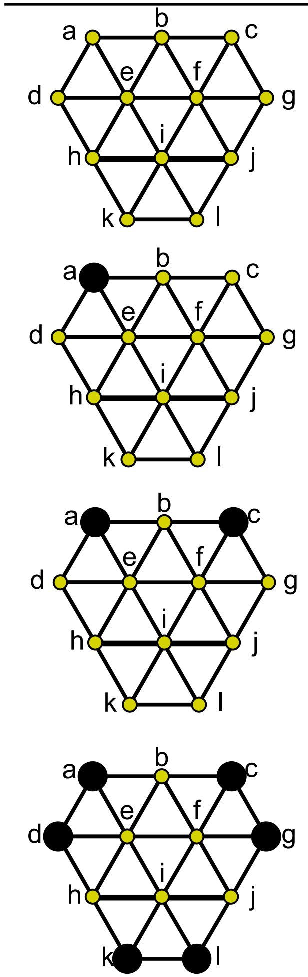
  $$x \text{ is } l\text{-dominated in } G\blacklozenge\{y \in V(G) \mid y \angle_s x\}$$

- **Theoreme [Desharnais, L, Marcoux, Polat]**
  A finite u-graphe is $\mathcal{N}$-cop-win iff it is l-dismountable

- Examples !!!

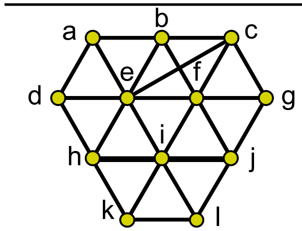
Example 1:

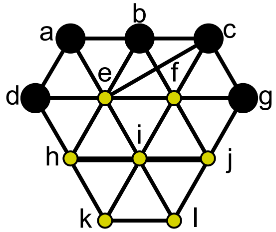
Easy to see: G is cop-win but not $\mathcal{N}$-cop-win

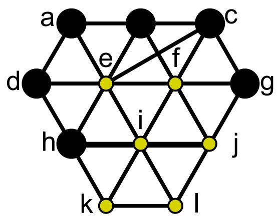


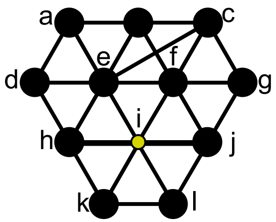
- a is *l*-dominated by e
- c is *l*-dominated by f
  - d is *l*-dominated by e
  - g is *l*-dominated by f
  - k is *l*-dominated by i
  - l is *l*-dominated by l
- The graph is not *l*-dismountable !

Example 2:
Easy to see : G is an $\mathcal{N}$-cop-win u-graph









- ✓ a,b,d are *l*-dominated by e
- ✓ c,g are *l*-dominated by e

✓ Then, h is *l*-dominated by i

✓ Then, e,f,j,k,l are *l*-dominated by i

✓ The graph is *l*-dismountable !

What about simplicially *l*-dismountable u-graphs?

- **Definition**
  a u-graph is *chordal* if it is simplicialement *l*-dismountable.

- **Lemma[Desharnais, L, Marcoux]**
  Every induced cycle of length>3 of a chordal u-graph has at most 2 light vertices.

- **Proposition[Desharnais, L, Marcoux]**
  A chordal u-graph whose all vertices is light is a chordal graph.

On u-graphs, the **clique-cop-win** game is:

- **Definition** The cop try to catch the robber, and they both alternately playing under the following rules:
  - The cop chooses a starting position x that is a light vertex **and a clique $\boldsymbol{K_x}$ that contains $\boldsymbol{x}$**;
  - Then, the robber also chooses a starting position that is a light vertex;
  - On its turn, the cop move to an adjacent light vertex x' **and then choose a new clique $\boldsymbol{K_{x'}}$**;
  - On its turn, supposing the robber position is u, it can move to any light vertex v for which there exists an uv-path P such that no vertices of P-$\{u\}$ (light or dark) belong to $\boldsymbol{K_{x'}}$.

The result

- **Theorem [Desharnais, L, Marcoux]**
  a u-graph is clique-cop-win iff it is chordal

- **Corollary** *(non undergrounded) chordal graphs are characterized by this game.*

- It seems that this corollary cannot be proved without this underground generalization.

May be this is the genesis of a new theory !