From Finite to Infinite

Lajos Soukup

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences

Infinite Graphs, 2007

Outline

- **theorems in finite combinatorics** vs **their infinite counterparts**
- the methods of generalizations
- **proof** of a "finite" theorem vs **proof** of its "infinite" version.
- basic proof methods
- set-theoretic tools
- nice problems

Outline

- **theorems in finite combinatorics** vs **their infinite counterparts**
- the methods of generalizations
- **proof** of a "finite" theorem vs **proof** of its "infinite" version.
- basic proof methods
- set-theoretic tools
- nice problems

Outline

- **theorems in finite combinatorics** vs **their infinite counterparts**
- the methods of generalizations
- **proof** of a "finite" theorem vs **proof** of its "infinite" version.
- basic proof methods
- set-theoretic tools
- nice problems

Outline

- **theorems in finite combinatorics** vs **their infinite counterparts**
- the methods of generalizations
- **proof** of a "finite" theorem vs **proof** of its "infinite" version.
- basic proof methods
- set-theoretic tools
- nice problems

Outline

- **theorems in finite combinatorics** vs **their infinite counterparts**
- the methods of generalizations
- **proof** of a “finite” theorem vs **proof** of its “infinite” version.
- basic proof methods
- set-theoretic tools
- nice problems

Outline

- **theorems in finite combinatorics** vs **their infinite counterparts**
- the methods of generalizations
- **proof** of a “finite” theorem vs **proof** of its “infinite” version.
- basic proof methods
- set-theoretic tools
- nice problems

Outline

- **theorems in finite combinatorics** vs **their infinite counterparts**
- the methods of generalizations
- **proof** of a “finite” theorem vs **proof** of its “infinite” version.
- basic proof methods
- set-theoretic tools
- nice problems

Connectedness

Theorem

A **finite** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Theorem

An **arbitrary** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Proof

Let $A = \{z \in V : \exists x\text{-}z\text{-path}\}$.

There is no edge between A and $V \setminus A$.

$A = V$.

Connectedness

Theorem

A **finite** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Theorem

An **arbitrary** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Proof

Let $A = \{z \in V : \exists x\text{-}z\text{-path}\}$.

There is no edge between A and $V \setminus A$.

$A = V$.

Connectedness

Theorem

A **finite** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Theorem

An **arbitrary** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Proof

Let $A = \{z \in V : \exists x\text{-}z\text{-path}\}$.

There is no edge between A and $V \setminus A$.

$A = V$.

Connectedness

Theorem

A **finite** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Theorem

An **arbitrary** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Proof

Let $A = \{z \in V : \exists x\text{-}z\text{-path}\}$.

There is no edge between A and $V \setminus A$.

$A = V$.

Connectedness

Theorem

A **finite** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Theorem

An **arbitrary** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Proof

Let $A = \{z \in V : \exists x\text{-}z\text{-path}\}$.

There is no edge between A and $V \setminus A$.

$A = V$.

Connectedness

Theorem

A **finite** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Theorem

An **arbitrary** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Proof

Let $A = \{z \in V : \exists x\text{-}z\text{-path}\}$.

There is no edge between A and $V \setminus A$.

$A = V$.

Theorem

A **finite** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Theorem

An **arbitrary** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Proof

Let $A = \{z \in V : \exists x\text{-}z\text{-path}\}$.

There is no edge between A and $V \setminus A$.

$A = V$.

Theorem

A **finite** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Theorem

An **arbitrary** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Proof

Let $A = \{z \in V : \exists x\text{-}z\text{-path}\}$.

There is no edge between A and $V \setminus A$.

$A = V$.

Connectedness

Theorem

A **finite** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Theorem

An **arbitrary** graph $G = (V, E)$ is **connected** iff given any **partition** (V_0, V_1) of the vertices into two non-empty sets there is an edge between V_0 and V_1.

Proof

Let $A = \{z \in V : \exists x\text{-}z\text{-path}\}$.

There is no edge between A and $V \setminus A$.

$A = V$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a **minimal** connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \ldots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\bigcap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a **minimal** connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \ldots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\bigcap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a **minimal** connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \ldots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\cap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a **minimal** connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \ldots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\bigcap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a minimal connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \ldots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\bigcap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a **minimal** connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \dots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\cap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a **minimal** connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \ldots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\cap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a **minimal** connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \dots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\cap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a **minimal** connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \dots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\cap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a **minimal** connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \dots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\cap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

First Proof

Let $T = \langle V, F \rangle$ be a **minimal** connected subgraph of G.

Then T can not contain a circle, so it is a spanning tree.

no infinite version

how to get a minimal connected subgraph of an infinite graph?

an infinite graph G may contain a decreasing chain $G_0, G_1, \ldots$ of connected subgraphs of G such that $V(G_i) = V(G)$ but $\cap_{i \in \mathbb{N}} E(G_i) = \emptyset$.

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

Second Proof

- $\mathcal{T} = \{$ connected subtrees of $G\}$
- $\langle \mathcal{T}, \subset \rangle$ has a maximal element T by **Zorn's lemma**
- Let T be a maximal connected subtree of G.
- There is no edge between $V(T)$ and $V \setminus V(T)$.
- $V(T) = V$.

Zorn's Lemma, Axiom of Choice. **Really need?**

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

Second Proof

$\mathcal{T} = \{$ connected subtrees of $G\}$

$\langle \mathcal{T}, \subset \rangle$ has a maximal element T by **Zorn's lemma**

Let T be a maximal connected subtree of G.

There is no edge between $V(T)$ and $V \setminus V(T)$.

$V(T) = V$.

Zorn's Lemma, Axiom of Choice. Really need?

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

Second Proof

$\mathcal{T} = \{$ connected subtrees of $G\}$

$\langle \mathcal{T}, \subset \rangle$ has a maximal element T by **Zorn's lemma**

Let T be a maximal connected subtree of G.

There is no edge between $V(T)$ and $V \setminus V(T)$.

$V(T) = V$.

Zorn's Lemma, Axiom of Choice. **Really need?**

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

Second Proof

$\mathcal{T} = \{$ connected subtrees of $G\}$

$\langle \mathcal{T}, \subset \rangle$ has a maximal element T by **Zorn's lemma**

Let T be a maximal connected subtree of G.

There is no edge between $V(T)$ and $V \setminus V(T)$.

$V(T) = V$.

Zorn's Lemma, Axiom of Choice. **Really need?**

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

Second Proof

$\mathcal{T} = \{$ connected subtrees of $G\}$

$\langle \mathcal{T}, \subset \rangle$ has a maximal element T by **Zorn's lemma**

Let T be a maximal connected subtree of G.

There is no edge between $V(T)$ and $V \setminus V(T)$.

$V(T) = V$.

Zorn's Lemma, Axiom of Choice. Really need?

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

Second Proof

$\mathcal{T} = \{$ connected subtrees of $G\}$

$\langle \mathcal{T}, \subset \rangle$ has a maximal element T by **Zorn's lemma**

Let T be a maximal connected subtree of G.

There is no edge between $V(T)$ and $V \setminus V(T)$.

$V(T) = V$.

Zorn's Lemma, Axiom of Choice. Really need?

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

Second Proof

$\mathcal{T} = \{$ connected subtrees of $G\}$

$\langle \mathcal{T}, \subset \rangle$ has a maximal element T by **Zorn's lemma**

Let T be a maximal connected subtree of G.

There is no edge between $V(T)$ and $V \setminus V(T)$.

$V(T) = V$.

Zorn's Lemma, Axiom of Choice. **Really need?**

Spanning trees

Finite case

Theorem

Every **finite** connected graph $G = (V, E)$ has a spanning tree.

General case

Theorem

Every connected graph $G = (V, E)$ has a spanning tree.

Second Proof

$\mathcal{T} = \{$ connected subtrees of $G\}$

$\langle \mathcal{T}, \subset \rangle$ has a maximal element T by **Zorn's lemma**

Let T be a maximal connected subtree of G.

There is no edge between $V(T)$ and $V \setminus V(T)$.

$V(T) = V$.

Zorn's Lemma, Axiom of Choice. **Really need?**

Spanning trees and AC

Theorem

If every ***connected graph*** *has a* ***spanning tree*** *then the* ***Axiom of Choice*** *holds.*

Proof

$\mathcal{A} = \{A_i : i \in I\}$ a family of non-empty sets. $A_i \cap A_j = \emptyset$

$V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \bigcup\{A_i : i \in I\}$,

$E = \{xy_i : i \in I\} \cup \bigcup_{i \in I}\{y_i a, az_i : a \in A_i\}$.

G is connected, $T = (V, F)$ spanning tree.

(i) $\{xy_i : i \in I\} \subset F$,

(ii) $\forall i \in I\ \exists! a_i \in A_i$ s.t. $y_i a_i, a_i z_i \in F$,

(iii) $\forall a \in A_i \setminus \{a_i\}$ ($y_i a \in F$ iff $az_i \notin F$).

$f(i) = a_i$ is a choice function for $\mathcal{A}$

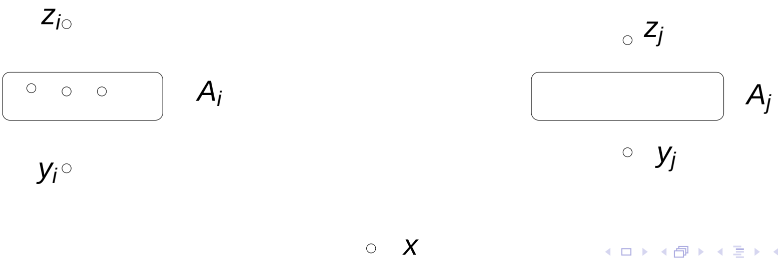
Proof

$\mathcal{A} = \{A_i : i \in I\}$ a family of non-empty sets. $A_i \cap A_j = \emptyset$

Proof

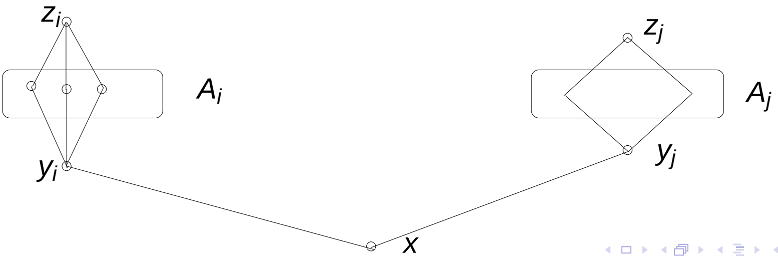
$\mathcal{A} = \{A_i : i \in I\}$ a family of non-empty sets. $A_i \cap A_j = \emptyset$

$V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \bigcup\{A_i : i \in I\}$,



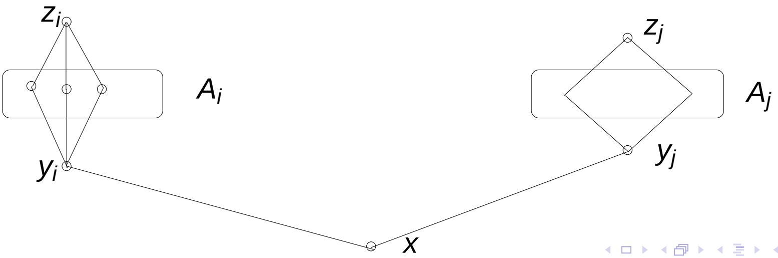
Proof

$\mathcal{A} = \{A_i : i \in I\}$ a family of non-empty sets. $A_i \cap A_j = \emptyset$
$V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \bigcup\{A_i : i \in I\}$,
$E = \{xy_i : i \in I\} \cup \bigcup_{i \in I}\{y_i a, az_i : a \in A_i\}$.



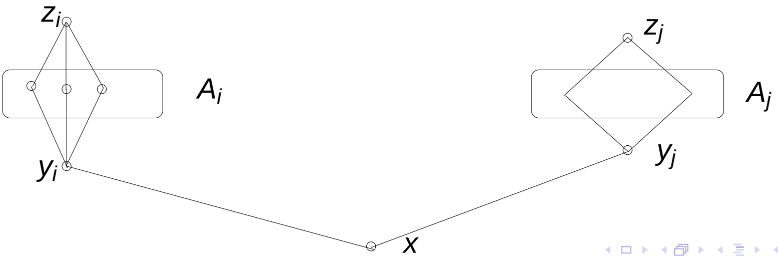
Proof

$\mathcal{A} = \{A_i : i \in I\}$ a family of non-empty sets. $A_i \cap A_j = \emptyset$
$V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \bigcup\{A_i : i \in I\}$,
$E = \{xy_i : i \in I\} \cup \bigcup_{i \in I}\{y_i a, a z_i : a \in A_i\}$.
G is connected,



Proof

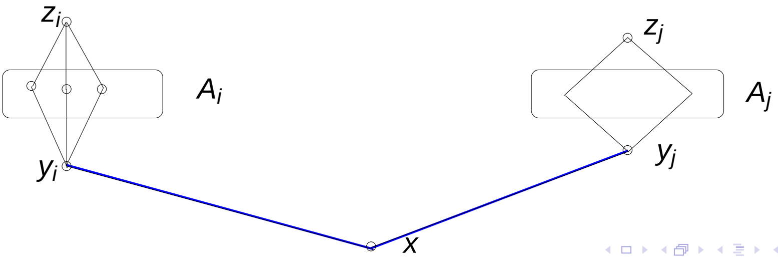
$\mathcal{A} = \{A_i : i \in I\}$ a family of non-empty sets. $A_i \cap A_j = \emptyset$
$V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \bigcup\{A_i : i \in I\}$,
$E = \{xy_i : i \in I\} \cup \bigcup_{i \in I}\{y_i a, az_i : a \in A_i\}$.
G is connected, $T = (V, F)$ spanning tree.



Proof

$\mathcal{A} = \{A_i : i \in I\}$ a family of non-empty sets. $A_i \cap A_j = \emptyset$
$V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \bigcup\{A_i : i \in I\}$,
$E = \{xy_i : i \in I\} \cup \bigcup_{i \in I}\{y_i a, az_i : a \in A_i\}$.
G is connected, $T = (V, F)$ spanning tree.

(i) $\{xy_i : i \in I\} \subset F$,

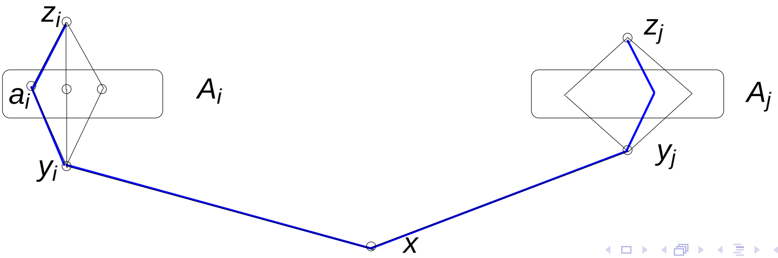


Proof

$\mathcal{A} = \{A_i : i \in I\}$ a family of non-empty sets. $A_i \cap A_j = \emptyset$
$V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \bigcup\{A_i : i \in I\}$,
$E = \{xy_i : i \in I\} \cup \bigcup_{i \in I}\{y_i a, az_i : a \in A_i\}$.
G is connected, $T = (V, F)$ spanning tree.

(i) $\{xy_i : i \in I\} \subset F$,

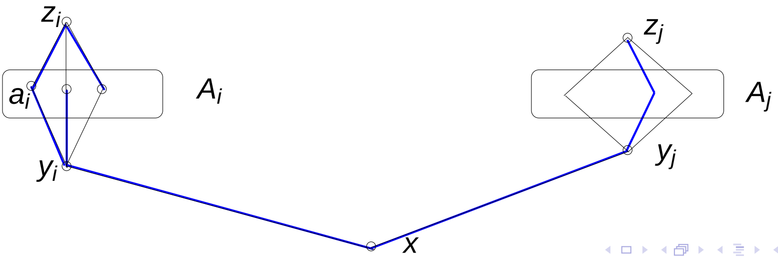
(ii) $\forall i \in I\ \exists! a_i \in A_i$ s.t. $y_i a_i, a_i z_i \in F$,



Proof

$\mathcal{A} = \{A_i : i \in I\}$ a family of non-empty sets. $A_i \cap A_j = \emptyset$
$V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \bigcup\{A_i : i \in I\}$,
$E = \{xy_i : i \in I\} \cup \bigcup_{i \in I}\{y_i a, az_i : a \in A_i\}$.
G is connected, $T = (V, F)$ spanning tree.

(i) $\{xy_i : i \in I\} \subset F$,
(ii) $\forall i \in I\ \exists! a_i \in A_i$ s.t. $y_i a_i, a_i z_i \in F$,
(iii) $\forall a \in A_i \setminus \{a_i\}$ ($y_i a \in F$ iff $az_i \notin F$).

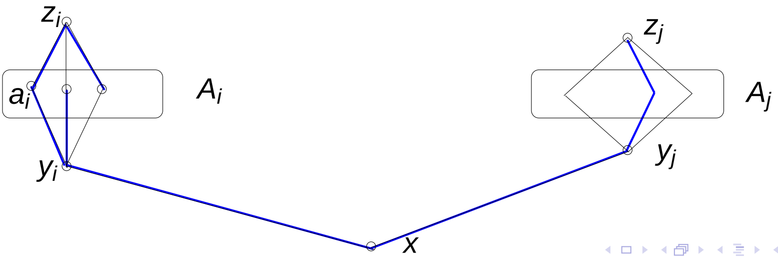


Proof

$\mathcal{A} = \{A_i : i \in I\}$ a family of non-empty sets. $A_i \cap A_j = \emptyset$
$V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \bigcup\{A_i : i \in I\}$,
$E = \{xy_i : i \in I\} \cup \bigcup_{i \in I}\{y_i a, a z_i : a \in A_i\}$.
G is connected, $T = (V, F)$ spanning tree.

(i) $\{xy_i : i \in I\} \subset F$,
(ii) $\forall i \in I \ \exists! a_i \in A_i$ s.t. $y_i a_i, a_i z_i \in F$,
(iii) $\forall a \in A_i \setminus \{a_i\}$ ($y_i a \in F$ iff $a z_i \notin F$).

$f(i) = a_i$ is a choice function for $\mathcal{A}$



Unfriendly Partitions

Unfriendly Partitions

Definition

Let $G = (V, E)$ be a graph. A **partition** (A, B) of V is called **unfriendly** iff every vertex has at least as many neighbor in the other class as in its own.

Unfriendly Partitions

Definition

Let $G = (V, E)$ be a graph. A **partition** (A, B) of V is called **unfriendly** iff every vertex has at least as many neighbor in the other class as in its own.

Observation

Every **finite** graph has an unfriendly partition.

Unfriendly Partitions

Definition

Let $G = (V, E)$ be a graph. A **partition** (A, B) of V is called **unfriendly** iff every vertex has at least as many neighbor in the other class as in its own.

Observation

Every **finite** graph has an unfriendly partition.

Unfriendly Partition Conjecture

Every graph has an unfriendly partition.

Unfriendly Partitions

Definition

Let $G = (V, E)$ be a graph. A **partition** (A, B) of V is called **unfriendly** iff every vertex has at least as many neighbor in the other class as in its own.

Observation

Every **finite** graph has an unfriendly partition.

Unfriendly Partition Conjecture

Every graph has an unfriendly partition.

Theorem (Shelah)

*There is an **uncountable** graph without an unfriendly partition.*

Unfriendly Partitions

Unfriendly Partitions

Theorem (Shelah)

Every graph has a ***partition into three pieces*** *such that every vertex has at least as many neighbor in the two other classes as in its own.*

Unfriendly Partitions

Theorem (Shelah)

Every graph has a ***partition into three pieces*** *such that every vertex has at least as many neighbor in the two other classes as in its own.*

Theorem

Every **locally finite** graph has an unfriendly partition.

Proof: locally finite graphs have unfriendly partitions

Proof: locally finite graphs have unfriendly partitions

Natural aproach:

Proof: locally finite graphs have unfriendly partitions

Natural aproach:**König's Lemma** instead of **Gödel's Theorem**

Proof: locally finite graphs have unfriendly partitions

Natural aproach:**König's Lemma** instead of **Gödel's Theorem**
$G = \langle V, E \rangle$ locally finite,

Proof: locally finite graphs have unfriendly partitions

Natural aproach:**König's Lemma** instead of **Gödel's Theorem**
$G = \langle V, E \rangle$ locally finite,
enumerate $V = \{x_0, x_1, \dots\}$

Proof: locally finite graphs have unfriendly partitions

Natural aproach:**König's Lemma** instead of **Gödel's Theorem**

$G = \langle V, E \rangle$ locally finite,

enumerate $V = \{x_0, x_1, \dots\}$

$T_n = \{(A, B) :$ an unfriendly partition of $G[x_0, \dots x_n]\}$

Proof: locally finite graphs have unfriendly partitions

Natural aproach:**König's Lemma** instead of **Gödel's Theorem**
$G = \langle V, E \rangle$ locally finite,
enumerate $V = \{x_0, x_1, \dots\}$
$T_n = \{(A, B) :$ an unfriendly partition of $G[x_0, \dots x_n]\}$
$\mathcal{T} = \{\cup T_n, \subset\}$

Proof: locally finite graphs have unfriendly partitions

Natural aproach:**König's Lemma** instead of **Gödel's Theorem**
$G = \langle V, E \rangle$ locally finite,
enumerate $V = \{x_0, x_1, \dots\}$
$T_n = \{(A, B):$ an unfriendly partition of $G[x_0, \dots x_n]\}$
$\mathcal{T} = \{\cup T_n, \subset\}$
can not apply König's Lemma

Proof: locally finite graphs have unfriendly partitions

Natural aproach:**König's Lemma** instead of **Gödel's Theorem**

$G = \langle V, E \rangle$ locally finite,

enumerate $V = \{x_0, x_1, \dots\}$

$T_n = \{(A, B) :$ an unfriendly partition of $G[x_0, \dots x_n]\}$

$\mathcal{T} = \{\cup T_n, \subset\}$

can not apply König's Lemma $T_n \neq$ **the** n^{th}**-level of** $\mathcal{T}$

Theorem (Gödel)

A theory T has a model provided every finite subset of T has a model.

Proof: locally finite graphs have unfriendly partitions

Gödel's Compactness Theorem

$G = (V, E)$ locally finite graph

.

$G = (V, E)$ locally finite graph
Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.

.

$G = (V, E)$ locally finite graph
Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.
Formulas: ψ: $\forall x \left(R_A(x) \leftrightarrow \neg R_B(x)\right)$

.

$G = (V, E)$ locally finite graph
Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.
Formulas: ψ: $\forall x \left(R_A(x) \leftrightarrow \neg R_B(x)\right)$
for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \geq |E(v)|/2\}$

.

$G = (V, E)$ locally finite graph
Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.
Formulas: ψ: $\forall x \, (R_A(x) \leftrightarrow \neg R_B(x))$
for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \geq |E(v)|/2\}$ and put
$\varphi_{v,A}$: $R_A(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_B(c_x)$

.

$G = (V, E)$ locally finite graph

Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.

Formulas: ψ: $\forall x\ (R_A(x) \leftrightarrow \neg R_B(x))$

for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \geq |E(v)|/2\}$ and put

$\varphi_{v,A}$: $R_A(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_B(c_x)$

$\varphi_{v,B}$: $R_B(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_A(c_x)$

.

$G = (V, E)$ locally finite graph
Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.
Formulas: ψ: $\forall x \left(R_A(x) \leftrightarrow \neg R_B(x)\right)$
for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \geq |E(v)|/2\}$ and put
$\varphi_{v,A}$: $R_A(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_B(c_x)$
$\varphi_{v,B}$: $R_B(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_A(c_x)$
Theory: $T = \{\psi, \varphi_{v,A}, \varphi_{v,B} : v \in V\}$

.

$G = (V, E)$ locally finite graph

Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.

Formulas: ψ: $\forall x \left(R_A(x) \leftrightarrow \neg R_B(x)\right)$

for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \geq |E(v)|/2\}$ and put

$\varphi_{v,A}$: $R_A(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_B(c_x)$

$\varphi_{v,B}$: $R_B(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_A(c_x)$

Theory: $T = \{\psi, \varphi_{v,A}, \varphi_{v,B} : v \in V\}$

Claim

Every $T' \in [T]^{<\omega}$ has a model.

.

Proof: locally finite graphs have unfriendly partitions

Gödel's Compactness Theorem

$G = (V, E)$ locally finite graph

Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.

Formulas: ψ: $\forall x \, (R_A(x) \leftrightarrow \neg R_B(x))$

for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \geq |E(v)|/2\}$ and put

$\varphi_{v,A}$: $R_A(c_v) \rightarrow \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_B(c_x)$

$\varphi_{v,B}$: $R_B(c_v) \rightarrow \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_A(c_x)$

Theory: $T = \{\psi, \varphi_{v,A}, \varphi_{v,B} : v \in V\}$

Claim

Every $T' \in [T]^{<\omega}$ has a model.

Let $W = \{v : c_v \text{ occurs in } T'\}$.

$G = (V, E)$ locally finite graph

Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.

Formulas: ψ: $\forall x\ (R_A(x) \leftrightarrow \neg R_B(x))$

for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \ge |E(v)|/2\}$ and put

$\varphi_{v,A}$: $R_A(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_B(c_x)$

$\varphi_{v,B}$: $R_B(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_A(c_x)$

Theory: $T = \{\psi, \varphi_{v,A}, \varphi_{v,B} : v \in V\}$

Claim

Every $T' \in [T]^{<\omega}$ has a model.

Let $W = \{v : c_v \text{ occurs in } T'\}$. Then $G[W]$ has an unfriendly partition (A, B).

$G = (V, E)$ locally finite graph
Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.
Formulas: ψ: $\forall x\ (R_A(x) \leftrightarrow \neg R_B(x))$
for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \geq |E(v)|/2\}$ and put
$\varphi_{v,A}$: $R_A(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_B(c_x)$
$\varphi_{v,B}$: $R_B(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_A(c_x)$
Theory: $T = \{\psi, \varphi_{v,A}, \varphi_{v,B} : v \in V\}$

Claim

Every $T' \in [T]^{<\omega}$ has a model.

Let $W = \{v : c_v$ occurs in $T'\}$. Then $G[W]$ has an unfriendly partition (A, B). Let M be the following model: the underlying set M is W, c_v is interpreted as v for $v \in W$, and R_A is interpreted as A and R_B is interpreted as B.

$G = (V, E)$ locally finite graph
Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.
Formulas: ψ: $\forall x\ (R_A(x) \leftrightarrow \neg R_B(x))$
for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \geq |E(v)|/2\}$ and put
$\varphi_{v,A}$: $R_A(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_B(c_x)$
$\varphi_{v,B}$: $R_B(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_A(c_x)$
Theory: $T = \{\psi, \varphi_{v,A}, \varphi_{v,B} : v \in V\}$

Claim

Every $T' \in [T]^{<\omega}$ has a model.

Let $W = \{v : c_v$ occurs in $T'\}$. Then $G[W]$ has an unfriendly partition (A, B). Let M be the following model: the underlying set M is W, c_v is interpreted as v for $v \in W$, and R_A is interpreted as A and R_B is interpreted as B. $M \models T'$.

$G = (V, E)$ locally finite graph

Language: $\{c_v : v \in V\}$ constant symbols, R_A and R_B are unary relation symbols.

Formulas: ψ: $\forall x \ (R_A(x) \leftrightarrow \neg R_B(x))$

for all $v \in V$ write $\mathcal{F}_v = \{C \subset E(v) : |F| \geq |E(v)|/2\}$ and put

$\varphi_{v,A}$: $R_A(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_B(c_x)$

$\varphi_{v,B}$: $R_B(c_v) \to \bigvee_{F \in \mathcal{F}_v} \bigwedge_{x \in F} R_A(c_x)$

Theory: $T = \{\psi, \varphi_{v,A}, \varphi_{v,B} : v \in V\}$

Claim

Every $T' \in [T]^{<\omega}$ has a model.

Let M be a model of T and let $A = \{v \in V : M \models R_A(c_v)\}$ and $B = \{v \in V : M \models R_B(c_v)\}$.

Unfriendly Partitions

Theorem

Every **locally finite** graph has an unfriendly partition.

Unfriendly Partitions

Theorem

Every **locally finite** graph has an unfriendly partition.

Fact

If $G = (V, E)$ is countable and every $v \in V$ has infinite degree then G has an unfriendly partition.

Unfriendly Partitions

Theorem

Every **locally finite** graph has an unfriendly partition.

Fact

If $G = (V, E)$ is countable and every $v \in V$ has infinite degree then G has an unfriendly partition.

Unfriendly Partition Conjecture, revised

Every countable graph has an unfriendly partition.

Unfriendly Partitions

Unfriendly Partitions

Question

Let $G = (V, E)$ be a **locally finite** graph and $V' \subset V$ such that V' is **"rare"** (e.g the distances are large between the elements of V' in G). Is it true that every partition (A', B') of V' can be **extended** to an unfriendly partition (A, B) of G?

Unfriendly Partitions

Question

Let $G = (V, E)$ be a **locally finite** graph and $V' \subset V$ such that V' is **"rare"** (e.g the distances are large between the elements of V' in G). Is it true that every partition (A', B') of V' can be **extended** to an unfriendly partition (A, B) of G?

Answer

No, V. Bonifaci gave a very strong counterexample.

Unfriendly partitions

Theorem (Bonifaci)

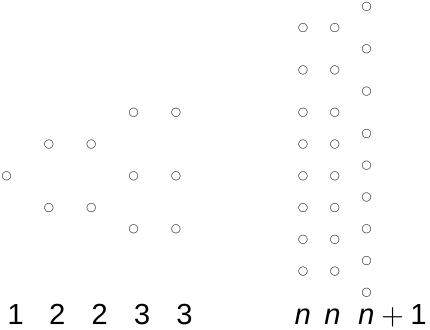
*There is a locally finite infinite graph with **exactly one** unfriendly partition.*

Unfriendly partitions

Theorem (Bonifaci)

There is a locally finite infinite graph with ***exactly one*** *unfriendly partition.*

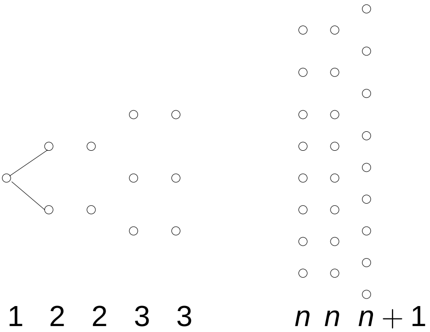
vertices: in columns



Unfriendly partitions

Theorem (Bonifaci)

There is a locally finite infinite graph with ***exactly one*** *unfriendly partition.*

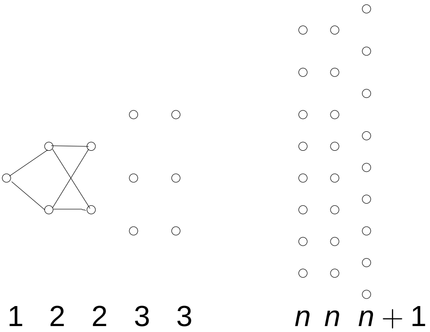


vertices: in columns
edges: between neighbouring columns

Unfriendly partitions

Theorem (Bonifaci)

*There is a locally finite infinite graph with **exactly one** unfriendly partition.*



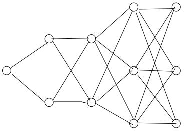
vertices: in columns
edges: between neighbouring columns

Unfriendly partitions

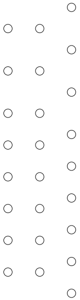
Theorem (Bonifaci)

There is a locally finite infinite graph with ***exactly one*** *unfriendly partition.*

vertices: in columns
edges: between neighbouring columns



1 2 2 3 3

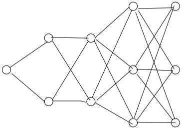


$n \quad n \quad n+1$

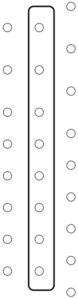
Unfriendly partitions

Theorem (Bonifaci)

There is a locally finite infinite graph with ***exactly one*** *unfriendly partition.*



1 2 2 3 3



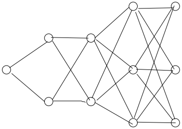
$n \quad n \quad n+1$

vertices: in columns
edges: between neighbouring columns
column of size n

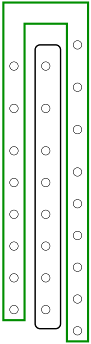
Unfriendly partitions

Theorem (Bonifaci)

*There is a locally finite infinite graph with **exactly one** unfriendly partition.*



1 2 2 3 3



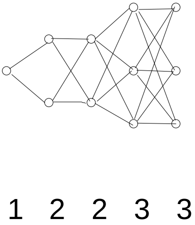
$n \quad n \quad n+1$

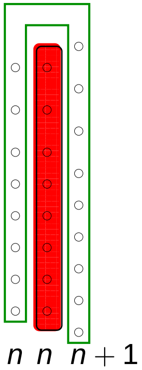
vertices: in columns
edges: between neighbouring columns
column of size n
red or **blue** majority in the neighbouring columns

Unfriendly partitions

Theorem (Bonifaci)

*There is a locally finite infinite graph with **exactly one** unfriendly partition.*



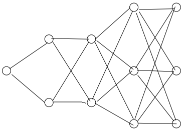


vertices: in columns
edges: between neighbouring columns
column of size n
red or **blue** majority in the neighbouring columns
blue majority $\Longrightarrow$ the column is **red**.

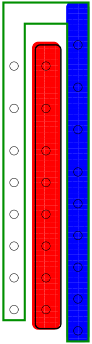
Unfriendly partitions

Theorem (Bonifaci)

There is a locally finite infinite graph with ***exactly one*** *unfriendly partition.*



1 2 2 3 3



n n $n+1$

vertices: in columns
edges: between neighbouring columns
column of size n
red or **blue** majority in the neighbouring columns
blue majority $\Longrightarrow$ the column is **red**.
next column is also monochromatic: it should be **blue**.

Pseudo-winners in tournaments

Pseudo-winners in tournaments

Definition

Let $T = (V, E)$ be a tournament and let $t \in V$.

Pseudo-winners in tournaments

Definition

Let $T = (V, E)$ be a tournament and let $t \in V$.
t is a ***pseudo-winner***

Pseudo-winners in tournaments

Definition

Let $T = (V, E)$ be a tournament and let $t \in V$.
t is a ***pseudo-winner*** iff for each $y \in V$ there is a **path of length at most** 2 which leads from t to y.

Pseudo-winners in tournaments

Pseudo-winners in tournaments

Finite case

Pseudo-winners in tournaments

Finite case

Theorem

Every **finite tournament** has a **pseudo-winner**.

Pseudo-winners in tournaments

Finite case

Theorem

Every **finite tournament** has a **pseudo-winner**.

Proof

If t has maximal out-degree then t is a pseudo-winner.

Pseudo-winners in tournaments

Finite case

Infinite case

Theorem

Every **finite tournament** has a **pseudo-winner**.

Proof

If t has maximal out-degree then t is a pseudo-winner.

Pseudo-winners in tournaments

Finite case

Theorem

Every **finite tournament** has a **pseudo-winner**.

Proof

If t has maximal out-degree then t is a pseudo-winner.

Infinite case

Observation

No pseudo-winner in $\langle \mathbb{Z}, < \rangle$.

Pseudo-winners in tournaments

Finite case

Theorem

Every **finite tournament** has a **pseudo-winner**.

Proof

If t has maximal out-degree then t is a pseudo-winner.

Infinite case

Observation

No pseudo-winner in $\langle \mathbb{Z}, < \rangle$.

Theorem

A tournament T contains a pseudo-winner

Pseudo-winners in tournaments

Finite case

Theorem

Every **finite tournament** has a **pseudo-winner**.

Proof

If t has maximal out-degree then t is a pseudo-winner.

Infinite case

Observation

No pseudo-winner in $\langle \mathbb{Z}, < \rangle$.

Theorem

A tournament T contains a pseudo-winner or $\exists x \neq y \in V$ s.t. $T = Out(x) \cup In(y)$.

Pseudo-winners in tournaments

Finite case

Theorem

Every **finite tournament** has a **pseudo-winner**.

Proof

If t has maximal out-degree then t is a pseudo-winner.

Infinite case

Observation

No pseudo-winner in $\langle \mathbb{Z}, < \rangle$.

Theorem

A tournament T contains a pseudo-winner or $\exists x \neq y \in V$ s.t. $T = Out(x) \cup In(y)$.

Proof

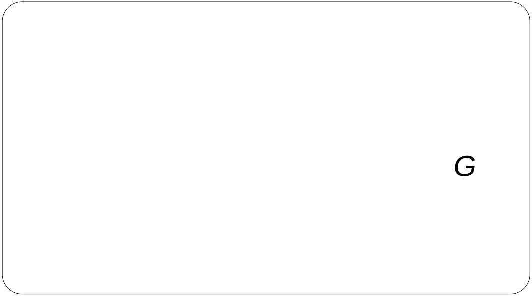
If y is not a pseudo-winner witnessed by x, then $T = Out(x) \cup In(y)$.

Quasi Kernels and Quasi Sinks

Quasi Kernels and Quasi Sinks

Theorem (Chvatal, Lovász)

Every finite ***digraph*** *(i.e. directed graph) contains a* ***quasi-kernel***



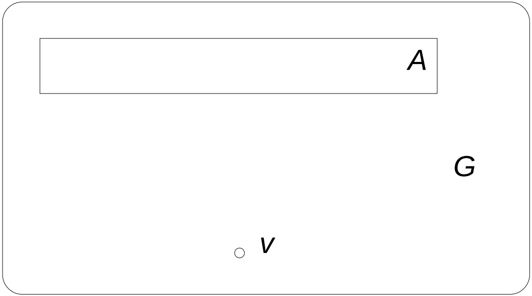
Quasi Kernels and Quasi Sinks

Theorem (Chvatal, Lovász)

Every finite ***digraph*** *(i.e. directed graph) contains a* ***quasi-kernel*** *(i.e it contains an* ***independent set*** *A*

Theorem (Chvatal, Lovász)

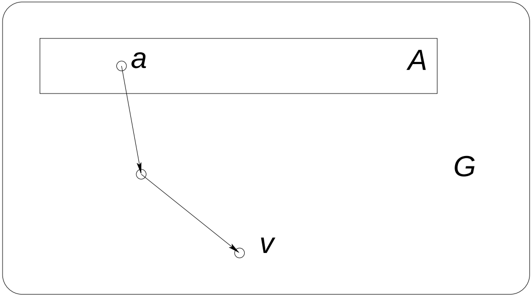
*Every finite **digraph** (i.e. directed graph) contains a **quasi-kernel** (i.e it contains an **independent set** A such that for each point v*



Quasi Kernels and Quasi Sinks

Theorem (Chvatal, Lovász)

*Every finite **digraph** (i.e. directed graph) contains a **quasi-kernel** (i.e it contains an **independent set** A such that for each point v there is a **path of length at most** 2 from some point of A to v.*



The original problem

joint work of P. L. Erdős, A. Hajnal and —

What is the right question?

The original problem

joint work of P. L. Erdős, A. Hajnal and —

What is the right question?

The original problem

joint work of P. L. Erdős, A. Hajnal and —

What is the right question?

Graphs with quasi-kernels

Theorem

*A **directed graph** $G = (V, E)$ has a **quasi-kernel**, provided (a) or (b) below holds:*

(a) $\text{In}(x)$ *is **finite** for each* $x \in V$,

(b) *the **chromatic number** of G is **finite**.*

Theorem

*A **directed graph** $G = (V, E)$ has a **quasi-kernel**, provided (a) or (b) below holds:*

(a) $\mathrm{In}(x)$ *is **finite** for each* $x \in V$,

(b) *the **chromatic number** of G is **finite**.*

Graphs with quasi-kernels

Theorem

*A **directed graph** $G = (V, E)$ has a **quasi-kernel**, provided (a) or (b) below holds:*

(a) $\mathrm{In}(x)$ *is **finite** for each* $x \in V$,

(b) *the **chromatic number** of G is **finite**.*

Graphs with quasi-kernels

Theorem

*A **directed graph** $G = (V, E)$ has a **quasi-kernel**, provided (a) or (b) below holds:*

(a) $\mathrm{In}(x)$ *is **finite** for each* $x \in V$,

(b) *the **chromatic number** of G is **finite**.*

Quasi Kernels and Quasi Sinks

Definition

Let $G = (V, E)$ be a digraph.
An **independent set** A is a **quasi-kernel** iff for each $v \in V$ there is a **path of length at most** 2 which leads **from some points of** A **to** v.
An **independent set** B is a **quasi-sink** iff for each $v \in V$ there is a path of length at most 2 which leads **from** v **to some points of** B.

Quasi Kernels and Quasi Sinks

Definition

Let $G = (V, E)$ be a digraph.
An **independent set** A is a **quasi-kernel** iff for each $v \in V$ there is a **path of length at most** 2 which leads **from some points of** A **to** v.
An **independent set** B is a **quasi-sink** iff for each $v \in V$ there is a path of length at most 2 which leads **from** v **to some points of** B.

Quasi Kernels and Quasi Sinks

Definition

Let $G = (V, E)$ be a digraph.
An **independent set** A is a **quasi-kernel** iff for each $v \in V$ there is a **path of length at most** 2 which leads **from some points of** A **to** v.
An **independent set** B is a **quasi-sink** iff for each $v \in V$ there is a path of length at most 2 which leads **from** v **to some points of** B.

Quasi Kernels and Quasi Sinks

Definition

Let $G = (V, E)$ be a digraph.
An **independent set** A is a **quasi-kernel** iff for each $v \in V$ there is a **path of length at most** 2 which leads **from some points of** A **to** v.
An **independent set** B is a **quasi-sink** iff for each $v \in V$ there is a path of length at most 2 which leads **from** v **to some points of** B.

Quasi Kernels and Quasi Sinks

Definition

Let $G = (V, E)$ be a digraph.
An **independent set** A is a **quasi-kernel** iff for each $v \in V$ there is a **path of length at most** 2 which leads **from some points of** A **to** v.
An **independent set** B is a **quasi-sink** iff for each $v \in V$ there is a path of length at most 2 which leads **from** v **to some points of** B.

Graphs with nice partitions

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

*Let $G = (V, E)$ be a directed graph. Then V has a **partition** (V_0, V_1) such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink** provided (a) or (b) below holds:*

(a) $K_n \not\subset \widetilde{G}$ *for some $n \geq 2$. (Especially, if the **chromatic number** of $\widetilde{G}$ is finite.)*

(b) $\widetilde{G}$ *is **locally finite**.*

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** **of the graph**, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

Let $G = (V, E)$ be a directed graph. Then V has a ***partition*** (V_0, V_1) *such that $G[V_0]$ has a* ***quasi-kernel****, and $G[V_1]$ has a* ***quasi-sink*** *provided (a) or (b) below holds:*

(a) $K_n \not\subset \widetilde{G}$ *for some $n \geq 2$. (Especially, if the* ***chromatic number*** *of $\widetilde{G}$ is finite.)*

(b) $\widetilde{G}$ *is* ***locally finite****.*

Graphs with nice partitions

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

*Let $G = (V, E)$ be a directed graph. Then V has a **partition** (V_0, V_1) such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink** provided (a) or (b) below holds:*

(a) $K_n \not\subset \widetilde{G}$ *for some $n \geq 2$. (Especially, if the **chromatic number** of $\widetilde{G}$ is finite.)*

(b) $\widetilde{G}$ *is **locally finite**.*

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

*Let $G = (V, E)$ be a directed graph. Then V has a **partition** (V_0, V_1) such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink** provided (a) or (b) below holds:*

(a) $K_n \not\subset \widetilde{G}$ *for some* $n \geq 2$. *(Especially, if the **chromatic number** of* $\widetilde{G}$ *is finite.)*

(b) $\widetilde{G}$ *is **locally finite**.*

Graphs with nice partitions

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

Let $G = (V, E)$ be a directed graph. Then V has a ***partition*** (V_0, V_1) *such that $G[V_0]$ has a* ***quasi-kernel****, and $G[V_1]$ has a* ***quasi-sink*** *provided (a) or (b) below holds:*

(a) $K_n \not\subset \widetilde{G}$ *for some $n \geq 2$. (Especially, if the* ***chromatic number*** *of $\widetilde{G}$ is finite.)*

(b) $\widetilde{G}$ *is* ***locally finite****.*

Graphs with nice partitions

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

*Let $G = (V, E)$ be a directed graph. Then V has a **partition** (V_0, V_1) such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink** provided (a) or (b) below holds:*

(a) $K_n \not\subset \widetilde{G}$ *for some* $n \geq 2$. *(Especially, if the **chromatic number** of* $\widetilde{G}$ *is finite.)*

(b) $\widetilde{G}$ *is **locally finite**.*

Graphs with nice partitions

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

*Let $G = (V, E)$ be a directed graph. Then V has a **partition** (V_0, V_1) such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink** provided (a) or (b) below holds:*

(a) $K_n \not\subset \widetilde{G}$ *for some $n \geq 2$. (Especially, if the **chromatic number** of $\widetilde{G}$ is finite.)*

(b) $\widetilde{G}$ *is **locally finite**.*

Graphs with nice partitions

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

Let $G = (V, E)$ be a directed graph. Then V has a ***partition*** (V_0, V_1) *such that $G[V_0]$ has a* ***quasi-kernel****, and $G[V_1]$ has a* ***quasi-sink*** *provided (a) or (b) below holds:*

(a) *$K_n \not\subset \widetilde{G}$ for some $n \geq 2$. (Especially, if the* ***chromatic number*** *of $\widetilde{G}$ is finite.)*

(b) *$\widetilde{G}$ is* ***locally finite****.*

Graphs with nice partitions

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

*Let $G = (V, E)$ be a directed graph. Then V has a **partition** (V_0, V_1) such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink** provided (a) or (b) below holds:*

(a) $K_n \not\subset \widetilde{G}$ *for some* $n \geq 2$. *(Especially, if the **chromatic number** of $\widetilde{G}$ is finite.)*

(b) $\widetilde{G}$ *is **locally finite**.*

Graphs with nice partitions

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

*Let $G = (V, E)$ be a directed graph. Then V has a **partition** (V_0, V_1) such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink** provided (a) or (b) below holds:*

(a) *$K_n \not\subset \widetilde{G}$ for some $n \geq 2$. (Especially, if the **chromatic number** of $\widetilde{G}$ is finite.)*

(b) *$\widetilde{G}$ is **locally finite**.*

Graphs with nice partitions

Definition

If $G = (V, E)$ is a digraph define the ***undirected complement*** of the graph, $\widetilde{G} = (V, \widetilde{E})$ as follows: $\{x, y\} \in \widetilde{E}$ if and only if $(x, y) \notin E$ and $(y, x) \notin E$.

Theorem

Let $G = (V, E)$ be a directed graph. Then V has a ***partition*** (V_0, V_1) *such that $G[V_0]$ has a* ***quasi-kernel****, and $G[V_1]$ has a* ***quasi-sink*** *provided (a) or (b) below holds:*

(a) $K_n \not\subset \widetilde{G}$ *for some $n \geq 2$. (Especially, if the* ***chromatic number*** *of $\widetilde{G}$ is finite.)*

(b) $\widetilde{G}$ *is* ***locally finite****.*

A theorem

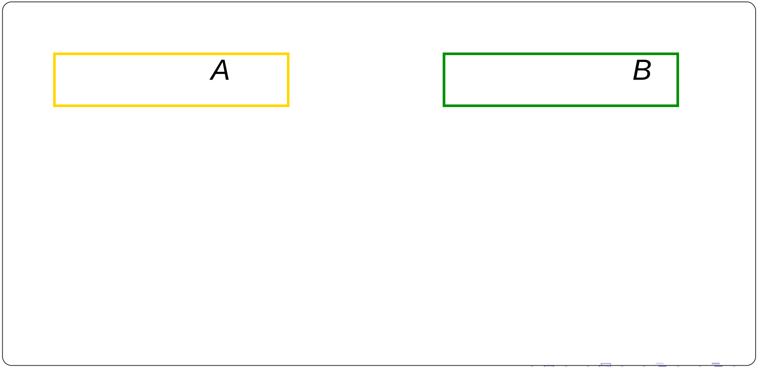
Theorem

For each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that for each $v \in V$ there is a path of length at most 2 which leads either from some points of A to v, or from v to some point of B.

A theorem

Theorem

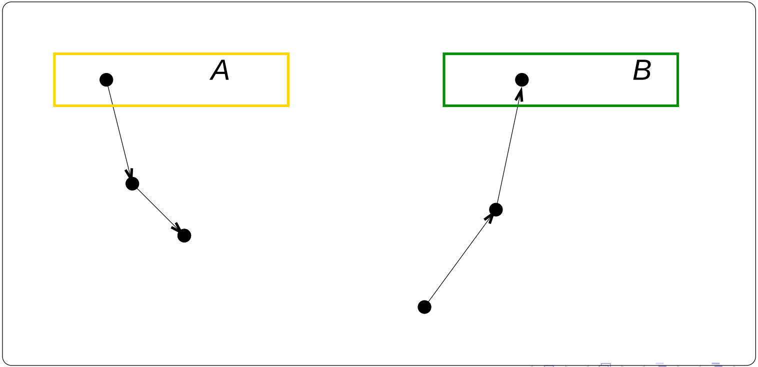
For each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that for each $v \in V$ there is a path of length at most 2 which leads either from some points of A to v, or from v to some point of B.



A theorem

Theorem

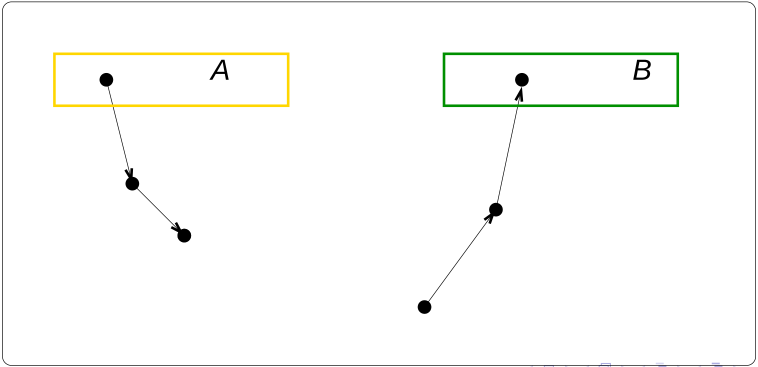
For each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that for each $v \in V$ there is a path of length at most 2 which leads either from some points of A to v, or from v to some point of B.



A theorem

Theorem

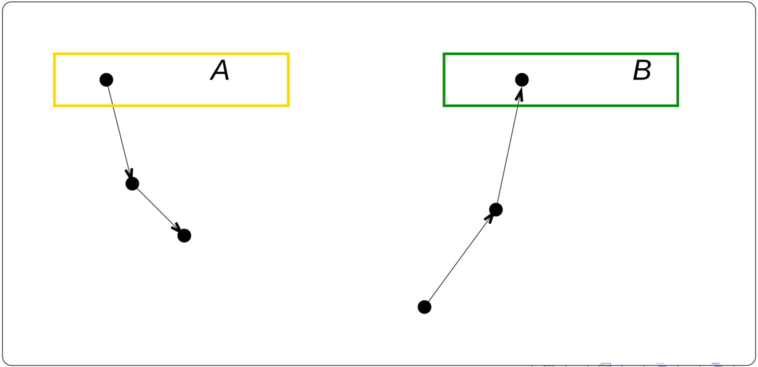
For each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that for each $v \in V$ there is a path of length at most 2 which leads either from some points of A to v, or from v to some point of B.



A theorem

Theorem

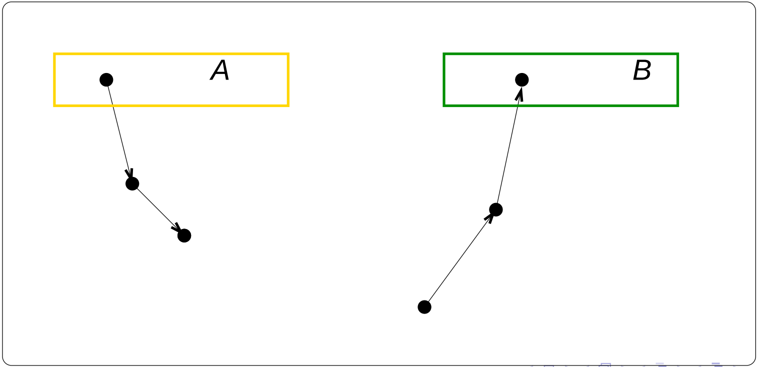
For each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that for each $v \in V$ there is a path of length at most 2 which leads either from some points of A to v, or from v to some point of B.



A theorem

Theorem

For each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that for each $v \in V$ there is a path of length at most 2 which leads either from some points of A to v, or from v to some point of B.



A theorem

Theorem

For each ***directed graph*** $G = (V, E)$ *there are* ***disjoint, independent subsets*** *A and B of V such that for each $v \in V$ there is a path of length at most 2 which leads either from some points of A to v, or from v to some point of B.*

Conjecture

For each digraph $G = (V, E)$ there is a **partition** (V_0, V_1) of V such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink**.

A theorem

Theorem

For each ***directed graph*** $G = (V, E)$ *there are* ***disjoint, independent subsets*** A *and* B *of* V *such that for each* $v \in V$ *there is a path of length at most 2 which leads either from some points of* A *to* v*, or from* v *to some point of* B*.*

Conjecture

For each digraph $G = (V, E)$ there is a **partition** (V_0, V_1) of V such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink**.

A theorem

Theorem

For each ***directed graph*** $G = (V, E)$ *there are* ***disjoint, independent subsets*** A *and* B *of* V *such that for each* $v \in V$ *there is a path of length at most 2 which leads either from some points of* A *to* v*, or from* v *to some point of* B.

Conjecture

For each digraph $G = (V, E)$ there is a **partition** (V_0, V_1) of V such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink**.

A theorem

Theorem

*For each **directed graph** $G = (V, E)$ there are **disjoint, independent subsets** A and B of V such that for each $v \in V$ there is a path of length at most 2 which leads either from some points of A to v, or from v to some point of B.*

Conjecture

For each digraph $G = (V, E)$ there is a **partition** (V_0, V_1) of V such that $G[V_0]$ has a **quasi-kernel**, and $G[V_1]$ has a **quasi-sink**.

Definition

Let $T = (V, E)$ be a tournament, $t \in V$ and $n \in \mathbb{N}$.
t is an ***n*-winner** iff for each $y \in V$ there is a **path of length at most n** which leads from t to y.

pseudo-winner = 2-winner

Theorem

Let $T = \langle V, E \rangle$ be an infinite tournament.
*(1) There is an infinite tournament $T = \langle V, E \rangle$ such that T has a **3-winner**, but there is **no 2-winner** in T.*
*(2) If T has an **n-winner** for some $n \geq 3$ then T has a 3-winner.*

Structure theorems for tournaments

Definition

Let $T = (V, E)$ be a tournament, $t \in V$ and $n \in \mathbb{N}$.
t is an ***n*-winner** iff for each $y \in V$ there is a **path of length at most** n which leads from t to y.

pseudo-winner = 2-winner

Theorem

Let $T = \langle V, E \rangle$ be an infinite tournament.
*(1) There is an infinite tournament $T = \langle V, E \rangle$ such that T has a 3-**winner**, but there is **no 2-winner** in T.*
*(2) If T has an **n-winner** for some $n \geq 3$ then T has a 3-winner.*

Definition

Let $T = (V, E)$ be a tournament, $t \in V$ and $n \in \mathbb{N}$.
t is an ***n*-winner** iff for each $y \in V$ there is a **path of length at most n** which leads from t to y.

pseudo-winner = 2-winner

Theorem

Let $T = \langle V, E\rangle$ be an infinite tournament.
*(1) There is an infinite tournament $T = \langle V, E\rangle$ such that T has a 3-**winner**, but there is **no 2-winner** in T.*
*(2) If T has an n-**winner** for some $n \geq 3$ then T has a 3-winner.*

Definition

Let $T = (V, E)$ be a tournament, $t \in V$ and $n \in \mathbb{N}$.
t is an ***n*-winner** iff for each $y \in V$ there is a **path of length at most n** which leads from t to y.

pseudo-winner = 2-winner

Theorem

Let $T = \langle V, E \rangle$ be an infinite tournament.
*(1) There is an infinite tournament $T = \langle V, E \rangle$ such that T has a **3-winner**, but there is **no 2-winner** in T.*
*(2) If T has an **n-winner** for some $n \geq 3$ then T has a 3-winner.*

Structure theorems for tournaments

Definition

Let $T = (V, E)$ be a tournament, $t \in V$ and $n \in \mathbb{N}$.
t is an ***n-winner*** iff for each $y \in V$ there is a **path of length at most** n which leads from t to y.

pseudo-winner = 2-winner

Theorem

Let $T = \langle V, E \rangle$ be an infinite tournament.
*(1) There is an infinite tournament $T = \langle V, E \rangle$ such that T has a 3-**winner**, but there is **no 2-winner** in T.*
*(2) If T has an n-**winner** for some $n \geq 3$ then T has a 3-**winner**.*

Definition

Let $T = (V, E)$ be a tournament, $t \in V$ and $n \in \mathbb{N}$.
t is an ***n*-winner** iff for each $y \in V$ there is a **path of length at most** n which leads from t to y.

pseudo-winner = 2-winner

Theorem

Let $T = \langle V, E \rangle$ be an infinite tournament.
*(1) There is an infinite tournament $T = \langle V, E \rangle$ such that T has a 3-**winner**, but there is **no** 2-**winner** in T.*
*(2) If T has an n-**winner** for some $n \geq 3$ then T has a 3-**winner**.*

Structure theorems for tournaments

Definition

Let $T = (V, E)$ be a tournament, $t \in V$ and $n \in \mathbb{N}$.
t is an ***n*-winner** iff for each $y \in V$ there is a **path of length at most n** which leads from t to y.

pseudo-winner = 2-winner

Theorem

Let $T = \langle V, E \rangle$ be an infinite tournament.
*(1) There is an infinite tournament $T = \langle V, E \rangle$ such that T has a **3-winner**, but there is **no 2-winner** in T.*
*(2) If T has an **n-winner** for some $n \geq 3$ then T has a **3-winner**.*

Definition

Let $T = (V, E)$ be a tournament, $t \in V$ and $n \in \mathbb{N}$.
t is an ***n-winner*** iff for each $y \in V$ there is a **path of length at most n** which leads from t to y.

pseudo-winner = 2-winner

Theorem

Let $T = \langle V, E \rangle$ be an infinite tournament.
*(1) There is an infinite tournament $T = \langle V, E \rangle$ such that T has a **3-winner**, but there is **no 2-winner** in T.*
*(2) If T has an **n-winner** for some $n \geq 3$ then T has a **3-winner**.*

Digraphs generated by finite structures

Definition

A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of G***, the elements of $N = V \setminus T$ are the ***nonterminal vertices of G***.

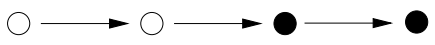
Construct $G \odot G = (W, F, S)$ from G as follows:
keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.
$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$
The edges are "inherited" from G in the natural way.

Digraphs generated by finite structures

Definition

A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of*** G, the elements of $N = V \setminus T$ are the ***nonterminal vertices of*** G.

Construct $G \odot G = (W, F, S)$ from G as follows:
keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.
$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$
The edges are "inherited" from G in the natural way.



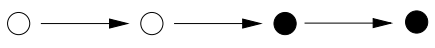
G

Digraphs generated by finite structures

Definition

A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of*** G, the elements of $N = V \setminus T$ are the ***nonterminal vertices of*** G.

Construct $G \odot G = (W, F, S)$ from G as follows:
keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.
$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$
The edges are “inherited” from G in the natural way.



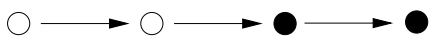
G

Digraphs generated by finite structures

Definition

A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of*** G, the elements of $N = V \setminus T$ are the ***nonterminal vertices of*** G.

Construct $G \odot G = (W, F, S)$ from G as follows:
keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.
$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$
The edges are “inherited” from G in the natural way.



G

Digraphs generated by finite structures

Definition

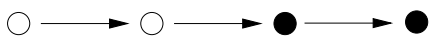
A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of*** G, the elements of $N = V \setminus T$ are the ***nonterminal vertices of*** G.

Construct $G \odot G = (W, F, S)$ from G as follows:

keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.

$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$

The edges are "inherited" from G in the natural way.



G

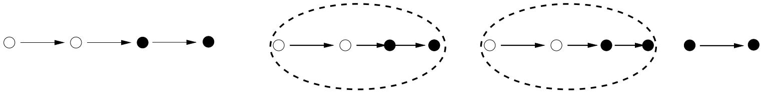
Digraphs generated by finite structures

Definition

A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of*** G, the elements of $N = V \setminus T$ are the ***nonterminal vertices of*** G.

Construct $G \odot G = (W, F, S)$ from G as follows:
keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.

$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$
The edges are “inherited” from G in the natural way.



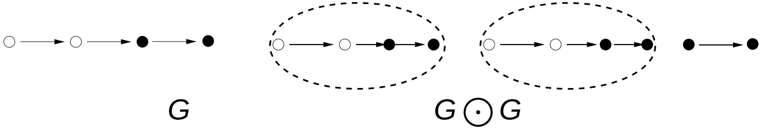
G $\qquad$ $G \odot G$

Digraphs generated by finite structures

Definition

A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of*** G, the elements of $N = V \setminus T$ are the ***nonterminal vertices of*** G.

Construct $G \odot G = (W, F, S)$ from G as follows:
keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.
$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$
The edges are "inherited" from G in the natural way.

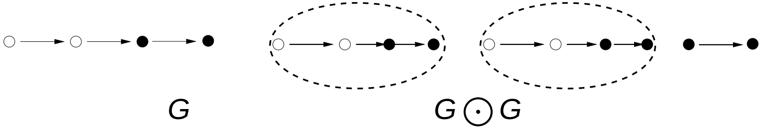


Digraphs generated by finite structures

Definition

A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of*** G, the elements of $N = V \setminus T$ are the ***nonterminal vertices of*** G.

Construct $G \odot G = (W, F, S)$ from G as follows:
keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.
$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$
The edges are "inherited" from G in the natural way.

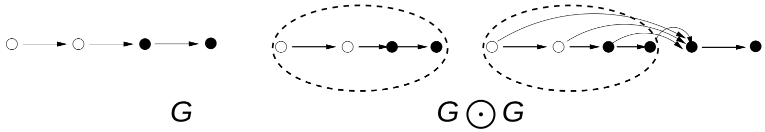


Digraphs generated by finite structures

Definition

A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of*** G, the elements of $N = V \setminus T$ are the ***nonterminal vertices of*** G.

Construct $G \odot G = (W, F, S)$ from G as follows:
keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.
$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$
The edges are "inherited" from G in the natural way.

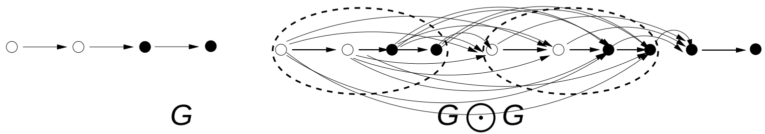


Digraphs generated by finite structures

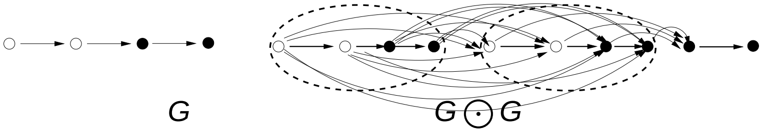
Definition

A ***digraph with terminal vertices*** is a triple $G = (V, E, T)$, where (V, E) is a digraph and $\emptyset \neq T \subset V$. The elements of T are the ***terminal vertices of*** G, the elements of $N = V \setminus T$ are the ***nonterminal vertices of*** G.

Construct $G \odot G = (W, F, S)$ from G as follows:
keep the terminal vertices and **blow up** each nonterminal vertex v to a (disjoint) copy G_v of G.
$T_{G \odot G} = T_G \cup \bigcup_{v \in V} T_{G_v}$, $N_{G \odot G} = \bigcup_{v \in V} N_{G_v}$
The edges are "inherited" from G in the natural way.



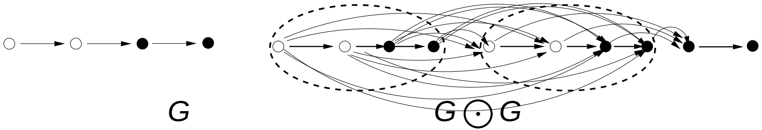
Digraphs generated by a finite structure



$G[T_G]$ is an **induced subgraph** of $(G \odot G)[T_{G \odot G}]$.
Now we can repeat the procedure above using $G \odot G$ instead of G to get $(G \odot G) \odot (G \odot G)$.
Hence we obtain a sequence $\langle G_n : n \in \mathbb{N} \rangle$ of digraphs with terminal vertices, $G_n = \langle V_n, E_n, T_n \rangle$ s. t. $G_0[T_0] \subset G_1[T_1] \subset G_2[T_2] \subset \dots$
Take

$$G^\infty = \bigcup \{ G_n[T_n] : n \in \mathbb{N} \}.$$

Digraphs generated by a finite structure



$G[T_G]$ is an **induced subgraph** of $(G \odot G)[T_{G \odot G}]$.

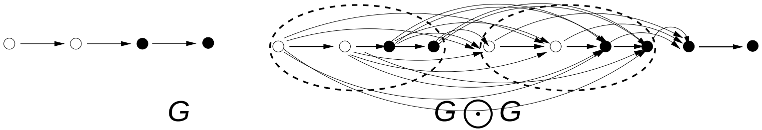
Now we can repeat the procedure above using $G \odot G$ instead of G to get $(G \odot G) \odot (G \odot G)$.

Hence we obtain a sequence $\langle G_n : n \in \mathbb{N} \rangle$ of digraphs with terminal vertices, $G_n = \langle V_n, E_n, T_n \rangle$ s. t. $G_0[T_0] \subset G_1[T_1] \subset G_2[T_2] \subset \dots$

Take

$$G^\infty = \bigcup \{G_n[T_n] : n \in \mathbb{N}\}.$$

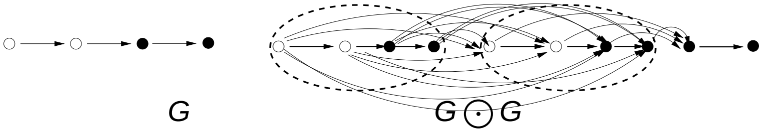
Digraphs generated by a finite structure



$G[T_G]$ is an **induced subgraph** of $(G \odot G)[T_{G \odot G}]$.
Now we can repeat the procedure above using $G \odot G$ instead of G to get $(G \odot G) \odot (G \odot G)$.
Hence we obtain a sequence $\langle G_n : n \in \mathbb{N} \rangle$ of digraphs with terminal vertices, $G_n = \langle V_n, E_n, T_n \rangle$ s. t. $G_0[T_0] \subset G_1[T_1] \subset G_2[T_2] \subset \dots$
Take

$$G^{\infty} = \bigcup \{ G_n[T_n] : n \in \mathbb{N} \}.$$

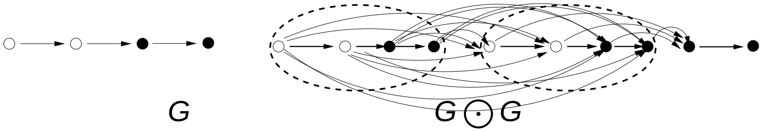
Digraphs generated by a finite structure



$G[T_G]$ is an **induced subgraph** of $(G \odot G)[T_{G \odot G}]$.
Now we can repeat the procedure above using $G \odot G$ instead of G to get $(G \odot G) \odot (G \odot G)$.
Hence we obtain a sequence $\langle G_n : n \in \mathbb{N} \rangle$ of digraphs with terminal vertices, $G_n = \langle V_n, E_n, T_n \rangle$ s. t. $G_0[T_0] \subset G_1[T_1] \subset G_2[T_2] \subset \ldots$
Take

$$G^\infty = \bigcup \{ G_n[T_n] : n \in \mathbb{N} \}.$$

Digraphs generated by a finite structure



$G[T_G]$ is an **induced subgraph** of $(G \odot G)[T_{G \odot G}]$.
Now we can repeat the procedure above using $G \odot G$ instead of G to get $(G \odot G) \odot (G \odot G)$.
Hence we obtain a sequence $\langle G_n : n \in \mathbb{N} \rangle$ of digraphs with terminal vertices, $G_n = \langle V_n, E_n, T_n \rangle$ s. t. $G_0[T_0] \subset G_1[T_1] \subset G_2[T_2] \subset \ldots$
Take

$$G^\infty = \bigcup \{G_n[T_n] : n \in \mathbb{N}\}.$$

Digraphs generated by a finite structure

Theorem

Let $G = (V, E, T)$ be a ***finite tournament with terminal vertices****.*
T. F. A. E:

(i) G^∞ *has a 3-winner,*

(ii) $\text{In}(v) \neq \emptyset$ *for each* $v \in V \setminus T$.

T. F. A. E.:

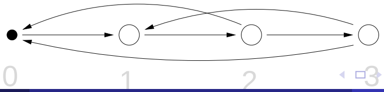
(a) G^∞ *has a 2-winner*

(b) *there is 2-winner* $v \in T$ *in* G.

Theorem

There is an finite tournament G such that G^∞ has a 3-winner, but no 2-winner.

G :



Digraphs generated by a finite structure

Theorem

Let $G = (V, E, T)$ be a ***finite tournament with terminal vertices****.*
T. F. A. E:

(i) G^∞ *has a 3-winner,*
(ii) $\text{In}(v) \neq \emptyset$ *for each* $v \in V \setminus T$.

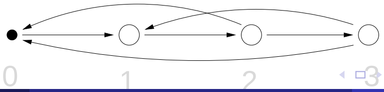
T. F. A. E.:

(a) G^∞ *has a 2-winner*
(b) *there is 2-winner* $v \in T$ *in* G.

Theorem

There is an finite tournament G *such that* G^∞ *has a 3-winner, but no 2-winner.*

G :



Digraphs generated by a finite structure

Theorem

Let $G = (V, E, T)$ be a ***finite tournament with terminal vertices****.*
T. F. A. E:

- *(i)* G^∞ *has a 3-winner,*
- *(ii)* $\mathrm{In}(v) \neq \emptyset$ *for each* $v \in V \setminus T$.

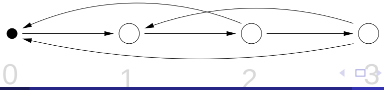
T. F. A. E.:

- *(a)* G^∞ *has a 2-winner*
- *(b)* *there is 2-winner* $v \in T$ *in* G.

Theorem

There is an finite tournament G such that G^∞ has a 3-winner, but no 2-winner.

G :



Digraphs generated by a finite structure

Theorem

Let $G = (V, E, T)$ be a ***finite tournament with terminal vertices****.*
T. F. A. E:

(i) G^∞ *has a 3-winner,*
(ii) $\mathrm{In}(v) \neq \emptyset$ *for each* $v \in V \setminus T$.

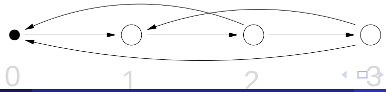
T. F. A. E.:

(a) G^∞ *has a 2-winner*
(b) *there is 2-winner* $v \in T$ *in* G.

Theorem

There is an finite tournament G such that G^∞ *has a 3-winner, but no 2-winner.*

G :



Digraphs generated by a finite structure

Theorem

Let $G = (V, E, T)$ be a ***finite tournament with terminal vertices****.*
T. F. A. E:

- *(i)* G^∞ *has a 3-winner,*
- *(ii)* $\mathrm{In}(v) \neq \emptyset$ *for each* $v \in V \setminus T$.

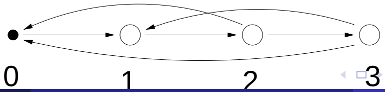
T. F. A. E.:

- *(a)* G^∞ *has a 2-winner*
- *(b)* *there is 2-winner* $v \in T$ *in* G.

Theorem

There is an finite tournament G such that G^∞ has a 3-winner, but no 2-winner.

G :



Digraphs generated by a finite structure

Theorem

If $G = \langle V, E, T\rangle$ is a finite digraph with terminal vertices $G^\infty = \langle V^\infty, E^\infty\rangle$, then there is a ***partition*** (V_0, V_1) *of V^∞ such that $G^\infty[V_0]$ has a* ***quasi-kernel****, and $G^\infty[V_1]$ has a* ***quasi-sink****.*

Digraphs generated by a finite structure

Theorem

*If $G = \langle V, E, T \rangle$ is a finite digraph with terminal vertices $G^\infty = \langle V^\infty, E^\infty \rangle$, then there is a **partition** (V_0, V_1) of V^∞ such that $G^\infty[V_0]$ has a **quasi-kernel**, and $G^\infty[V_1]$ has a **quasi-sink**.*

Digraphs generated by a finite structure

Theorem

If $G = \langle V, E, T \rangle$ is a finite digraph with terminal vertices $G^\infty = \langle V^\infty, E^\infty \rangle$, then there is a ***partition*** *(V_0, V_1) of V^∞ such that $G^\infty[V_0]$ has a* ***quasi-kernel****, and $G^\infty[V_1]$ has a* ***quasi-sink****.*

Multi-way cuts

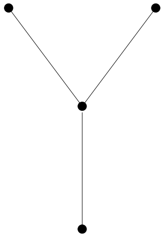
Multiway Cut Problem

Fix a graph $G = (V, E)$ and a subset S of vertices called terminals. A multiway cut is a set of edges whose removal disconnects each terminal from the others. The multiway cut problem is to find the minimal size of a multiway cut denoted by $\pi_{G,S}$.

Multi-way cuts

Multiway Cut Problem

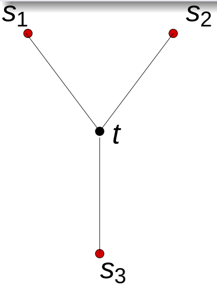
Fix a graph $G = (V, E)$



Multi-way cuts

Multiway Cut Problem

Fix a graph $G = (V, E)$ and a subset S of vertices called **terminals**.

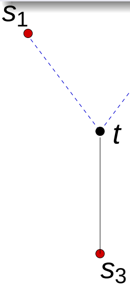


$S = \{s_1, s_2, s_3\}$

Multi-way cuts

Multiway Cut Problem

Fix a graph $G = (V, E)$ and a subset S of vertices called **terminals**. A **multiway cut** is a **set of edges** whose removal disconnects each terminal from the others.

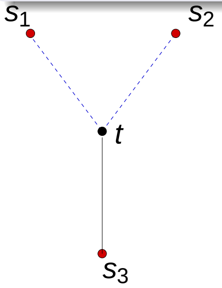


$$S = \{s_1, s_2, s_3\}$$

Multi-way cuts

Multiway Cut Problem

Fix a graph $G = (V, E)$ and a subset S of vertices called **terminals**. A **multiway cut** is a **set of edges** whose removal disconnects each terminal from the others. The **multiway cut problem** is to find the **minimal size** of a multiway cut denoted by $\pi_{G,S}$.



$$S = \{s_1, s_2, s_3\}$$

$$\pi_{G,S} = 2$$

Multiway cuts

Definition

If $\vec{G} = (V, E)$ is a directed graph and $A, B \subset V$ let $\lambda(\vec{G}, A, B)$ be the maximal number of **edge-disjoint directed paths** from some element of A into some element of B.

Multiway cuts

Definition

If $\vec{G} = (V, E)$ is a directed graph and $A, B \subset V$ let $\lambda(\vec{G}, A, B)$ be the maximal number of **edge-disjoint directed paths** from some element of A into some element of B.

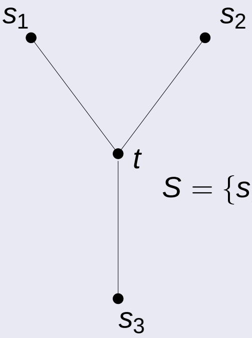
Definition

Definition

If $\vec{G} = (V, E)$ is a directed graph and $A, B \subset V$ let $\lambda(\vec{G}, A, B)$ be the maximal number of **edge-disjoint directed paths** from some element of A into some element of B.

Definition

If $G = (V, E)$ is a finite graph, $S \subset V$,



$S = \{s_1, s_2, s_3\}$

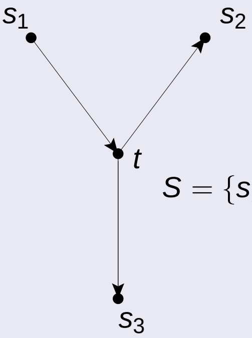
Multiway cuts

Definition

If $\vec{G} = (V, E)$ is a directed graph and $A, B \subset V$ let $\lambda(\vec{G}, A, B)$ be the maximal number of **edge-disjoint directed paths** from some element of A into some element of B.

Definition

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges,



$S = \{s_1, s_2, s_3\}$

Multiway cuts

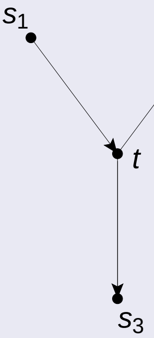
Definition

If $\vec{G} = (V, E)$ is a directed graph and $A, B \subset V$ let $\lambda(\vec{G}, A, B)$ be the maximal number of **edge-disjoint directed paths** from some element of A into some element of B.

Definition

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges, then let

$$\nu_{\vec{G},S} = \sum_{s \in S} \lambda(\vec{G}, S - s, s)$$



$S = \{s_1, s_2, s_3\}$

Multiway cuts

Definition

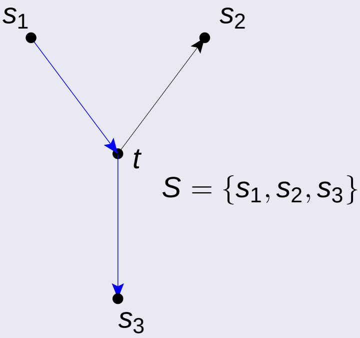
If $\vec{G} = (V, E)$ is a directed graph and $A, B \subset V$ let $\lambda(\vec{G}, A, B)$ be the maximal number of **edge-disjoint directed paths** from some element of A into some element of B.

Definition

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges, then let

$$\nu_{\vec{G},S} = \sum_{s \in S} \lambda(\vec{G}, S - s, s)$$

$\lambda(\vec{G}, S - s_3, s_3) = 1,$



Multiway cuts

Definition

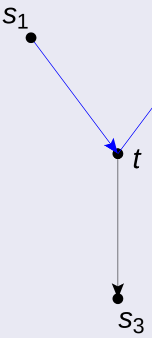
If $\vec{G} = (V, E)$ is a directed graph and $A, B \subset V$ let $\lambda(\vec{G}, A, B)$ be the maximal number of **edge-disjoint directed paths** from some element of A into some element of B.

Definition

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges, then let

$$\nu_{\vec{G},S} = \sum_{s \in S} \lambda(\vec{G}, S - s, s)$$

$\lambda(\vec{G}, S - s_3, s_3) = 1$, $\lambda(\vec{G}, S - s_2, s_2) = 1$,



$S = \{s_1, s_2, s_3\}$

Multiway cuts

Definition

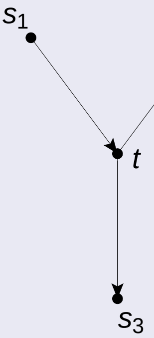
If $\vec{G} = (V, E)$ is a directed graph and $A, B \subset V$ let $\lambda(\vec{G}, A, B)$ be the maximal number of **edge-disjoint directed paths** from some element of A into some element of B.

Definition

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges, then let

$$\nu_{\vec{G},S} = \sum_{s \in S} \lambda(\vec{G}, S - s, s)$$

$\lambda(\vec{G}, S - s_3, s_3) = 1$, $\lambda(\vec{G}, S - s_2, s_2) = 1$,
$\lambda(\vec{G}, S - s_1, s_1) = 0$,



$S = \{s_1, s_2, s_3\}$

Multiway cuts

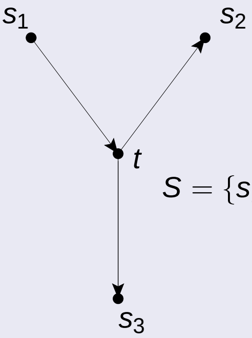
Definition

If $\vec{G} = (V, E)$ is a directed graph and $A, B \subset V$ let $\lambda(\vec{G}, A, B)$ be the maximal number of **edge-disjoint directed paths** from some element of A into some element of B.

Definition

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges, then let

$$\nu_{\vec{G},S} = \sum_{s \in S} \lambda(\vec{G}, S - s, s)$$



$$S = \{s_1, s_2, s_3\}$$

$\lambda(\vec{G}, S - s_3, s_3) = 1$, $\lambda(\vec{G}, S - s_2, s_2) = 1$, $\lambda(\vec{G}, S - s_1, s_1) = 0$, $\nu_{\vec{G},S} = 2$

Multiway cuts

Theorem (P. L. Erdős, A. Frank, L. Székely)

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges,

Multiway cuts

Theorem (P. L. Erdős, A. Frank, L. Székely)

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges, then $\nu_{\vec{G},S} \leq \pi_{G,S}$.

Multiway cuts

Theorem (P. L. Erdős, A. Frank, L. Székely)

If $G = (V, E)$ is a finite graph, $S \subset V$, and $\vec{G}$ is obtained from G by an orientation of the edges, then $\nu_{\vec{G},S} \leq \pi_{G,S}$.

Theorem (E. Dahjhaus, D. S. Johson, C. H. Papadimitriou, P.D. Seymout, M. Yannakakis)

The multiway cut problem is ***NP-complete***.

Multiway cuts

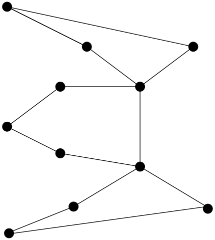
Special case:

$G - S$ is a **tree**.

Multiway cuts

Special case:

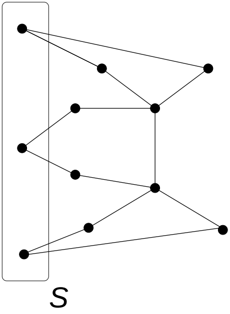
$G - S$ is a **tree**.



Multiway cuts

Special case:

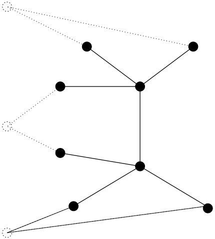
$G - S$ is a **tree**.



Multiway cuts

Special case:

$G - S$ is a **tree**.



Multiway cuts

Theorem (P. L. Erdős, L. Székely)

If $G = (V, E)$ is a finite graph, $S \subset V$ such that $G - S$ is tree, then

$$\max_{\vec{G}} \nu_{\vec{G},S} = \pi_{G,S}.$$

where the maximum is taken over all orientations $\vec{G}$ of G.

Multiway cuts

Theorem (P. L. Erdős, A. Frank, L. Székely, reformulated)

Multiway cuts

Theorem (P. L. Erdős, A. Frank, L. Székely, reformulated)

If $G = (V, E)$ is a ***finite graph***,

Multiway cuts

Theorem (P. L. Erdős, A. Frank, L. Székely, reformulated)

If $G = (V, E)$ *is a* ***finite graph****,* $S \subset V$ *such that* $G - S$ *is* ***tree,***

Theorem (P. L. Erdős, A. Frank, L. Székely, reformulated)

If $G = (V, E)$ is a ***finite graph****, $S \subset V$ such that $G - S$ is* ***tree,*** *then there is an* ***orientation*** $\vec{G}$ *of G,*

Multiway cuts

Theorem (P. L. Erdős, A. Frank, L. Székely, reformulated)

*If $G = (V, E)$ is a **finite graph**, $S \subset V$ such that $G - S$ is **tree,** then there is an **orientation** $\vec{G}$ of G, and for each $s \in S$ there is an **edge-disjoint family $\mathcal{P}_s$ of $(S - s, s)$-paths** in $\vec{G}$*

Multiway cuts

Theorem (P. L. Erdős, A. Frank, L. Székely, reformulated)

*If $G = (V, E)$ is a **finite graph**, $S \subset V$ such that $G - S$ is **tree,** then there is an **orientation** $\vec{G}$ of G, and for each $s \in S$ there is an **edge-disjoint family** $\mathcal{P}_s$ **of** $(S - s, s)$**-paths** in $\vec{G}$ and for each $P \in \mathcal{P}_s$ we can pick an edge $e_P \in P$*

Multiway cuts

Theorem (P. L. Erdős, A. Frank, L. Székely, reformulated)

If $G = (V, E)$ is a ***finite graph****, $S \subset V$ such that $G - S$ is* ***tree,*** *then there is an* ***orientation*** *$\vec{G}$ of G, and for each $s \in S$ there is an* ***edge-disjoint family*** *$\mathcal{P}_s$* ***of*** *$(S - s, s)$****-paths*** *in $\vec{G}$ and for each $P \in \mathcal{P}_s$ we can pick an edge $e_P \in P$ such that*

$$\{e_P : P \in \mathcal{P}_s \text{ for some } s \in S\}$$

is a ***multiway cut*** *(in G for S).*

Multiway cuts

Theorem (–)

,

Multiway cuts

Theorem (–)

If $G = (V, E)$ is a graph,

,

Multiway cuts

Theorem (–)

If $G = (V, E)$ is a graph, $S \subset V$ is finite such that $G - S$ is tree

,

Multiway cuts

Theorem (–)

If $G = (V, E)$ is a graph, $S \subset V$ is finite such that $G - S$ is tree ***without infinite trails****,*

Multiway cuts

Theorem (–)

If $G = (V, E)$ is a graph, $S \subset V$ is finite such that $G - S$ is tree ***without infinite trails****, then there is an orientation $\vec{G}$ of G,*

Multiway cuts

Theorem (–)

If $G = (V, E)$ is a graph, $S \subset V$ is finite such that $G - S$ is tree ***without infinite trails****, then there is an orientation $\vec{G}$ of G, and for each $s \in S$ there is an edge-disjoint family $\mathcal{P}_s$ of $(S - s, s)$-paths in $\vec{G}$*

Multiway cuts

Theorem (–)

*If $G = (V, E)$ is a graph, $S \subset V$ is finite such that $G - S$ is tree **without infinite trails**, then there is an orientation $\vec{G}$ of G, and for each $s \in S$ there is an edge-disjoint family $\mathcal{P}_s$ of $(S - s, s)$-paths in $\vec{G}$ and for each $P \in \mathcal{P}_s$ we can pick an edge $e_P \in P$*

Theorem (–)

If $G = (V, E)$ is a graph, $S \subset V$ is finite such that $G - S$ is tree ***without infinite trails****, then there is an orientation $\vec{G}$ of G, and for each $s \in S$ there is an edge-disjoint family $\mathcal{P}_s$ of $(S - s, s)$-paths in $\vec{G}$ and for each $P \in \mathcal{P}_s$ we can pick an edge $e_P \in P$ such that*

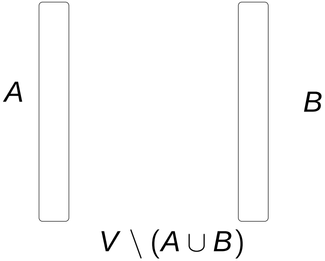
$$\{e_P : P \in \mathcal{P}_s \text{ for some } s \in S\}$$

is a multiway cut (in G for S).

Multiway cuts

Proposition

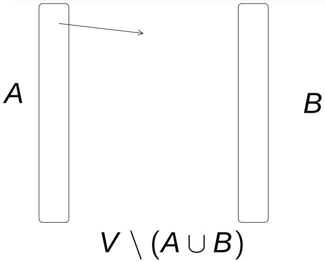
Let $G = (V, E)$ be a finite directed graph, and $A, B \subset V$ s.t



Proposition

Let $G = (V, E)$ be a finite directed graph, and $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,



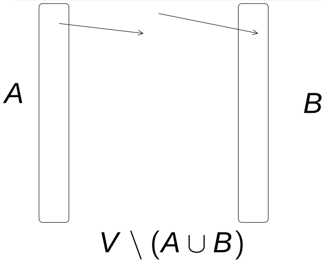
Multiway cuts

Proposition

Let $G = (V, E)$ be a finite directed graph, and $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,



Multiway cuts

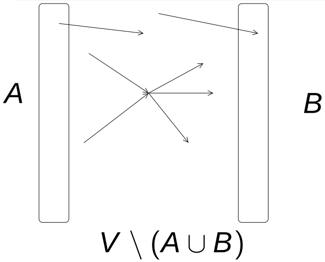
Proposition

Let $G = (V, E)$ be a finite directed graph, and $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup b)$.



Multiway cuts

Proposition

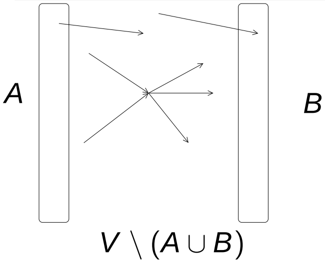
Let $G = (V, E)$ be a finite directed graph, and $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup b)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.



Multiway cuts

Proposition

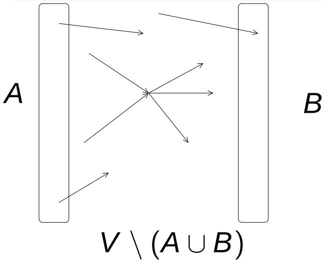
Let $G = (V, E)$ be a finite directed graph, and $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup b)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.



Multiway cuts

Proposition

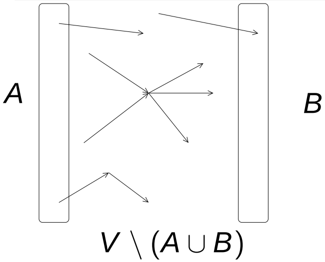
Let $G = (V, E)$ be a finite directed graph, and $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \le out(x)$ for each $x \in V \setminus (A \cup b)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.



Multiway cuts

Proposition

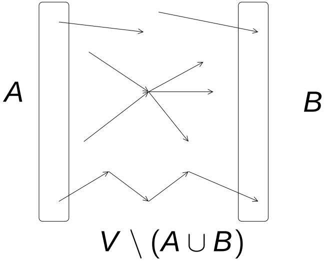
Let $G = (V, E)$ be a finite directed graph, and $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup b)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.



Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed trail, and let $A, B \subset V$ s.t

Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed trail, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed trail, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed trail, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup B)$.

Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed trail, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup B)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.

Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed trail, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup B)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.

Proof.

G is countable:

□

Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed trail, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup B)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.

Proof.

G is countable: easy induction:

□

Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed trail, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup B)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.

Proof.

G is countable: easy induction: if P is an A-B-path then $G - P$ satisfies (1)–(3)

□

Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed trail, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup B)$.

Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.

Proof.

G is countable: easy induction: if P is an A-B-path then $G - P$ satisfies (1)–(3)
If G is uncountable

□

Theorem

Let $G = (V, E)$ be a directed graph which does not contain infinite directed trail, and let $A, B \subset V$ s.t

(1) $in(a) = 0$ and $out(a) = 1$ for each $a \in A$,

(2) $in(b) = 1$ and $out(b) = 0$ for each $b \in B$,

(3) $in(x) \leq out(x)$ for each $x \in V \setminus (A \cup B)$.

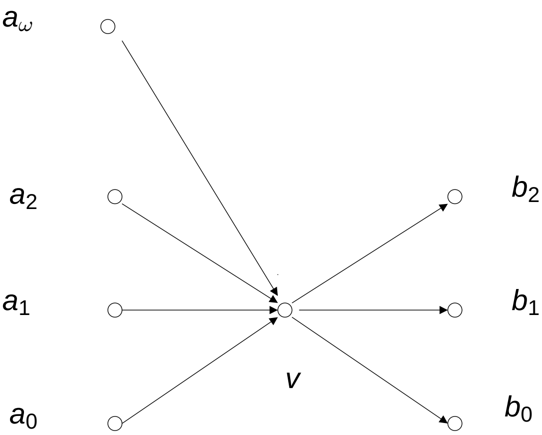
Then there is a family $\mathcal{P}$ of edge-disjoint A-B-paths s .t. $\mathcal{P}$ covers A.

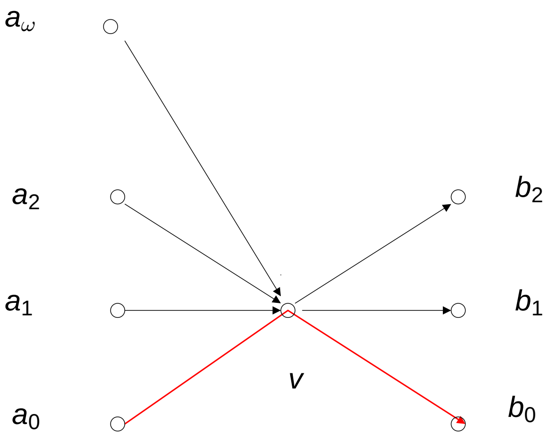
Proof.

G is countable: easy induction: if P is an A-B-path then $G - P$ satisfies (1)–(3)
If G is uncountable then we may got stuck at some point

□

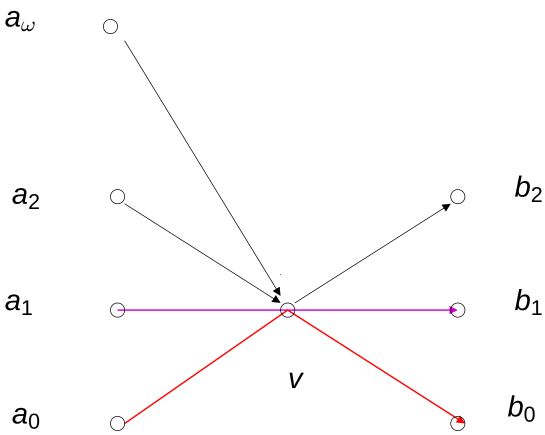
Multiway cuts
Infinite case



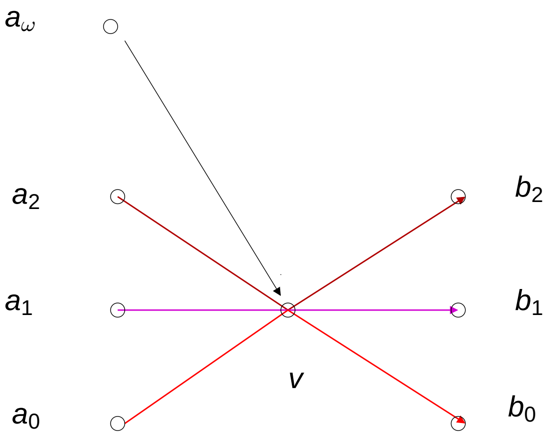


a_ω
a_2
a_1
a_0
v
b_2
b_1
b_0

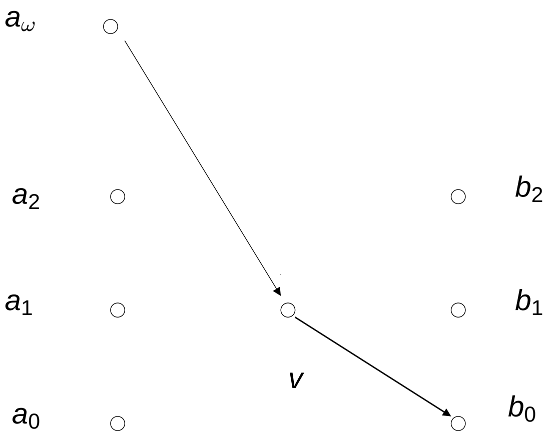
Multiway cuts
Infinite case

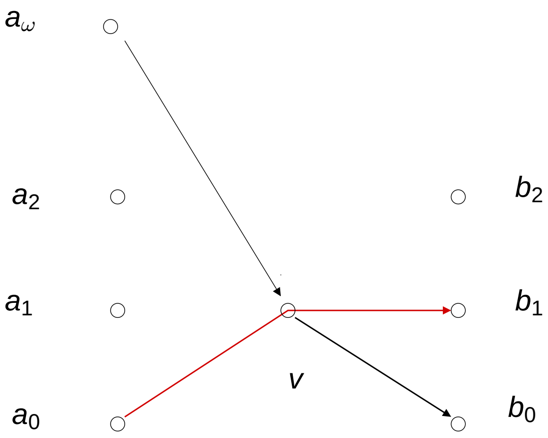


Multiway cuts
Infinite case



Multiway cuts
Infinite case





a_ω
a_2
a_1
a_0
b_2
b_1
b_0
v

Multi-way cuts

Uncountable case

$G = (V, E)$, $A, B \subset V$, $|V| = |A| = \omega_1$

Multi-way cuts
Uncountable case

$G = (V, E)$, $A, B \subset V$, $|V| = |A| = \omega_1$
Inductive construction, but using the **right enumeration**

$G = (V, E)$, $A, B \subset V$, $|V| = |A| = \omega_1$

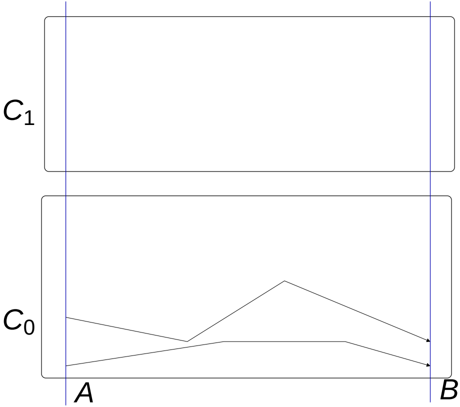
Inductive construction, but using the **right enumeration**

Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$

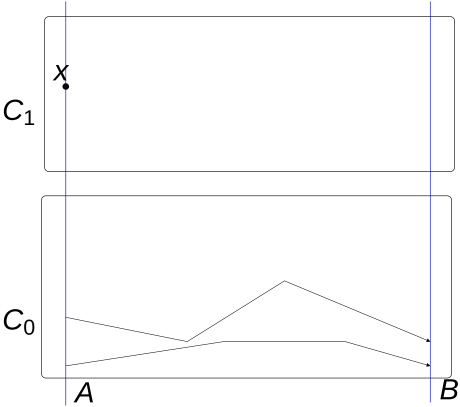
$G = (V, E)$, $A, B \subset V$, $|V| = |A| = \omega_1$
Inductive construction, but using the **right enumeration**
Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$
Enumerate $A = \{a_\xi : \xi < \omega_1\}$ such that $C_0 \cap A = \{a_0, a_1, \dots\}$,
$C_1 \cap A = \{a_\omega, a_{\omega+1}, \dots\}$,

$G = (V, E)$, $A, B \subset V$, $|V| = |A| = \omega_1$
Inductive construction, but using the **right enumeration**
Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$
Enumerate $A = \{a_\xi : \xi < \omega_1\}$ such that $C_0 \cap A = \{a_0, a_1, \dots\}$,
$C_1 \cap A = \{a_\omega, a_{\omega+1}, \dots\}$,
By transfinite induction find edge-disjoint families $\mathcal{P}_\alpha$ of A-B paths in $G[\cup\{C_\xi : \xi \leq \alpha\}]$ such that $\mathcal{P}_\alpha$ covers $C_\alpha \cap A$.

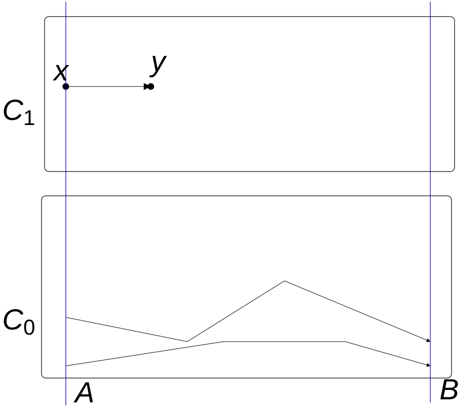
Multi-way cuts



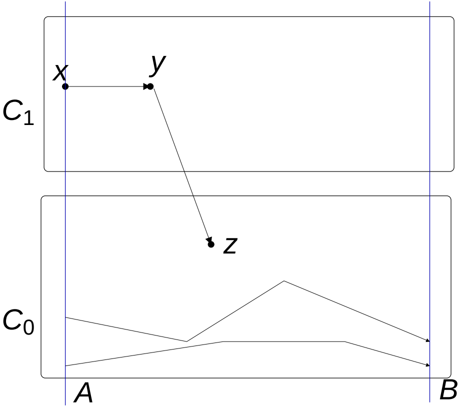
Multi-way cuts



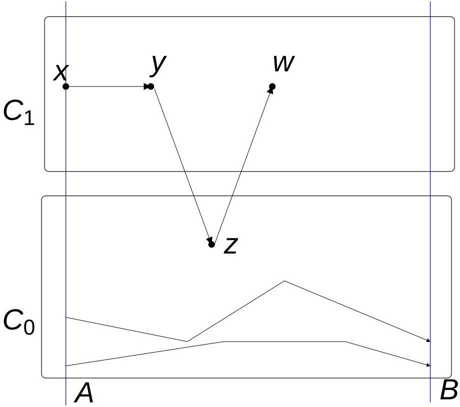
Multi-way cuts



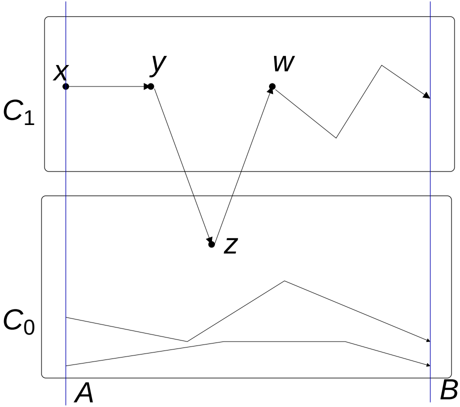
Multi-way cuts



Multi-way cuts



Multi-way cuts



Multi-way cuts
Uncountable case

Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$ s.t

Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$ s.t

- if $x \in C_\alpha$ with $|In(x)| \leq \omega$ then $In(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.

Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$ s.t

- if $x \in C_\alpha$ with $|In(x)| \leq \omega$ then $In(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.
- if $x \in C_\alpha$ with $|Out(x)| \leq \omega$ then $Out(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.

Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$ s.t

- if $x \in C_\alpha$ with $|In(x)| \le \omega$ then $In(x) \subset \cup\{C_\xi : \xi \le \alpha\}$.
- if $x \in C_\alpha$ with $|Out(x)| \le \omega$ then $Out(x) \subset \cup\{C_\xi : \xi \le \alpha\}$.
- if $x \in C_\alpha$ with $|Out(x)| > \omega$ then $Out(x) \cap C_\beta$ is infinite for each $\alpha \le \beta < \omega_1$.

Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$ s.t

- if $x \in C_\alpha$ with $|In(x)| \leq \omega$ then $In(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.
- if $x \in C_\alpha$ with $|Out(x)| \leq \omega$ then $Out(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.
- if $x \in C_\alpha$ with $|Out(x)| > \omega$ then $Out(x) \cap C_\beta$ is infinite for each $\alpha \leq \beta < \omega_1$.

How to get such a partition?

Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$ s.t

- if $x \in C_\alpha$ with $|In(x)| \leq \omega$ then $In(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.
- if $x \in C_\alpha$ with $|Out(x)| \leq \omega$ then $Out(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.
- if $x \in C_\alpha$ with $|Out(x)| > \omega$ then $Out(x) \cap C_\beta$ is infinite for each $\alpha \leq \beta < \omega_1$.

How to get such a partition? How to get the right properties of such a partition?

Partition V into countable sets $\{C_\alpha : \alpha < \omega_1\}$ s.t

- if $x \in C_\alpha$ with $|In(x)| \le \omega$ then $In(x) \subset \cup\{C_\xi : \xi \le \alpha\}$.
- if $x \in C_\alpha$ with $|Out(x)| \le \omega$ then $Out(x) \subset \cup\{C_\xi : \xi \le \alpha\}$.
- if $x \in C_\alpha$ with $|Out(x)| > \omega$ then $Out(x) \cap C_\beta$ is infinite for each $\alpha \le \beta < \omega_1$.

How to get such a partition? How to get the right properties of such a partition?

Elementary submodels

Let θ be a large regular cardinals.

Let θ be a large regular cardinals. $\theta = (2^{|G|})^+$

Let θ be a large regular cardinals. $\theta = (2^{|G|})^{+}$
transitive closure of a set x is $x \cup (\cup x) \cup (\cup \cup x) \cup \dots$

Let θ be a large regular cardinals. $\theta = (2^{|G|})^+$
transitive closure of a set x is $x \cup (\cup x) \cup (\cup \cup x) \cup \ldots$
Let $H(\theta)$ be the family of sets whose transitive closure has cardinality less than θ.

Let θ be a large regular cardinals. $\theta = (2^{|G|})^+$
transitive closure of a set x is $x \cup (\cup x) \cup (\cup \cup x) \cup \ldots$
Let $H(\theta)$ be the family of sets whose transitive closure has cardinality less than θ.
$\mathcal{H}(\theta) = \langle H(\theta), \in, \prec \rangle$, where $\prec$ is a well-ordering

Let θ be a large regular cardinals. $\theta = (2^{|G|})^{+}$
transitive closure of a set x is $x \cup (\cup x) \cup (\cup \cup x) \cup \ldots$
Let $H(\theta)$ be the family of sets whose transitive closure has cardinality less than θ.
$\mathcal{H}(\theta) = \langle H(\theta), \in, \prec \rangle$, where $\prec$ is a well-ordering
Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be an **increasing continuous chain of countable elementary submodels** of $\mathcal{H}(\theta)$ with $G, A \in M_0$.

Let θ be a large regular cardinals. $\theta = (2^{|G|})^+$

transitive closure of a set x is $x \cup (\cup x) \cup (\cup \cup x) \cup \dots$

Let $H(\theta)$ be the family of sets whose transitive closure has cardinality less than θ.

$\mathcal{H}(\theta) = \langle H(\theta), \in, \prec \rangle$, where $\prec$ is a well-ordering

Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be an **increasing continuous chain of countable elementary submodels** of $\mathcal{H}(\theta)$ with $G, A \in M_0$. i.e.

(1) M_α is a countable elementary submodel of $\mathcal{H}(\theta)$ for $\alpha < \omega_1$

Let θ be a large regular cardinals. $\theta = (2^{|G|})^+$
transitive closure of a set x is $x \cup (\cup x) \cup (\cup \cup x) \cup \dots$
Let $H(\theta)$ be the family of sets whose transitive closure has cardinality less than θ.
$\mathcal{H}(\theta) = \langle H(\theta), \in, \prec \rangle$, where $\prec$ is a well-ordering
Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be an **increasing continuous chain of countable elementary submodels** of $\mathcal{H}(\theta)$ with $G, A \in M_0$. i.e.

(1) M_α is a countable elementary submodel of $\mathcal{H}(\theta)$ for $\alpha < \omega_1$

(2) $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$,

Let θ be a large regular cardinals. $\theta = (2^{|G|})^+$
transitive closure of a set x is $x \cup (\cup x) \cup (\cup \cup x) \cup \ldots$
Let $H(\theta)$ be the family of sets whose transitive closure has cardinality less than θ.
$\mathcal{H}(\theta) = \langle H(\theta), \in, \prec \rangle$, where $\prec$ is a well-ordering
Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be an **increasing continuous chain of countable elementary submodels** of $\mathcal{H}(\theta)$ with $G, A \in M_0$. i.e.

(1) M_α is a countable elementary submodel of $\mathcal{H}(\theta)$ for $\alpha < \omega_1$
(2) $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$,
(3) $M_\alpha = \bigcup \{M_\beta : \beta < \alpha\}$ provided α is limit

Let θ be a large regular cardinals. $\theta = (2^{|G|})^+$
transitive closure of a set x is $x \cup (\cup x) \cup (\cup \cup x) \cup \ldots$
Let $H(\theta)$ be the family of sets whose transitive closure has cardinality less than θ.
$\mathcal{H}(\theta) = \langle H(\theta), \in, \prec \rangle$, where $\prec$ is a well-ordering
Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be an **increasing continuous chain of countable elementary submodels** of $\mathcal{H}(\theta)$ with $G, A \in M_0$. i.e.

(1) M_α is a countable elementary submodel of $\mathcal{H}(\theta)$ for $\alpha < \omega_1$
(2) $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$,
(3) $M_\alpha = \bigcup \{M_\beta : \beta < \alpha\}$ provided α is limit
(4) $G, A \in M_0$.

Let θ be a large regular cardinals. $\theta = (2^{|G|})^+$

transitive closure of a set x is $x \cup (\cup x) \cup (\cup \cup x) \cup \dots$

Let $H(\theta)$ be the family of sets whose transitive closure has cardinality less than θ.

$\mathcal{H}(\theta) = \langle H(\theta), \in, \prec \rangle$, where $\prec$ is a well-ordering

Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be an **increasing continuous chain of countable elementary submodels** of $\mathcal{H}(\theta)$ with $G, A \in M_0$. i.e.

(1) M_α is a countable elementary submodel of $\mathcal{H}(\theta)$ for $\alpha < \omega_1$

(2) $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$,

(3) $M_\alpha = \bigcup \{M_\beta : \beta < \alpha\}$ provided α is limit

(4) $G, A \in M_0$.

Let $C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \leq \alpha < \omega_1$.

$\langle M_\alpha : \alpha < \omega_1 \rangle$ is an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in \rangle$ with $G, A \in M_0$.
$C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \leq \alpha < \omega_1$.

$\langle M_\alpha : \alpha < \omega_1 \rangle$ is an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in \rangle$ with $G, A \in M_0$.
$C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \leq \alpha < \omega_1$.

(*) if $x \in C_\alpha$ with $|In(x)| \leq \omega$ then $In(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.

$\langle M_\alpha : \alpha < \omega_1\rangle$ is an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in\rangle$ with $G, A \in M_0$.
$C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \leq \alpha < \omega_1$.

(*) if $x \in C_\alpha$ with $|In(x)| \leq \omega$ then $In(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.

$\mathcal{H}(\theta) \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega\rangle$"

$\langle M_\alpha : \alpha < \omega_1 \rangle$ is an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in \rangle$ with $G, A \in M_0$.
$C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \leq \alpha < \omega_1$.

(*) if $x \in C_\alpha$ with $|In(x)| \leq \omega$ then $In(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.

$\mathcal{H}(\theta) \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega \rangle$"
$M_{\alpha+1} \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega \rangle$"

$\langle M_\alpha : \alpha < \omega_1 \rangle$ is an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in \rangle$ with $G, A \in M_0$.
$C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \le \alpha < \omega_1$.

(*) if $x \in C_\alpha$ with $|In(x)| \le \omega$ then $In(x) \subset \cup\{C_\xi : \xi \le \alpha\}$.

$\mathcal{H}(\theta) \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega \rangle$"
$M_{\alpha+1} \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega \rangle$"
there is $\vec{x} \in M_{\alpha+1}$ s.t. $M_{\alpha+1} \models$ "$\vec{x}$ is an enumeration of $In(x)$"

$\langle M_\alpha : \alpha < \omega_1 \rangle$ is an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in \rangle$ with $G, A \in M_0$.
$C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \leq \alpha < \omega_1$.

(*) if $x \in C_\alpha$ with $|In(x)| \leq \omega$ then $In(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.

$\mathcal{H}(\theta) \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega \rangle$"
$M_{\alpha+1} \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega \rangle$"
there is $\vec{x} \in M_{\alpha+1}$ s.t. $M_{\alpha+1} \models$ "$\vec{x}$ is an enumeration of $In(x)$"
$\mathcal{H}(\theta) \models \vec{x}$ is an enumeration of $In(x)$

$\langle M_\alpha : \alpha < \omega_1\rangle$ is an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in\rangle$ with $G, A \in M_0$.
$C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \le \alpha < \omega_1$.

(*) if $x \in C_\alpha$ with $|In(x)| \le \omega$ then $In(x) \subset \cup\{C_\xi : \xi \le \alpha\}$.

$\mathcal{H}(\theta) \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega\rangle$"
$M_{\alpha+1} \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega\rangle$"
there is $\vec{x} \in M_{\alpha+1}$ s.t. $M_{\alpha+1} \models$ "$\vec{x}$ is an enumeration of $In(x)$"
$\mathcal{H}(\theta) \models \vec{x}$ is an enumeration of $In(x)$
$\vec{x}$ is an enumeration of $In(x)$

$\langle M_\alpha : \alpha < \omega_1\rangle$ is an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in\rangle$ with $G, A \in M_0$.
$C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \le \alpha < \omega_1$.

(*) if $x \in C_\alpha$ with $|In(x)| \le \omega$ then $In(x) \subset \cup\{C_\xi : \xi \le \alpha\}$.

$\mathcal{H}(\theta) \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega\rangle$"
$M_{\alpha+1} \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega\rangle$"
there is $\vec{x} \in M_{\alpha+1}$ s.t. $M_{\alpha+1} \models$ "$\vec{x}$ is an enumeration of $In(x)$"
$\mathcal{H}(\theta) \models \vec{x}$ is an enumeration of $In(x)$
$\vec{x}$ is an enumeration of $In(x)$
Since $\omega \subset M_{\alpha+1}$ we have $\vec{x}(n) \in M_{\alpha+1}$ for each $n \in \omega$

$\langle M_\alpha : \alpha < \omega_1 \rangle$ is an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in \rangle$ with $G, A \in M_0$.
$C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \leq \alpha < \omega_1$.

(*) if $x \in C_\alpha$ with $|In(x)| \leq \omega$ then $In(x) \subset \cup\{C_\xi : \xi \leq \alpha\}$.

$\mathcal{H}(\theta) \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega \rangle$"
$M_{\alpha+1} \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega \rangle$"
there is $\vec{x} \in M_{\alpha+1}$ s.t. $M_{\alpha+1} \models$ "$\vec{x}$ is an enumeration of $In(x)$"
$\mathcal{H}(\theta) \models \vec{x}$ is an enumeration of $In(x)$
$\vec{x}$ is an enumeration of $In(x)$
Since $\omega \subset M_{\alpha+1}$ we have $\vec{x}(n) \in M_{\alpha+1}$ for each $n \in \omega$
So $In(x) \subset M_{\alpha+1}$.

$\langle M_\alpha : \alpha < \omega_1 \rangle$ is an **increasing continuous chain of countable elementary submodels** of $\langle H(\theta), \in \rangle$ with $G, A \in M_0$.
$C_0 = M_0 \cap V$ and $C_n = (M_{n+1} \setminus M_n) \cap V$ for $0 < n < \omega$ and $C_\alpha = (M_{\alpha+1} \setminus M_\alpha) \cap V$ for $\omega \le \alpha < \omega_1$.

(*) if $x \in C_\alpha$ with $|In(x)| \le \omega$ then $In(x) \subset \cup\{C_\xi : \xi \le \alpha\}$.

$\mathcal{H}(\theta) \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega \rangle$"
$M_{\alpha+1} \models$ " $In(x)$ has an enumeration $\vec{x} = \langle x_n : n < \omega \rangle$"
there is $\vec{x} \in M_{\alpha+1}$ s.t. $M_{\alpha+1} \models$ "$\vec{x}$ is an enumeration of $In(x)$"
$\mathcal{H}(\theta) \models \vec{x}$ is an enumeration of $In(x)$
$\vec{x}$ is an enumeration of $In(x)$
Since $\omega \subset M_{\alpha+1}$ we have $\vec{x}(n) \in M_{\alpha+1}$ for each $n \in \omega$
So $In(x) \subset M_{\alpha+1}$.
$\cup\{C_\xi : \xi \le \alpha\} = V \cap M_{\alpha+1}$.

Chromatic number of product of graphs

Hedetniemi's Conjecture

If $\min\{\chi(G), \chi(H)\} \geq n \in \mathbb{N}$ then $\chi(G \times H) \geq n$.

Chromatic number of product of graphs

Hedetniemi's Conjecture

If $\min\{\chi(G), \chi(H)\} \geq n \in \mathbb{N}$ then $\chi(G \times H) \geq n$.

Theorem (El-Sahar, Sauer)

If $\min\{\chi(G), \chi(H)\} \geq 4$ *then* $\chi(G \times H) \geq 4$.

Chromatic number of product of graphs

Hedetniemi's Conjecture

If $\min\{\chi(G), \chi(H)\} \geq n \in \mathbb{N}$ then $\chi(G \times H) \geq n$.

Theorem (El-Sahar, Sauer)

If $\min\{\chi(G), \chi(H)\} \geq 4$ *then* $\chi(G \times H) \geq 4$.

Theorem (Hajnal)

If $\chi(G), \chi(H) \geq \omega$ *then* $\chi(G \times H) \geq \omega$.

If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f: V(G) \times V(H) \to n$.

If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

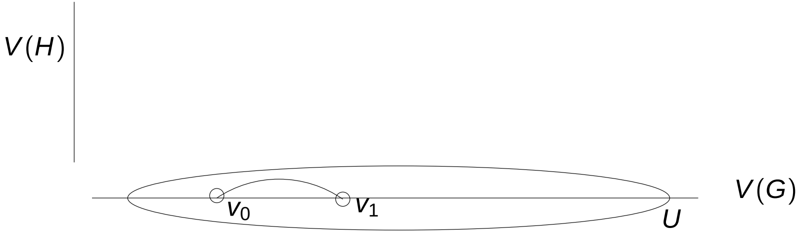
$f: V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.

If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f \colon V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.
There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.

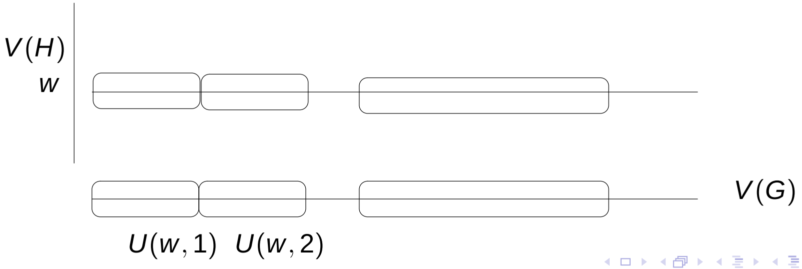
If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f: V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.
There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.
$(*)$ $\forall U \in \mathcal{U}$ $(\exists v_0, v_1 \in U)$ $v_0 v_1 \in E(G)$.



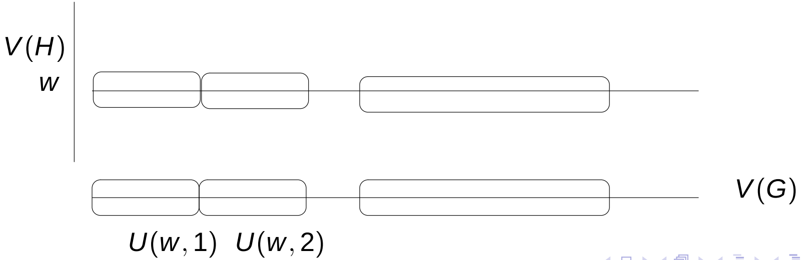
If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f : V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.
There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.
$(*)\ \forall U \in \mathcal{U}\ (\exists v_0, v_1 \in U)\ v_0 v_1 \in E(G)$.
For $w \in V(H)$ and $i < n$ let $U(w,i) = \{v \in V(G) : f(v,w) = i\}$



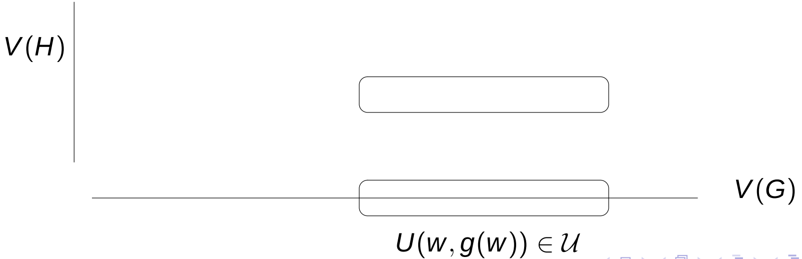
If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f\colon V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.
There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.
$(*)\ \forall U \in \mathcal{U}\ (\exists v_0, v_1 \in U)\ v_0 v_1 \in E(G)$.
For $w \in V(H)$ and $i < n$ let $U(w,i) = \{v \in V(G) : f(v,w) = i\}$
$V(G) = U(w,0) \cup \cdots \cup U(w,n-1)$.



If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f: V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.
There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.
$(*)\ \forall U \in \mathcal{U}\ (\exists v_0, v_1 \in U)\ v_0 v_1 \in E(G)$.
For $w \in V(H)$ and $i < n$ let $U(w,i) = \{v \in V(G) : f(v,w) = i\}$
$V(G) = U(w,0) \cup \cdots \cup U(w,n-1)$.
$\forall w \in V(H)\ \exists g(w) < n\ U(w,g(w)) \in \mathcal{U}$



If $\chi(G) \ge \omega$ and $\chi(H) \ge n+1$ then $\chi(G \times H) \ge n+1$.

$f: V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.

There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.

$(*)\ \forall U \in \mathcal{U}\ (\exists v_0, v_1 \in U)\ v_0 v_1 \in E(G)$.

For $w \in V(H)$ and $i < n$ let $U(w,i) = \{v \in V(G) : f(v,w) = i\}$

$V(G) = U(w,0) \cup \cdots \cup U(w,n-1)$.

$\forall w \in V(H)\ \exists g(w) < n\ U(w, g(w)) \in \mathcal{U}$

$g : V(H) \to n$

If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f: V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.

There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.

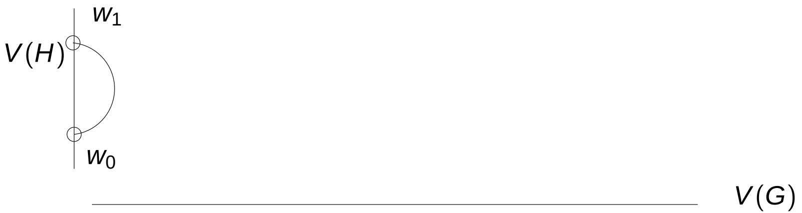
$(*)\ \forall U \in \mathcal{U}\ (\exists v_0, v_1 \in U)\ v_0 v_1 \in E(G)$.

For $w \in V(H)$ and $i < n$ let $U(w,i) = \{v \in V(G) : f(v,w) = i\}$

$V(G) = U(w,0) \cup \cdots \cup U(w, n-1)$.

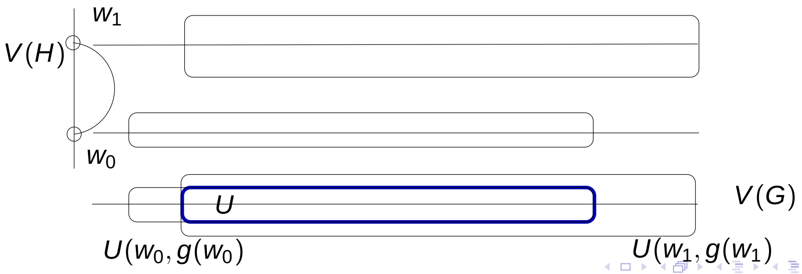
$\forall w \in V(H)\ \exists g(w) < n\ U(w, g(w)) \in \mathcal{U}$

$g : V(H) \to n$ $\exists w_0 w_1 \in E(H)$ s.t. $g(w_0) = g(w_1)$.



If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f: V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.
There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.
$(*)\ \forall U \in \mathcal{U}\ (\exists v_0, v_1 \in U)\ v_0 v_1 \in E(G)$.
For $w \in V(H)$ and $i < n$ let $U(w,i) = \{v \in V(G) : f(v,w) = i\}$
$V(G) = U(w,0) \cup \cdots \cup U(w, n-1)$.
$\forall w \in V(H)\ \exists g(w) < n\ U(w, g(w)) \in \mathcal{U}$
$g : V(H) \to n\ \ \exists w_0 w_1 \in E(H)$ s.t. $g(w_0) = g(w_1)$.
$U = U(w_0, g(w_0)) \cap U(w_1, g(w_1)) \in \mathcal{U}$



If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f: V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.

There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.

$(*)\ \forall U \in \mathcal{U}\ (\exists v_0, v_1 \in U)\ v_0 v_1 \in E(G)$.

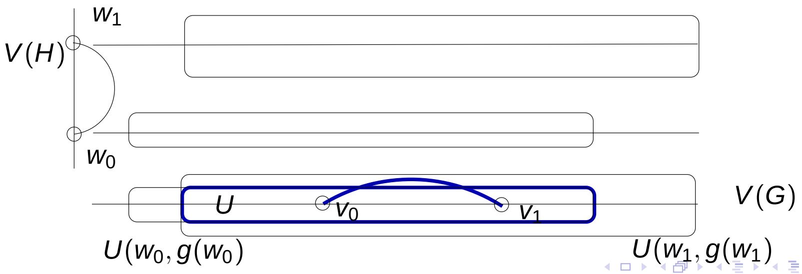
For $w \in V(H)$ and $i < n$ let $U(w,i) = \{v \in V(G) : f(v,w) = i\}$

$V(G) = U(w,0) \cup \cdots \cup U(w,n-1)$.

$\forall w \in V(H)\ \exists g(w) < n\ U(w, g(w)) \in \mathcal{U}$

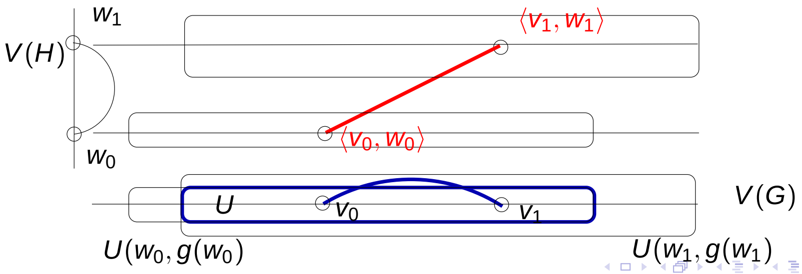
$g : V(H) \to n$ $\exists w_0 w_1 \in E(H)$ s.t. $g(w_0) = g(w_1)$.

$U = U(w_0, g(w_0)) \cap U(w_1, g(w_1)) \in \mathcal{U}$ $\exists v_0, v_1 \in U$ s.t. $v_0 v_1 \in E(G)$.



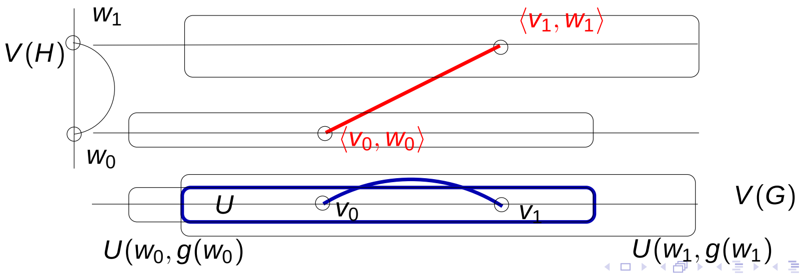
If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f : V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.
There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.
$(*)\ \forall U \in \mathcal{U}\ (\exists v_0, v_1 \in U)\ v_0 v_1 \in E(G)$.
For $w \in V(H)$ and $i < n$ let $U(w,i) = \{v \in V(G) : f(v,w) = i\}$
$V(G) = U(w,0) \cup \cdots \cup U(w,n-1)$.
$\forall w \in V(H)\ \exists g(w) < n\ U(w, g(w)) \in \mathcal{U}$
$g : V(H) \to n\ \ \exists w_0 w_1 \in E(H)$ s.t. $g(w_0) = g(w_1)$.
$U = U(w_0, g(w_0)) \cap U(w_1, g(w_1)) \in \mathcal{U}\ \ \exists v_0, v_1 \in U$ s.t. $v_0 v_1 \in E(G)$.
$\langle v_0, w_0 \rangle \langle v_1, w_1 \rangle \in E(G \times H)$.



If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f : V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.
There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.
$(*)\ \forall U \in \mathcal{U}\ (\exists v_0, v_1 \in U)\ v_0v_1 \in E(G)$.
For $w \in V(H)$ and $i < n$ let $U(w,i) = \{v \in V(G) : f(v,w) = i\}$
$V(G) = U(w,0) \cup \cdots \cup U(w,n-1)$.
$\forall w \in V(H)\ \exists g(w) < n\ U(w,g(w)) \in \mathcal{U}$
$g : V(H) \to n\ \ \exists w_0w_1 \in E(H)$ s.t. $g(w_0) = g(w_1)$.
$U = U(w_0,g(w_0)) \cap U(w_1,g(w_1)) \in \mathcal{U}\ \ \exists v_0, v_1 \in U$ s.t. $v_0v_1 \in E(G)$.
$\langle v_0,w_0\rangle\ \langle v_1,w_1\rangle \in E(G \times H)$. $f(\langle v_0,w_0\rangle) = g(w_0) = g(w_1) = f(\langle v_1,w_1\rangle)$.



If $\chi(G) \geq \omega$ and $\chi(H) \geq n+1$ then $\chi(G \times H) \geq n+1$.

$f: V(G) \times V(H) \to n$. $\mathcal{I} = \{V' \subset V(G) : \chi(G[V']) < \omega\}$.
There is an **ultrafilter** $\mathcal{U}$ on $V(G)$ s.t. $\mathcal{U} \cap \mathcal{I} = \emptyset$.
$(*)$ $\forall U \in \mathcal{U}$ $(\exists v_0, v_1 \in U)$ $v_0 v_1 \in E(G)$.
For $w \in V(H)$ and $i < n$ let $U(w,i) = \{v \in V(G) : f(v,w) = i\}$
$V(G) = U(w,0) \cup \cdots \cup U(w, n-1)$.
$\forall w \in V(H)$ $\exists g(w) < n$ $U(w, g(w)) \in \mathcal{U}$
$g : V(H) \to n$ $\exists w_0 w_1 \in E(H)$ s.t. $g(w_0) = g(w_1)$.
$U = U(w_0, g(w_0)) \cap U(w_1, g(w_1)) \in \mathcal{U}$ $\exists v_0, v_1 \in U$ s.t. $v_0 v_1 \in E(G)$.
$\langle v_0, w_0 \rangle \langle v_1, w_1 \rangle \in E(G \times H)$. $f(\langle v_0, w_0 \rangle) = g(w_0) = g(w_1) = f(\langle v_1, w_1 \rangle)$.

Chromatic number of product of graphs

Theorem (Hajnal)

There are two ω_1-chromatic graphs G and H on ω_1 such that $\chi(G \times H) = \omega$.

Chromatic number of product of graphs

Theorem (Hajnal)

There are two ω_1-chromatic graphs G and H on ω_1 such that $\chi(G \times H) = \omega$.

Theorem (–)

It is consistent with GCH that there are two ω_2-chromatic graphs G and H on ω_2 s. t. $\chi(G \times H) = \omega$.

Chromatic number of product of graphs

Theorem (Hajnal)

There are two ω_1-chromatic graphs G and H on ω_1 such that $\chi(G \times H) = \omega$.

Theorem (–)

It is consistent with GCH that there are two ω_2-chromatic graphs G and H on ω_2 s. t. $\chi(G \times H) = \omega$.

Problem

Is it consistent with GCH that there are two ω_3-chromatic graphs G and H on ω_3 s. t. $\chi(G \times H) = \omega$?

Consistency proofs are unavoidable

Consistency proofs are unavoidable

- independence proofs are rather sophisticated

Consistency proofs are unavoidable

- independence proofs are rather sophisticated
- the results themselves are usually of interest to “ordinary” mathematicians

Consistency proofs are unavoidable

- independence proofs are rather sophisticated
- the results themselves are usually of interest to “ordinary” mathematicians

Solution:

Consistency proofs are unavoidable

- independence proofs are rather sophisticated
- the results themselves are usually of interest to “ordinary” mathematicians

Solution: isolate a relatively small number of **principles**, i.e. independent statements

Consistency proofs are unavoidable

- independence proofs are rather sophisticated
- the results themselves are usually of interest to “ordinary” mathematicians

Solution: isolate a relatively small number of **principles**, i.e. independent statements

- that are **simple** to formulate

Consistency proofs are unavoidable

- independence proofs are rather sophisticated
- the results themselves are usually of interest to “ordinary” mathematicians

Solution: isolate a relatively small number of **principles**, i.e. independent statements

- that are **simple** to formulate
- that are **useful** in the sense that they have many interesting consequences.

Consistency proofs are unavoidable

- independence proofs are rather sophisticated
- the results themselves are usually of interest to "ordinary" mathematicians

Solution: isolate a relatively small number of **principles**, i.e. independent statements

- that are **simple** to formulate
- that are **useful** in the sense that they have many interesting consequences.

combinatorial principles

Consistency proofs are unavoidable

- independence proofs are rather sophisticated
- the results themselves are usually of interest to “ordinary” mathematicians

Solution: isolate a relatively small number of **principles**, i.e. independent statements

- that are **simple** to formulate
- that are **useful** in the sense that they have many interesting consequences.

combinatorial principles
Continuum Hypothesis,

Consistency proofs are unavoidable

- independence proofs are rather sophisticated
- the results themselves are usually of interest to “ordinary” mathematicians

Solution: isolate a relatively small number of **principles**, i.e. independent statements

- that are **simple** to formulate
- that are **useful** in the sense that they have many interesting consequences.

combinatorial principles
Continuum Hypothesis, Martin’s Axiom

Consistency proofs are unavoidable

- independence proofs are rather sophisticated
- the results themselves are usually of interest to “ordinary” mathematicians

Solution: isolate a relatively small number of **principles**, i.e. independent statements

- that are **simple** to formulate
- that are **useful** in the sense that they have many interesting consequences.

combinatorial principles
Continuum Hypothesis, Martin’s Axiom
Other models?

Consistency proofs are unavoidable

- independence proofs are rather sophisticated
- the results themselves are usually of interest to “ordinary” mathematicians

Solution: isolate a relatively small number of **principles**, i.e. independent statements

- that are **simple** to formulate
- that are **useful** in the sense that they have many interesting consequences.

combinatorial principles
Continuum Hypothesis, Martin’s Axiom
Other models?
principles which describe the **Cohen Model**

Covers of $\mathbb{R}^n$

Covers of $\mathbb{R}^n$

Theorem (Aharoni, R.; Hajnal, A.; Milner, E. C.)

Any κ-fold cover of $\mathbb{R}$ by intervals can be partitioned into κ subcovers.

Covers of $\mathbb{R}^n$

Theorem (Aharoni, R.; Hajnal, A.; Milner, E. C.)

Any κ-fold cover of $\mathbb{R}$ by intervals can be partitioned into κ subcovers.

Theorem (M. Elekes, T. Matrai, –)

Covers of $\mathbb{R}^n$

Theorem (Aharoni, R.; Hajnal, A.; Milner, E. C.)

Any κ-fold cover of $\mathbb{R}$ by intervals can be partitioned into κ subcovers.

Theorem (M. Elekes, T. Matrai, –)

- *any ω_1-fold cover of $\mathbb{R}^n$ by polytopes can be partitioned into ω_1 subcovers.*

Covers of $\mathbb{R}^n$

Theorem (Aharoni, R.; Hajnal, A.; Milner, E. C.)

Any κ-fold cover of $\mathbb{R}$ by intervals can be partitioned into κ subcovers.

Theorem (M. Elekes, T. Matrai, –)

- *any ω_1-fold cover of $\mathbb{R}^n$ by polytopes can be partitioned into ω_1 subcovers.*
- *R^2 has an ω-fold cover by rectangles which can not be partitioned into two subcovers .*

Covers of $\mathbb{R}^n$

Theorem (Aharoni, R.; Hajnal, A.; Milner, E. C.)

Any κ-fold cover of $\mathbb{R}$ by intervals can be partitioned into κ subcovers.

Theorem (M. Elekes, T. Matrai, –)

- *any ω_1-fold cover of $\mathbb{R}^n$ by polytopes can be partitioned into ω_1 subcovers.*
- *R^2 has an ω-fold cover by rectangles which can not be partitioned into two subcovers .*
- *CH $\Longrightarrow$ any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.*

Covers of $\mathbb{R}^n$

Theorem (Aharoni, R.; Hajnal, A.; Milner, E. C.)

Any κ-fold cover of $\mathbb{R}$ by intervals can be partitioned into κ subcovers.

Theorem (M. Elekes, T. Matrai, –)

- *any ω_1-fold cover of $\mathbb{R}^n$ by polytopes can be partitioned into ω_1 subcovers.*
- *R^2 has an ω-fold cover by rectangles which can not be partitioned into two subcovers .*
- *CH $\Longrightarrow$ any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.*
- *If MA_{ω_1} then there is an ω_1-fold cover of $\mathbb{R}^n$ by closed sets which can not be partitioned into ω_1 subcovers.*

Covers of $\mathbb{R}^n$

$(*)$: any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.

Covers of $\mathbb{R}^n$

(∗): any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.

(1) $CH \implies (*)$. (2) If MA_{ω_1} then $\neg(*)$.

Covers of $\mathbb{R}^n$

$(*)$: any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.

(1) $CH \implies (*)$. (2) If MA_{ω_1} then $\neg(*)$.

Definition

A poset P has the **weak Freese-Nation property**

Covers of $\mathbb{R}^n$

$(*)$: any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.

(1) $CH \implies (*)$. (2) If MA_{ω_1} then $\neg(*)$.

Definition

A poset P has the **weak Freese-Nation property** iff $\exists f : P \to [P]^{\leq\omega}$

$(*)$: any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.

(1) $CH \implies (*)$. (2) If MA_{ω_1} then $\neg(*)$.

Definition

A poset P has the **weak Freese-Nation property** iff $\exists f : P \to [P]^{\leq\omega}$ s.t. $\forall \{p, q\} \in [P]^2$, $p \leq_P q$, $\exists r \in f(p) \cap f(q)$ with $p \leq_P r \leq_P q$.

Covers of $\mathbb{R}^n$

$(*)$: any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.

(1) $CH \implies (*)$. (2) If MA_{ω_1} then $\neg(*)$.

Definition

A poset P has the **weak Freese-Nation property** iff $\exists f : P \to [P]^{\leq\omega}$ s.t. $\forall \{p, q\} \in [P]^2, p \leq_P q, \exists r \in f(p) \cap f(q)$ with $p \leq_P r \leq_P q$.

Theorem (Fuchino, –)

In the ***Cohen modell*** $\langle P(\omega), \subset \rangle$ *has the weak Freese-Nation property*

Covers of $\mathbb{R}^n$

$(*)$: any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.

(1) $CH \implies (*)$. (2) If MA_{ω_1} then $\neg(*)$.

Definition

A poset P has the **weak Freese-Nation property** iff $\exists f : P \to [P]^{\leq\omega}$ s.t. $\forall \{p, q\} \in [P]^2, p \leq_P q, \exists r \in f(p) \cap f(q)$ with $p \leq_P r \leq_P q$.

Theorem (Fuchino, –)

In the ***Cohen modell*** $\langle P(\omega), \subset\rangle$ *has the weak Freese-Nation property*

Covers of $\mathbb{R}^n$

$(*)$: any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.

(1) $CH \implies (*)$. (2) If MA_{ω_1} then $\neg(*)$.

Definition

A poset P has the **weak Freese-Nation property** iff $\exists f : P \to [P]^{\leq\omega}$ s.t. $\forall \{p, q\} \in [P]^2, p \leq_P q, \exists r \in f(p) \cap f(q)$ with $p \leq_P r \leq_P q$.

Theorem (Fuchino, –)

*In the **Cohen modell** $\langle P(\omega), \subset \rangle$ has the weak Freese-Nation property*

Theorem (M. Elekes, T. Matrai, –)

*If $\langle P(\omega), \subset \rangle$ has the weak Freese-Nation property then $(*)$ **holds**.*

Covers of $\mathbb{R}^n$

$(*)$: any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.

(1) $CH \implies (*)$. (2) If MA_{ω_1} then $\neg(*)$.

Definition

A poset P has the **weak Freese-Nation property** iff $\exists f : P \to [P]^{\leq\omega}$ s.t. $\forall \{p, q\} \in [P]^2$, $p \leq_P q$, $\exists r \in f(p) \cap f(q)$ with $p \leq_P r \leq_P q$.

Theorem (Fuchino, –)

*In the **Cohen modell** $\langle P(\omega), \subset \rangle$ has the weak Freese-Nation property*

Theorem (M. Elekes, T. Matrai, –)

*If $\langle P(\omega), \subset \rangle$ has the weak Freese-Nation property then $(*)$ **holds**.*

Covers of $\mathbb{R}^n$

$(*)$: any ω_1-fold cover of $\mathbb{R}^n$ by closed sets can be partitioned into ω_1 subcovers.

(1) $CH \implies (*)$. (2) If MA_{ω_1} then $\neg(*)$.

Definition

A poset P has the **weak Freese-Nation property** iff $\exists f : P \to [P]^{\leq\omega}$ s.t. $\forall \{p, q\} \in [P]^2$, $p \leq_P q$, $\exists r \in f(p) \cap f(q)$ with $p \leq_P r \leq_P q$.

Theorem (Fuchino, –)

*In the **Cohen modell** $\langle P(\omega), \subset\rangle$ has the weak Freese-Nation property*

Theorem (M. Elekes, T. Matrai, –)

*If $\langle P(\omega), \subset\rangle$ has the weak Freese-Nation property then $(*)$ **holds**.*
So $(*)$+ $\neg CH$ is consistent.

When a principle fails

When a principle fails

T: statement

When a principle fails

T: statement
Con(T)?

When a principle fails

T: statement

Con(T)?

Plan: Pick a principle P and prove that P implies T.

When a principle fails

T: statement
Con(T)?
Plan: Pick a principle P and prove that P implies T.
can't prove that P implies T

When a principle fails

T: statement
Con(T)?
Plan: Pick a principle P and prove that P implies T.
can't prove that P implies T
Problem: Prove that P does not imply T

Smooth graphs

K.A.Kierstead and P.J.Nyikos: Are there infinite graphs which are **very homogeneous**

Smooth graphs

K.A.Kierstead and P.J.Nyikos: Are there infinite graphs which are **very homogeneous** in a non-trivial way?

Smooth graphs

K.A.Kierstead and P.J.Nyikos: Are there infinite graphs which are **very homogeneous** in a non-trivial way?

smooth=homogeneous

Smooth graphs

K.A.Kierstead and P.J.Nyikos: Are there infinite graphs which are **very homogeneous** in a non-trivial way?

smooth=homogeneous

$G = \langle \omega_1, E \rangle$

Smooth graphs

K.A.Kierstead and P.J.Nyikos: Are there infinite graphs which are **very homogeneous** in a non-trivial way?

smooth=homogeneous

$G = \langle \omega_1, E \rangle$

How to measure homogeneity of a graph G?

Smooth graphs

K.A.Kierstead and P.J.Nyikos: Are there infinite graphs which are **very homogeneous** in a non-trivial way?

smooth=homogeneous

$G = \langle \omega_1, E \rangle$

How to measure homogeneity of a graph G?

I(G): isomorphism classes of induced uncountable subgraphs of G.

Smooth graphs

K.A.Kierstead and P.J.Nyikos: Are there infinite graphs which are **very homogeneous** in a non-trivial way?

smooth=homogeneous

$G = \langle \omega_1, E \rangle$

How to measure homogeneity of a graph G?

I(G): isomorphism classes of induced uncountable subgraphs of G.

1. $|\mathrm{I}(G)|$ is small,

Smooth graphs

K.A.Kierstead and P.J.Nyikos: Are there infinite graphs which are **very homogeneous** in a non-trivial way?

smooth=homogeneous

$G = \langle \omega_1, E \rangle$

How to measure homogeneity of a graph G?

I(G): isomorphism classes of induced uncountable subgraphs of G.

1. $|\mathrm{I}(G)|$ is small,
2. $G \cong G[A]$ for many $A \subset \omega_1$.

Smooth graphs

G is **smooth** iff $|\,\mathrm{I}(G)| = 1$

Smooth graphs

G is **smooth** iff $|\mathrm{I}(G)| = 1$

Fact

A smooth graph is either complete or empty.

Smooth graphs

G is **smooth** iff $|\,\mathrm{I}(G)| = 1$

Fact

A smooth graph is either complete or empty.

Proof.

$\square$

Smooth graphs

G is **smooth** iff $|\mathrm{I}(G)| = 1$

Fact

A smooth graph is either complete or empty.

Proof.

- $x \in \omega_1$, w.l.o.g $|E(x)| = \omega_1$.

□

Smooth graphs

G is **smooth** iff $|\,\mathrm{I}(G)| = 1$

Fact

A smooth graph is either complete or empty.

Proof.

- $x \in \omega_1$, w.l.o.g $|E(x)| = \omega_1$.
- $G \cong G[\{x\} \cup E(x)]$

□

Smooth graphs

G is **smooth** iff $|\,\mathrm{I}(G)| = 1$

Fact

A smooth graph is either complete or empty.

Proof.

- $x \in \omega_1$, w.l.o.g $|E(x)| = \omega_1$.
- $G \cong G[\{x\} \cup E(x)]$
- $\exists v \in \omega_1 \; \omega_1 = \{v\} \cup E(v)$

□

Smooth graphs

G is **smooth** iff $|\mathrm{I}(G)| = 1$

Fact

A smooth graph is either complete or empty.

Proof.

- $x \in \omega_1$, w.l.o.g $|E(x)| = \omega_1$.
- $G \cong G[\{x\} \cup E(x)]$
- $\exists v \in \omega_1$ $\omega_1 = \{v\} \cup E(v)$
- $\forall W \in [V]^{\omega_1}$ $\exists w \in W$ $W \subset \{w\} \cup E(w)$

□

Smooth graphs

G is **smooth** iff $|\mathrm{I}(G)| = 1$

Fact

A smooth graph is either complete or empty.

Proof.

- $x \in \omega_1$, w.l.o.g $|E(x)| = \omega_1$.
- $G \cong G[\{x\} \cup E(x)]$
- $\exists v \in \omega_1 \; \omega_1 = \{v\} \cup E(v)$
- $\forall W \in [V]^{\omega_1} \; \exists w \in W \; W \subset \{w\} \cup E(w)$
- G is complete

$\square$

Smooth graphs

G is **non-trivial** iff there are no uncountable **cliques** or **independent subsets** in G.

Smooth graphs

G is **non-trivial** iff there are no uncountable **cliques** or **independent subsets** in G.

Theorem (K.A.Kierstead and P.J.Nyikos)

If G is a non-trivial graph on ω_1 then $|\mathrm{I}(G)| \geq \omega$.

Smooth graphs

G is **non-trivial** iff there are no uncountable **cliques** or **independent subsets** in G.

Theorem (K.A.Kierstead and P.J.Nyikos)

If G is a non-trivial graph on ω_1 then $|\mathrm{I}(G)| \geq \omega$.

Theorem (Hajnal, Nagy, –)

(1) $|\mathrm{I}(G)| \geq 2^{\omega}$ for each non-trivial graph G on ω_1.

Smooth graphs

G is **non-trivial** iff there are no uncountable **cliques** or **independent subsets** in G.

Theorem (K.A.Kierstead and P.J.Nyikos)

If G is a non-trivial graph on ω_1 then $|\mathrm{I}(G)| \geq \omega$.

Theorem (Hajnal, Nagy, –)

(1) $|\mathrm{I}(G)| \geq 2^\omega$ for each non-trivial graph G on ω_1.
(2) Under $\diamondsuit^+$ there exists a non-trivial graph G on ω_1 with $|\mathrm{I}(G)| = \omega_1$.

Smooth graphs

G is **non-trivial** iff there are no uncountable **cliques** or **independent subsets** in G.

Theorem (K.A.Kierstead and P.J.Nyikos)

If G is a non-trivial graph on ω_1 then $|\mathrm{I}(G)| \geq \omega$.

Theorem (Hajnal, Nagy, –)

(1) $|\mathrm{I}(G)| \geq 2^\omega$ for each non-trivial graph G on ω_1.
(2) Under $\diamondsuit^+$ there exists a non-trivial graph G on ω_1 with $|\mathrm{I}(G)| = \omega_1$.

Theorem (Shelah, –)

Assume that GCH holds and every Aronszajn-tree is special. Then $|\mathrm{I}(G)| = 2^{\omega_1}$ for each non-trivial graph $G = \langle \omega_1, E \rangle$.

Smooth graphs

G is ***almost smooth*** iff $G \cong G[\omega_1 \setminus A]$ for each $A \in [\omega_1]^{\omega}$.

Smooth graphs

G is ***almost smooth*** iff $G \cong G[\omega_1 \setminus A]$ for each $A \in [\omega_1]^{\omega}$.

Theorem (Hajnal, Nagy, –)

If CH holds then there is a non-trivial, almost smooth graph on ω_1.

Smooth graphs

G is ***almost smooth*** iff $G \cong G[\omega_1 \setminus A]$ for each $A \in [\omega_1]^{\omega}$.

Theorem (Hajnal, Nagy, –)

If CH holds then there is a non-trivial, almost smooth graph on ω_1.

Problem

Is there a non-trivial, almost smooth graph on ω_1?

Smooth graphs

G is ***almost smooth*** iff $G \cong G[\omega_1 \setminus A]$ for each $A \in [\omega_1]^{\omega}$.

Theorem (Hajnal, Nagy, –)

If CH holds then there is a non-trivial, almost smooth graph on ω_1.

Problem

Is there a non-trivial, almost smooth graph on ω_1?

Smooth graphs

G is ***almost smooth*** iff $G \cong G[\omega_1 \setminus A]$ for each $A \in [\omega_1]^{\omega}$.

Theorem (Hajnal, Nagy, –)

If CH holds then there is a non-trivial, almost smooth graph on ω_1.

Problem

Is there a non-trivial, almost smooth graph on ω_1?

Does Martin's Axiom imply that there is no non-trivial, almost smooth graph on ω_1?

Smooth graphs

Theorem (–)

It is consistent that $MA_{\aleph_1}$ holds and there is a non-trivial, almost smooth graph on ω_1.

A blackbox theorem

A blackbox theorem

$\mathrm{Fn}_m(\omega_1, K) = \{s : s \text{ is a function, } \mathrm{dom}(s) \in [\omega_1]^m, \mathrm{ran}(s) \subset K\}$

A blackbox theorem

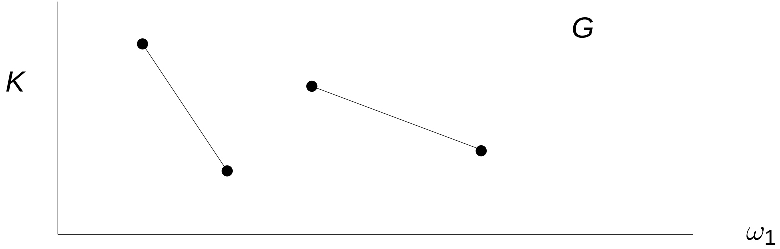
$\mathrm{Fn}_m(\omega_1, K) = \{s : s \text{ is a function, } \mathrm{dom}(s) \in [\omega_1]^m, \mathrm{ran}(s) \subset K\}$

$\langle s_\alpha : \alpha < \omega_1 \rangle \subset \mathrm{Fn}_m(\omega_1, K)$ is ***dom-disjoint*** iff $\mathrm{dom}(s_\alpha) \cap \mathrm{dom}(s_\beta) = \emptyset$

A blackbox theorem

Definition

Let G be a graph on $\omega_1 \times K$, $m \in \omega$.



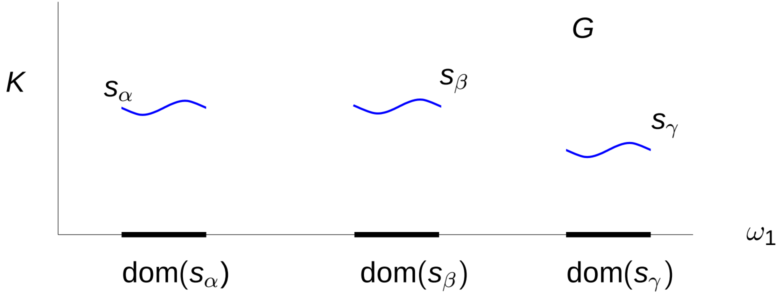
A blackbox theorem

Definition

Let G be a graph on $\omega_1 \times K$, $m \in \omega$. We say that G is ***m*-solid** if given any **dom-disjoint sequence** $\langle s_\alpha : \alpha < \omega_1 \rangle \subset \mathrm{Fn}_m(\omega_1, K)$ there are $\alpha < \beta < \omega_1$ such that

$$[s_\alpha, s_\beta] \subset G.$$

G is called ***strongly solid*** iff it is m-solid for each $m \in \omega$.



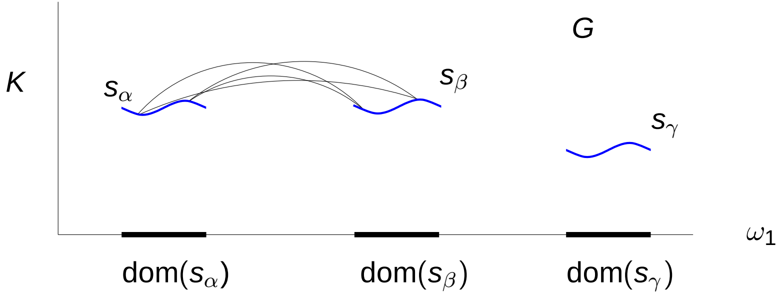
A blackbox theorem

Definition

Let G be a graph on $\omega_1 \times K$, $m \in \omega$. We say that G is ***m*-solid** if given any **dom-disjoint sequence** $\langle s_\alpha : \alpha < \omega_1 \rangle \subset \mathrm{Fn}_m(\omega_1, K)$ there are $\alpha < \beta < \omega_1$ such that

$$[s_\alpha, s_\beta] \subset G.$$

G is called ***strongly solid*** iff it is m-solid for each $m \in \omega$.



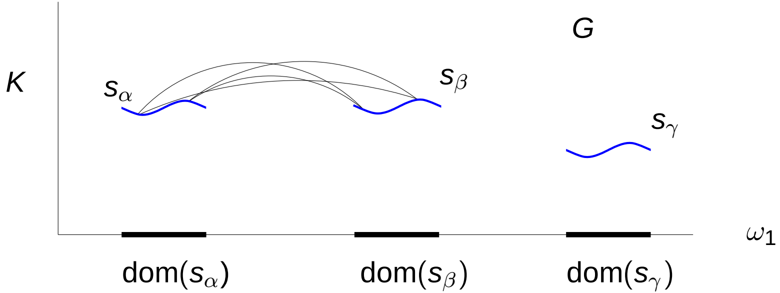
A blackbox theorem

Definition

Let G be a graph on $\omega_1 \times K$, $m \in \omega$. We say that G is ***m*-solid** if given any **dom-disjoint sequence** $\langle s_\alpha : \alpha < \omega_1 \rangle \subset Fn_m(\omega_1, K)$ there are $\alpha < \beta < \omega_1$ such that

$$[s_\alpha, s_\beta] \subset G.$$

G is called ***strongly solid*** iff it is m-solid for each $m \in \omega$.



A blackbox theorem

Let G be a graph on $\omega_1 \times K$, $m \in \omega$. We say that G is ***m*-solid** if given any dom-disjoint sequence $\langle s_\alpha : \alpha < \omega_1 \rangle \subset \mathrm{Fn}_m(\omega_1, K)$ there are $\alpha < \beta < \omega_1$ such that

$$[s_\alpha, s_\beta] \subset G.$$

G is called ***strongly solid*** iff it is m-solid for each $m \in \omega$.

Theorem (–)

*Assume $2^{\omega_1} = \omega_2$. If G is a **strongly solid** graph on $\omega_1 \times K$, then for each $m \in \omega$ in some (c.c.c. generic) extension W of V we have*

$$W \models \text{“}G \text{ is } m\text{-solid} + \mathrm{MA}_{\omega_1} \text{ holds”}$$

A blackbox theorem

Let G be a graph on $\omega_1 \times K$, $m \in \omega$. We say that G is ***m*-solid** if given any dom-disjoint sequence $\langle s_\alpha : \alpha < \omega_1 \rangle \subset \mathrm{Fn}_m(\omega_1, K)$ there are $\alpha < \beta < \omega_1$ such that

$$[s_\alpha, s_\beta] \subset G.$$

G is called ***strongly solid*** iff it is m-solid for each $m \in \omega$.

Theorem (–)

*Assume $2^{\omega_1} = \omega_2$. If G is a **strongly solid** graph on $\omega_1 \times K$, then for each $m \in \omega$ in some (c.c.c. generic) extension W of V we have*

$$W \models \text{“}G \text{ is } m\text{-solid} + \mathrm{MA}_{\omega_1} \text{ holds”}$$

Smooth graphs

Theorem (–)

It is consistent that $MA_{\aleph_1}$ holds and there is a non-trivial, almost smooth graph on ω_1.

Proof

- **Coding:** Given a graph C on ω_1 define a suitable K and a graph $G(C)$ on $\omega_1 \times K$ s. t.
  - (a) If $G(C)$ is 1-solid then C is non-trivial.
  - (b) If $G(C)$ is 1-solid and $MA_{\aleph_1}$ holds then C is almost smooth.
  - (c) $G(C)$ is strongly solid provided C has some property (P).
- Using GCH construct a graph on ω_1 with property (P)
- Black Box Theorem:
  $W \models$ "$G(C)$ is 1-solid and MA_{ω_1} holds."
- Theorem:
  $W \models$ "C is non-trivial, almost smooth and MA_{ω_1} holds."

Smooth graphs

Theorem (–)

It is consistent that $MA_{\aleph_1}$ holds and there is a non-trivial, almost smooth graph on ω_1.

Proof

- **Coding:** Given a graph C on ω_1 define a suitable K and a graph $G(C)$ on $\omega_1 \times K$ s. t.
  - (a) If $G(C)$ is 1-solid then C is non-trivial
  - (b) If $G(C)$ is 1-solid and $MA_{\aleph_1}$ holds then C is almost smooth.
  - (c) $G(C)$ is strongly solid provided C has some property (P)
- Using GCH construct a graph on ω_1 with property (P)
- Black Box Theorem:
  $W \models$ "$G(C)$ is 1-solid and MA_{ω_1} holds."
- Theorem:
  $W \models$ "C is non-trivial, almost smooth and MA_{ω_1} holds."

Smooth graphs

Theorem (–)

It is consistent that $MA_{\aleph_1}$ holds and there is a non-trivial, almost smooth graph on ω_1.

Proof

- **Coding:** Given a graph C on ω_1 define a suitable K and a graph $G(C)$ on $\omega_1 \times K$ s. t.
  - (a) If $G(C)$ is 1-solid then C is non-trivial
  - (b) If $G(C)$ is 1-solid and $MA_{\aleph_1}$ holds then C is almost smooth.
  - (c) $G(C)$ is strongly solid provided C has some property (P)
- Using GCH construct a graph on ω_1 with property (P)
- Black Box Theorem:
  $W \models$ "$G(C)$ is 1-solid and MA_{ω_1} holds."
- Theorem:
  $W \models$ "C is non-trivial, almost smooth and MA_{ω_1} holds."

Smooth graphs

Theorem (–)

It is consistent that $MA_{\aleph_1}$ holds and there is a non-trivial, almost smooth graph on ω_1.

Proof

- **Coding:** Given a graph C on ω_1 define a suitable K and a graph $G(C)$ on $\omega_1 \times K$ s. t.
  - (a) If $G(C)$ is 1-solid then C is non-trivial
  - (b) If $G(C)$ is 1-solid and $MA_{\aleph_1}$ holds then C is almost smooth.
  - (c) $G(C)$ is strongly solid provided C has some property (P)
- Using GCH construct a graph on ω_1 with property (P)
- Black Box Theorem:
  $W \models$ "$G(C)$ is 1-solid and MA_{ω_1} holds."
- Theorem:
  $W \models$ "C is non-trivial, almost smooth and MA_{ω_1} holds."

Smooth graphs

Theorem (–)

It is consistent that $MA_{\aleph_1}$ holds and there is a non-trivial, almost smooth graph on ω_1.

Proof

- **Coding:** Given a graph C on ω_1 define a suitable K and a graph $G(C)$ on $\omega_1 \times K$ s. t.
  - (a) If $G(C)$ is 1-solid then C is non-trivial
  - (b) If $G(C)$ is 1-solid and $MA_{\aleph_1}$ holds then C is almost smooth.
  - (c) $G(C)$ is strongly solid provided C has some property (P)
- Using GCH construct a graph on ω_1 with property (P)
- Black Box Theorem:
  $W \models$ "$G(C)$ is 1-solid and MA_{ω_1} holds."
- Theorem:
  $W \models$ "C is non-trivial, almost smooth and MA_{ω_1} holds."

Smooth graphs

Theorem (–)

It is consistent that $MA_{\aleph_1}$ holds and there is a non-trivial, almost smooth graph on ω_1.

Proof

- **Coding:** Given a graph C on ω_1 define a suitable K and a graph $G(C)$ on $\omega_1 \times K$ s. t.
  - (a) If $G(C)$ is 1-solid then C is non-trivial
  - (b) If $G(C)$ is 1-solid and $MA_{\aleph_1}$ holds then C is almost smooth.
  - (c) $G(C)$ is strongly solid provided C has some property (P)
- Using GCH construct a graph on ω_1 with property (P)
- Black Box Theorem: $W \models$ "$G(C)$ is 1-solid and MA_{ω_1} holds."
- Theorem: $W \models$ "C is non-trivial, almost smooth and MA_{ω_1} holds."

Smooth graphs

Theorem (–)

It is consistent that $MA_{\aleph_1}$ holds and there is a non-trivial, almost smooth graph on ω_1.

Proof

- **Coding:** Given a graph C on ω_1 define a suitable K and a graph $G(C)$ on $\omega_1 \times K$ s. t.
  - (a) If $G(C)$ is 1-solid then C is non-trivial
  - (b) If $G(C)$ is 1-solid and $MA_{\aleph_1}$ holds then C is almost smooth.
  - (c) $G(C)$ is strongly solid provided C has some property (P)
- Using GCH construct a graph on ω_1 with property (P)
- Black Box Theorem:
  $W \models$ "$G(C)$ is 1-solid and MA_{ω_1} holds."
- Theorem:
  $W \models$ "C is non-trivial, almost smooth and MA_{ω_1} holds."

Smooth graphs

Theorem (–)

It is consistent that $MA_{\aleph_1}$ holds and there is a non-trivial, almost smooth graph on ω_1.

Proof

- **Coding:** Given a graph C on ω_1 define a suitable K and a graph $G(C)$ on $\omega_1 \times K$ s. t.
  - (a) If $G(C)$ is 1-solid then C is non-trivial
  - (b) If $G(C)$ is 1-solid and $MA_{\aleph_1}$ holds then C is almost smooth.
  - (c) $G(C)$ is strongly solid provided C has some property (P)
- Using GCH construct a graph on ω_1 with property (P)
- Black Box Theorem:
  $W \models$ "$G(C)$ is 1-solid and MA_{ω_1} holds."
- Theorem:
  $W \models$ "C is non-trivial, almost smooth and MA_{ω_1} holds."

Let G and H be graphs or di-graphs.

Definition

$G \leq H$ iff that there is a **homomorphism** from G to H

$\leq$ is a **quasi-order** and so it induces an equivalence relation:
$G \sim H$ if and only if $G \leq H$ and $H \leq G$.

Definition

The **homomorphism posets** $\mathbb{G}$ and $\mathbb{D}$ are the partially ordered sets of all equivalence classes of **finite undirected** and **directed graphs**, respectively, **ordered by the** $\leq$.

Let G and H be graphs or di-graphs.

Definition

$G \leq H$ iff that there is a **homomorphism** from G to H

$\leq$ is a **quasi-order** and so it induces an equivalence relation:
$G \sim H$ if and only if $G \leq H$ and $H \leq G$.

Definition

The **homomorphism posets** $\mathbb{G}$ and $\mathbb{D}$ are the partially ordered sets of all equivalence classes of **finite undirected** and **directed graphs**, respectively, **ordered by the** $\leq$.

Let G and H be graphs or di-graphs.

Definition

$G \leq H$ iff that there is a **homomorphism** from G to H

$\leq$ is a **quasi-order** and so it induces an equivalence relation:
$G \sim H$ if and only if $G \leq H$ and $H \leq G$.

Definition

The **homomorphism posets** $\mathbb{G}$ and $\mathbb{D}$ are the partially ordered sets of all equivalence classes of **finite undirected** and **directed graphs**, respectively, **ordered by the** $\leq$.

Let G and H be graphs or di-graphs.

Definition

$G \leq H$ iff that there is a **homomorphism** from G to H

$\leq$ is a **quasi-order** and so it induces an equivalence relation:
$G \sim H$ if and only if $G \leq H$ and $H \leq G$.

Definition

The **homomorphism posets** $\mathbb{G}$ and $\mathbb{D}$ are the partially ordered sets of all equivalence classes of **finite undirected** and **directed graphs**, respectively, **ordered by the** $\leq$.

Let G and H be graphs or di-graphs.

Definition

$G \leq H$ iff that there is a **homomorphism** from G to H

$\leq$ is a **quasi-order** and so it induces an equivalence relation:
$G \sim H$ if and only if $G \leq H$ and $H \leq G$.

Definition

The **homomorphism posets** $\mathbb{G}$ and $\mathbb{D}$ are the partially ordered sets of all equivalence classes of **finite undirected** and **directed graphs**, respectively, **ordered by the** $\leq$.

Definition

A **maximal antichain** A of a poset P **splits** if A can be **partitioned** into two subsets B and C such that $P = B^{\uparrow} \cup C^{\downarrow}$.

In a homomorphism poset
a maximal antichain A **splits**: **structure theorem on (di)graphs**

Definition

A **maximal antichain** A of a poset P **splits** if A can be **partitioned** into two subsets B and C such that $P = B^{\uparrow} \cup C^{\downarrow}$.

In a homomorphism poset

a maximal antichain A **splits**: **structure theorem on (di)graphs**

Definition

A **maximal antichain** A of a poset P **splits** if A can be **partitioned** into two subsets B and C such that $P = B^{\uparrow} \cup C^{\downarrow}$.

In a homomorphism poset

a maximal antichain A **splits**: **structure theorem on (di)graphs**

Definition

A **maximal antichain** A of a poset P **splits** if A can be **partitioned** into two subsets B and C such that $P = B^{\uparrow} \cup C^{\downarrow}$.

In a homomorphism poset
a maximal antichain A **splits**: **structure theorem on (di)graphs**

Definition

A **maximal antichain** A of a poset P **splits** if A can be **partitioned** into two subsets B and C such that $P = B^{\uparrow} \cup C^{\downarrow}$.

In a homomorphism poset
a maximal antichain A splits: **structure theorem on (di)graphs**

$A \subset P$ splits iff $A = B \cup^* C$ s.t. $P = B^{\uparrow} \cup C^{\downarrow}$

$\mathbb{G}$ is the homomorphism posets of all finite undirected graphs.

Theorem

$\mathbb{G}$ has only two finite maximal antichains: $\{K_1\}$ and $\{K_2\}$.

Let $\mathbb{G}' = \mathbb{G} \setminus \{K_1, K_2\}$.

Theorem (Duffus D., Erdos P.L., Nesetril J., Soukup, L.)

*For each **finite antichain** $A \subseteq \mathbb{G}'$ there are **maximal antichains** $A_0, A_1 \supset A$ such that A_0 **splits** and A_1 **does not split**.*

$A \subset P$ splits iff $A = B \cup^* C$ s.t. $P = B^{\uparrow} \cup C^{\downarrow}$

$\mathbb{G}$ is the homomorphism posets of all finite undirected graphs.

Theorem

$\mathbb{G}$ has only two finite maximal antichains: $\{K_1\}$ and $\{K_2\}$.

Let $\mathbb{G}' = \mathbb{G} \setminus \{K_1, K_2\}$.

Theorem (Duffus D., Erdos P.L., Nesetril J., Soukup, L.)

*For each **finite antichain** $A \subseteq \mathbb{G}'$ there are **maximal antichains** $A_0, A_1 \supset A$ such that A_0 **splits** and A_1 **does not split**.*

$A \subset P$ splits iff $A = B \cup^* C$ s.t. $P = B^{\uparrow} \cup C^{\downarrow}$

$\mathbb{G}$ is the homomorphism posets of all finite undirected graphs.

Theorem

$\mathbb{G}$ has only two finite maximal antichains: {K_1} and {K_2}.

Let $\mathbb{G}' = \mathbb{G} \setminus \{K_1, K_2\}$.

Theorem (Duffus D., Erdos P.L., Nesetril J., Soukup, L.)

*For each **finite antichain** $A \subseteq \mathbb{G}'$ there are **maximal antichains** $A_0, A_1 \supset A$ such that A_0 **splits** and A_1 **does not split**.*

$A \subset P$ splits iff $A = B \cup^* C$ s.t. $P = B^{\uparrow} \cup C^{\downarrow}$

$\mathbb{G}$ is the homomorphism posets of all finite undirected graphs.

Theorem

$\mathbb{G}$ has only two finite maximal antichains: $\{K_1\}$ and $\{K_2\}$.

Let $\mathbb{G}' = \mathbb{G} \setminus \{K_1, K_2\}$.

Theorem (Duffus D., Erdos P.L., Nesetril J., Soukup, L.)

*For each **finite antichain** $A \subseteq \mathbb{G}'$ there are **maximal antichains** $A_0, A_1 \supset A$ such that A_0 **splits** and A_1 **does not split**.*

$A \subset P$ splits iff $A = B \cup^* C$ s.t. $P = B^{\uparrow} \cup C^{\downarrow}$

$\mathbb{G}$ is the homomorphism posets of all finite undirected graphs.

Theorem

$\mathbb{G}$ has only two finite maximal antichains: $\{K_1\}$ and $\{K_2\}$.

Let $\mathbb{G}' = \mathbb{G} \setminus \{K_1, K_2\}$.

Theorem (Duffus D., Erdos P.L., Nesetril J., Soukup, L.)

*For each **finite antichain** $A \subseteq \mathbb{G}'$ there are **maximal antichains** $A_0, A_1 \supset A$ such that A_0 **splits** and A_1 **does not split**.*

$A \subset P$ splits iff $A = B \cup^* C$ s.t. $P = B^{\uparrow} \cup C^{\downarrow}$

$\mathbb{G}$ is the homomorphism posets of all finite undirected graphs.

Theorem

$\mathbb{G}$ has only two finite maximal antichains: $\{K_1\}$ and $\{K_2\}$.

Let $\mathbb{G}' = \mathbb{G} \setminus \{K_1, K_2\}$.

Theorem (Duffus D., Erdos P.L., Nesetril J., Soukup, L.)

*For each **finite antichain** $A \subseteq \mathbb{G}'$ there are **maximal antichains** $A_0, A_1 \supset A$ such that A_0 **splits** and A_1 **does not split.***

$A \subset P$ splits iff $A = B \cup^* C$ s.t. $P = B^{\uparrow} \cup C^{\downarrow}$

$\mathbb{G}$ is the homomorphism posets of all finite undirected graphs.

Theorem

$\mathbb{G}$ has only two finite maximal antichains: $\{K_1\}$ and $\{K_2\}$.

Let $\mathbb{G}' = \mathbb{G} \setminus \{K_1, K_2\}$.

Theorem (Duffus D., Erdos P.L., Nesetril J., Soukup, L.)

For each ***finite antichain*** *$A \subseteq \mathbb{G}'$ there are* ***maximal antichains*** *$A_0, A_1 \supset A$ such that A_0* ***splits*** *and A_1* ***does not split****.*

$\mathbb{D}$ is the homomorphism posets of all finite directed graphs.

Theorem (Foniok-Nešetřil-Tardif)

a full description of the finite maximal antichains in $\mathbb{D}$.

Corollary (Foniok-Nešetřil-Tardif)

*Every **finite** maximal antichain splits in $\mathbb{D}$.*

What about infinite antichains in $\mathbb{D}$?

Theorem (Duffus, D., Erdos P.L., Nesetril J, Soukup, L)

There are both splitting and non-splitting maximal infinite antichains in $\mathbb{D}$.

$\mathbb{D}$ is the homomorphism posets of all finite directed graphs.

Theorem (Foniok-Nešetřil-Tardif)

a full description of the finite maximal antichains in $\mathbb{D}$.

Corollary (Foniok-Nešetřil-Tardif)

*Every **finite** maximal antichain splits in $\mathbb{D}$.*

What about infinite antichains in $\mathbb{D}$?

Theorem (Duffus, D., Erdos P.L., Nesetril J, Soukup, L)

There are both splitting and non-splitting maximal infinite antichains in $\mathbb{D}$.

$\mathbb{D}$ is the homomorphism posets of all finite directed graphs.

Theorem (Foniok-Nešetřil-Tardif)

a full description of the finite maximal antichains in $\mathbb{D}$.

Corollary (Foniok-Nešetřil-Tardif)

*Every **finite** maximal antichain splits in $\mathbb{D}$.*

What about infinite antichains in $\mathbb{D}$?

Theorem (Duffus, D., Erdos P.L., Nesetril J, Soukup, L)

There are both splitting and non-splitting maximal infinite antichains in $\mathbb{D}$.

$\mathbb{D}$ is the homomorphism posets of all finite directed graphs.

Theorem (Foniok-Nešetřil-Tardif)

a full description of the finite maximal antichains in $\mathbb{D}$.

Corollary (Foniok-Nešetřil-Tardif)

*Every **finite** maximal antichain splits in $\mathbb{D}$.*

What about infinite antichains in $\mathbb{D}$?

Theorem (Duffus, D., Erdos P.L., Nesetril J, Soukup, L)

There are both splitting and non-splitting maximal infinite antichains in $\mathbb{D}$.

$\mathbb{D}$ is the homomorphism posets of all finite directed graphs.

Theorem (Foniok-Nešetřil-Tardif)

a full description of the finite maximal antichains in $\mathbb{D}$.

Corollary (Foniok-Nešetřil-Tardif)

*Every **finite** maximal antichain splits in $\mathbb{D}$.*

What about infinite antichains in $\mathbb{D}$?

Theorem (Duffus, D., Erdos P.L., Nesetril J, Soukup, L)

There are both splitting and non-splitting maximal infinite antichains in $\mathbb{D}$.

There are both splitting and non-splitting maximal infinite antichains in $\mathbb{D}$.

Question

Assume that $A \subset \mathbb{D}$ is a maximal antichain, $A = B \cup^* C$, $A = B^{\uparrow} \cup C^{\downarrow}$. Is it true that $|B| = |C| = \omega$?

Theorem (Bodirsky M., Erdos L. P., Schahcht M., Soukup L.)

Assume that $A \subset \mathbb{D}$ is an infinite maximal antichain, $A = B \cup^* C$, $A = B^{\uparrow} \cup C^{\downarrow}$. Then $|B| = \omega$.

There are both splitting and non-splitting maximal infinite antichains in $\mathbb{D}$.

Question

Assume that $A \subset \mathbb{D}$ is a maximal antichain, $A = B \cup^* C$, $A = B^{\uparrow} \cup C^{\downarrow}$. Is it true that $|B| = |C| = \omega$?

Theorem (Bodirsky M., Erdos L. P., Schahcht M., Soukup L.)

Assume that $A \subset \mathbb{D}$ is an infinite maximal antichain, $A = B \cup^* C$, $A = B^{\uparrow} \cup C^{\downarrow}$. Then $|B| = \omega$.

There are both splitting and non-splitting maximal infinite antichains in $\mathbb{D}$.

Question

Assume that $A \subset \mathbb{D}$ is a maximal antichain, $A = B \cup^* C$, $A = B^{\uparrow} \cup C^{\downarrow}$. Is it true that $|B| = |C| = \omega$?

Theorem (Bodirsky M., Erdos L. P., Schahcht M., Soukup L.)

Assume that $A \subset \mathbb{D}$ is an infinite maximal antichain, $A = B \cup^* C$, $A = B^{\uparrow} \cup C^{\downarrow}$. Then $|B| = \omega$.

the **girth** of a graph is the **length of a shortest cycle** contained in the graph.

Theorem (Paul Erdős, 1959)

$\forall k, \ell \in \mathbb{N}\ \exists G$ *s. t.* $\chi(G) > k$ *and* $\text{girth}(G) > \ell$.

Definition

The **homomorphism poset** $\mathbb{G}_\omega$ is the partially ordered set of all equivalence classes of **countable undirected graphs ordered by the** $\leq$.

the **girth** of a graph is the **length of a shortest cycle** contained in the graph.

Theorem (Paul Erdős, 1959)

$\forall k, \ell \in \mathbb{N}\ \exists G$ *s. t.* $\chi(G) > k$ *and* $\mathrm{girth}(G) > \ell$.

Definition

The **homomorphism poset** $\mathbb{G}_\omega$ is the partially ordered set of all equivalence classes of **countable undirected graphs ordered by the** $\leq$.

the **girth** of a graph is the **length of a shortest cycle** contained in the graph.

Theorem (Paul Erdős, 1959)

$\forall k, \ell \in \mathbb{N}\ \exists G$ *s. t.* $\chi(G) > k$ *and* $\text{girth}(G) > \ell$.

Definition

The **homomorphism poset** $\mathbb{G}_\omega$ is the partially ordered set of all equivalence classes of **countable undirected graphs ordered by the** $\leq$.

Theorem (Nesetril, Shelah)

*If A is a **1-element maximal antichain** in $\mathbb{G}_\omega$ then $A = \{K_1\}, \{K_2\}$ or $\{K_\omega\}$.*

Conjecture

Conjecture: If A is a finite maximal antichain in $\mathbb{G}_\omega$ then $A \cap \mathbb{G} \neq \emptyset$.

Need: infinite version of Erdős theorem.

$\forall k, \ell \in \mathbb{N}\ \exists G$ s. t. $\chi(G) > k$ and girth$(G) > \ell$.

$\chi(G) > k$ iff $G \not\leq K_k$.

Conjecture

If $H \in \mathbb{G}_\omega$, $K_\omega \not\leq H$ and $\ell \in \mathbb{N}$ then $\exists G \in \mathbb{G}_\omega$ s.t. $G \not\leq H$ and girth$(G) > \ell$

Theorem (Nesetril, Shelah)

If A is a ***1-element maximal antichain*** *in* $\mathbb{G}_\omega$ *then* $A = \{K_1\}, \{K_2\}$ *or* $\{K_\omega\}$.

Conjecture

Conjecture: If A is a finite maximal antichain in $\mathbb{G}_\omega$ then $A \cap \mathbb{G} \neq \emptyset$.

Need: infinite version of Erdős theorem.

$\forall k, \ell \in \mathbb{N}\ \exists G$ s. t. $\chi(G) > k$ and girth$(G) > \ell$.

$\chi(G) > k$ iff $G \not\leq K_k$.

Conjecture

If $H \in \mathbb{G}_\omega$, $K_\omega \not\leq H$ and $\ell \in \mathbb{N}$ then $\exists G \in \mathbb{G}_\omega$ s.t. $G \not\leq H$ and girth$(G) > \ell$

Theorem (Nesetril, Shelah)

If A is a ***1-element maximal antichain*** *in* $\mathbb{G}_\omega$ *then* $A = \{K_1\}, \{K_2\}$ *or* $\{K_\omega\}$.

Conjecture

Conjecture: If A is a finite maximal antichain in $\mathbb{G}_\omega$ then $A \cap \mathbb{G} \neq \emptyset$.

Need: infinite version of Erdős theorem.

$\forall k, \ell \in \mathbb{N}\ \exists G$ s. t. $\chi(G) > k$ and girth$(G) > \ell$.

$\chi(G) > k$ iff $G \not\leq K_k$.

Conjecture

If $H \in \mathbb{G}_\omega$, $K_\omega \not\leq H$ and $\ell \in \mathbb{N}$ then $\exists G \in \mathbb{G}_\omega$ s.t. $G \not\leq H$ and girth$(G) > \ell$

Theorem (Nesetril, Shelah)

If A is a ***1-element maximal antichain*** *in $\mathbb{G}_\omega$ then $A = \{K_1\}, \{K_2\}$ or $\{K_\omega\}$.*

Conjecture

Conjecture: If A is a finite maximal antichain in $\mathbb{G}_\omega$ then $A \cap \mathbb{G} \neq \emptyset$.

Need: infinite version of Erdős theorem.

$\forall k, \ell \in \mathbb{N}\ \exists G$ s. t. $\chi(G) > k$ and girth$(G) > \ell$.

$\chi(G) > k$ iff $G \not\leq K_k$.

Conjecture

If $H \in \mathbb{G}_\omega$, $K_\omega \not\leq H$ and $\ell \in \mathbb{N}$ then $\exists G \in \mathbb{G}_\omega$ s.t. $G \not\leq H$ and girth$(G) > \ell$

Theorem (Nesetril, Shelah)

If A is a ***1-element maximal antichain*** *in $\mathbb{G}_\omega$ then $A = \{K_1\}, \{K_2\}$ or $\{K_\omega\}$.*

Conjecture

Conjecture: If A is a finite maximal antichain in $\mathbb{G}_\omega$ then $A \cap \mathbb{G} \neq \emptyset$.

Need: infinite version of Erdős theorem.

$\forall k, \ell \in \mathbb{N}\ \exists G$ s. t. $\chi(G) > k$ and girth$(G) > \ell$.

$\chi(G) > k$ iff $G \not\leq K_k$.

Conjecture

If $H \in \mathbb{G}_\omega$, $K_\omega \not\leq H$ and $\ell \in \mathbb{N}$ then $\exists G \in \mathbb{G}_\omega$ s.t. $G \not\leq H$ and girth$(G) > \ell$

Theorem (Nesetril, Shelah)

If A is a ***1-element maximal antichain*** *in $\mathbb{G}_\omega$ then $A = \{K_1\}, \{K_2\}$ or $\{K_\omega\}$.*

Conjecture

Conjecture: If A is a finite maximal antichain in $\mathbb{G}_\omega$ then $A \cap \mathbb{G} \neq \emptyset$.

Need: infinite version of Erdős theorem.

$\forall k, \ell \in \mathbb{N}\ \exists G$ s. t. $\chi(G) > k$ and girth$(G) > \ell$.

$\chi(G) > k$ iff $G \not\leq K_k$.

Conjecture

If $H \in \mathbb{G}_\omega$, $K_\omega \not\leq H$ and $\ell \in \mathbb{N}$ then $\exists G \in \mathbb{G}_\omega$ s.t. $G \not\leq H$ and girth$(G) > \ell$

Theorem (Nesetril, Shelah)

If A is a ***1-element maximal antichain*** *in $\mathbb{G}_\omega$ then $A = \{K_1\}, \{K_2\}$ or $\{K_\omega\}$.*

Conjecture

Conjecture: If A is a finite maximal antichain in $\mathbb{G}_\omega$ then $A \cap \mathbb{G} \neq \emptyset$.

Need: infinite version of Erdős theorem.

$\forall k, \ell \in \mathbb{N}\ \exists G$ s. t. $\chi(G) > k$ and girth$(G) > \ell$.

$\chi(G) > k$ iff $G \not\leq K_k$.

Conjecture

If $H \in \mathbb{G}_\omega$, $K_\omega \not\leq H$ and $\ell \in \mathbb{N}$ then $\exists G \in \mathbb{G}_\omega$ s.t. $G \not\leq H$ and girth$(G) > \ell$

Perm(λ): the group of all permutations of a cardinal λ.

$G \leq \text{Perm}(\lambda)$ is κ**-homogeneous** iff for all $X, Y \in [\lambda]^{\kappa}$ there is a $g \in G$ with $g''X = Y$.

$G \leq \text{Perm}(\lambda)$ is κ**-transitive** iff for all 1-1 functions $x, y : \kappa \to \lambda$, there is a $g \in G$ s.t. $g(x(\alpha)) = y(\alpha)$ for all $\alpha < \kappa$

Theorem

*A finite **n-homogeneous** permutation group is **$n-1$-homogeneous**.*

Theorem

An n-homogeneous group is not necesserily n-transitive.

Proof.

Continuous automorphisms of the circle. □

Perm(λ): the group of all permutations of a cardinal λ.
$G \leq \mathrm{Perm}(\lambda)$ is **κ-homogeneous** iff for all $X, Y \in [\lambda]^\kappa$ there is a $g \in G$ with $g''X = Y$.
$G \leq \mathrm{Perm}(\lambda)$ is **κ-transitive** iff for all 1-1 functions $x, y : \kappa \to \lambda$, there is a $g \in G$ s.t. $g(x(\alpha)) = y(\alpha)$ for all $\alpha < \kappa$

Theorem

*A finite **n-homogeneous** permutation group is **$n - 1$-homogeneous**.*

Theorem

An n-homogeneous group is not necesserily n-transitive.

Proof.

Continuous automorphisms of the circle. □

Perm(λ): the group of all permutations of a cardinal λ.
$G \leq \text{Perm}(\lambda)$ is **κ-homogeneous** iff for all $X, Y \in [\lambda]^\kappa$ there is a $g \in G$ with $g''X = Y$.
$G \leq \text{Perm}(\lambda)$ is **κ-transitive** iff for all 1-1 functions $x, y : \kappa \to \lambda$, there is a $g \in G$ s.t. $g(x(\alpha)) = y(\alpha)$ for all $\alpha < \kappa$

Theorem

*A finite **n-homogeneous** permutation group is **$n-1$-homogeneous**.*

Theorem

An n-homogeneous group is not necesserily n-transitive.

Proof.

Continuous automorphisms of the circle. □

Perm(λ): the group of all permutations of a cardinal λ.
$G \leq \text{Perm}(\lambda)$ is **κ-homogeneous** iff for all $X, Y \in [\lambda]^{\kappa}$ there is a $g \in G$ with $g''X = Y$.
$G \leq \text{Perm}(\lambda)$ is **κ-transitive** iff for all 1-1 functions $x, y : \kappa \to \lambda$, there is a $g \in G$ s.t. $g(x(\alpha)) = y(\alpha)$ for all $\alpha < \kappa$

Theorem

*A finite **n-homogeneous** permutation group is **$n-1$-homogeneous**.*

Theorem

An n-homogeneous group is not necesserily n-transitive.

Proof.

Continuous automorphisms of the circle. □

Perm(λ): the group of all permutations of a cardinal λ.
$G \leq \mathrm{Perm}(\lambda)$ is κ**-homogeneous** iff for all $X, Y \in [\lambda]^{\kappa}$ there is a $g \in G$ with $g''X = Y$.
$G \leq \mathrm{Perm}(\lambda)$ is κ**-transitive** iff for all 1-1 functions $x, y : \kappa \to \lambda$, there is a $g \in G$ s.t. $g(x(\alpha)) = y(\alpha)$ for all $\alpha < \kappa$

Theorem

A finite n***-homogeneous*** *permutation group is* $n-1$***-homogeneous****.*

Theorem

An n-homogeneous group is not necesserily n-transitive.

Proof.

Continuous automorphisms of the circle. □

Perm(λ): the group of all permutations of a cardinal λ.
$G \leq \mathrm{Perm}(\lambda)$ is **κ-homogeneous** iff for all $X, Y \in [\lambda]^\kappa$ there is a $g \in G$ with $g''X = Y$.
$G \leq \mathrm{Perm}(\lambda)$ is **κ-transitive** iff for all 1-1 functions $x, y : \kappa \to \lambda$, there is a $g \in G$ s.t. $g(x(\alpha)) = y(\alpha)$ for all $\alpha < \kappa$

Theorem

*A finite **n-homogeneous** permutation group is **$n-1$-homogeneous**.*

Theorem

An n-homogeneous group is not necesserily n-transitive.

Proof.

Continuous automorphisms of the circle. □

Theorem (Hajnal)

If $\square_{\omega_1}$ holds then $\exists G \leq \textbf{Perm}(\omega_2)$ ω_1-homog, but not ω-homog.

Theorem (Shelah, –)

Con($\forall \lambda \geq \omega_1$ $\exists G \leq \textbf{Perm}(\lambda)$ ω_1-homog, but not ω-homog.)

$\exists G \leq \mathrm{Perm}(\omega_1)$ ω-**homogeneous**, but **not** ω-**transitive**.

Theorem (–)

Con($\forall \lambda \geq \omega_1$ $\exists G \leq \textbf{Perm}(\lambda)$ ω-homog, but not ω-transitive.)

Theorem (Hajnal)

If $\square_{\omega_1}$ holds then $\exists G \leq \mathbf{Perm}(\omega_2)$ ω_1-homog, but not ω-homog.

Theorem (Shelah, –)

Con($\forall\lambda \geq \omega_1$ $\exists G \leq \mathbf{Perm}(\lambda)$ ω_1-homog, but not ω-homog.)

$\exists G \leq \mathrm{Perm}(\omega_1)$ ω-**homogeneous**, but **not** ω-**transitive**.

Theorem (–)

Con($\forall\lambda \geq \omega_1$ $\exists G \leq \mathbf{Perm}(\lambda)$ ω-homog, but not ω-transitive.)

Theorem (Hajnal)

*If $\square_{\omega_1}$ holds then $\exists G \leq$ **Perm**(ω_2) ω_1-homog, but not ω-homog.*

Theorem (Shelah, –)

*Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ **Perm**(λ) ω_1-homog, but not ω-homog.)*

$\exists G \leq \mathrm{Perm}(\omega_1)$ ω-**homogeneous**, but **not** ω-**transitive**.

Theorem (–)

*Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ **Perm**(λ) ω-homog, but not ω-transitive.)*

Theorem (Hajnal)

If $\square_{\omega_1}$ *holds then* $\exists G \leq$ ***Perm***(ω_2) ω_1*-homog, but not* ω*-homog.*

Theorem (Shelah, –)

Con($\forall \lambda \geq \omega_1$ $\exists G \leq$ ***Perm***(λ) ω_1*-homog, but not* ω*-homog.)*

$\exists G \leq$ Perm(ω_1) ω**-homogeneous**, but **not** ω**-transitive**.

Theorem (–)

Con($\forall \lambda \geq \omega_1$ $\exists G \leq$ ***Perm***(λ) ω*-homog, but not* ω*-transitive.)*

Theorem (Hajnal)

*If $\square_{\omega_1}$ holds then $\exists G \leq$ **Perm**(ω_2) ω_1-homog, but not ω-homog.*

Theorem (Shelah, –)

*Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ **Perm**(λ) ω_1-homog, but not ω-homog.)*

$\exists G \leq \mathrm{Perm}(\omega_1)$ **ω-homogeneous**, but **not ω-transitive**.

Theorem (–)

*Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ **Perm**(λ) ω-homog, but not ω-transitive.)*

Theorem (Hajnal)

*If $\square_{\omega_1}$ holds then $\exists G \leq$ **Perm**(ω_2) ω_1-homog, but not ω-homog.*

Theorem (Shelah, –)

*Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ **Perm**(λ) ω_1-homog, but not ω-homog.)*

$\exists G \leq$ Perm(ω_1) **ω-homogeneous**, but **not ω-transitive**.

Theorem (–)

*Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ **Perm**(λ) ω-homog, but not ω-transitive.)*

Theorem (Hajnal)

If $\square_{\omega_1}$ *holds then* $\exists G \leq$ ***Perm***(ω_2) ω_1*-homog, but not* ω*-homog.*

Theorem (Shelah, –)

Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ ***Perm***(λ) ω_1*-homog, but not* ω*-homog.)*

$\exists G \leq \text{Perm}(\omega_1)$ **ω-homogeneous**, but **not ω-transitive**.

Theorem (–)

Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ ***Perm***(λ) ω*-homog, but not* ω*-transitive.)*

Theorem (Hajnal)

*If $\square_{\omega_1}$ holds then $\exists G \leq$ **Perm**(ω_2) ω_1-homog, but not ω-homog.*

Theorem (Shelah, –)

*Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ **Perm**(λ) ω_1-homog, but not ω-homog.)*

$\exists G \leq \mathrm{Perm}(\omega_1)$ **ω-homogeneous**, but **not ω-transitive**.

Theorem (–)

*Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ **Perm**(λ) ω-homog, but not ω-transitive.)*

Theorem (Hajnal)

If $\square_{\omega_1}$ *holds then* $\exists G \leq$ ***Perm***(ω_2) ω_1*-homog, but not* ω*-homog.*

Theorem (Shelah, –)

Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ ***Perm***(λ) ω_1*-homog, but not* ω*-homog.)*

$\exists G \leq \text{Perm}(\omega_1)$ ω**-homogeneous**, but **not** ω**-transitive**.

Theorem (–)

Con($\forall\lambda \geq \omega_1$ $\exists G \leq$ ***Perm***(λ) ω*-homog, but not* ω*-transitive.)*

Euler Theorem

Theorem

*(1) A **finite** connected graph has an **Euler-circle** iff the graph is **Eulerian**, i.e. each vertex has even degree.*
*(2) A **finite** connected graph has an **Euler-trail** with end-vertices $v \neq w$ iff v and w are the only vertices of odd degree.*

A **one-way infinite Euler trail** T: a **one-way infinite sequence** $T = (x_0, x_1 \dots,)$ of vertices such that $\{x_i x_{i+1} : i \in \mathbb{N}\}$ is a 1–1 enumeration of the edges of G. x_0 is the *end-vertex* of the trail.
A **two-way infinite Euler trail** T: a **two-way infinite sequence** $T = (\dots, x_{-2}, x_{-1}, x_0, x_1 \dots,)$ of vertices such that $\{x_i x_{i+1} : i \in \mathbb{Z}\}$ is a 1–1 enumeration of the edges of G.

Problem (König)

When does an infinite graph G contain a one/two-way infinite Euler trail?

Euler Theorem

Theorem

(1) A ***finite*** *connected graph has an* ***Euler-circle*** *iff the graph is* ***Eulerian****, i.e. each vertex has even degree.*
(2) A ***finite*** *connected graph has an* ***Euler-trail*** *with end-vertices $v \neq w$ iff v and w are the only vertices of odd degree.*

A **one-way infinite Euler trail** T: a **one-way infinite sequence** $T = (x_0, x_1 \dots,)$ of vertices such that $\{x_i x_{i+1} : i \in \mathbb{N}\}$ is a 1–1 enumeration of the edges of G. x_0 is the *end-vertex* of the trail.
A **two-way infinite Euler trail** T: a **two-way infinite sequence** $T = (\dots, x_{-2}, x_{-1}, x_0, x_1 \dots,)$ of vertices such that $\{x_i x_{i+1} : i \in \mathbb{Z}\}$ is a 1–1 enumeration of the edges of G.

Problem (König)

When does an infinite graph G contain a one/two-way infinite Euler trail?

Euler Theorem

Theorem

*(1) A **finite** connected graph has an **Euler-circle** iff the graph is **Eulerian**, i.e. each vertex has even degree.*
*(2) A **finite** connected graph has an **Euler-trail** with end-vertices $v \neq w$ iff v and w are the only vertices of odd degree.*

A **one-way infinite Euler trail** T: a **one-way infinite sequence** $T = (x_0, x_1 \ldots,)$ of vertices such that $\{x_i x_{i+1} : i \in \mathbb{N}\}$ is a 1–1 enumeration of the edges of G. x_0 is the *end-vertex* of the trail.
A **two-way infinite Euler trail** T: a **two-way infinite sequence** $T = (\ldots, x_{-2}, x_{-1}, x_0, x_1 \ldots,)$ of vertices such that $\{x_i x_{i+1} : i \in \mathbb{Z}\}$ is a 1–1 enumeration of the edges of G.

Problem (König)

When does an infinite graph G contain a one/two-way infinite Euler trail?

Euler Theorem

Theorem

*(1) A **finite** connected graph has an **Euler-circle** iff the graph is **Eulerian**, i.e. each vertex has even degree.*
*(2) A **finite** connected graph has an **Euler-trail** with end-vertices $v \neq w$ iff v and w are the only vertices of odd degree.*

A **one-way infinite Euler trail** T: a **one-way infinite sequence** $T = (x_0, x_1 \ldots ,)$ of vertices such that $\{x_i x_{i+1} : i \in \mathbb{N}\}$ is a 1–1 enumeration of the edges of G. x_0 is the *end-vertex* of the trail.
A **two-way infinite Euler trail** T: a **two-way infinite sequence** $T = (\ldots, x_{-2}, x_{-1}, x_0, x_1 \ldots ,)$ of vertices such that $\{x_i x_{i+1} : i \in \mathbb{Z}\}$ is a 1–1 enumeration of the edges of G.

Problem (König)

When does an infinite graph G contain a one/two-way infinite Euler trail?

Euler Theorem

Observation

The plain generalization fails for infinite graphs:

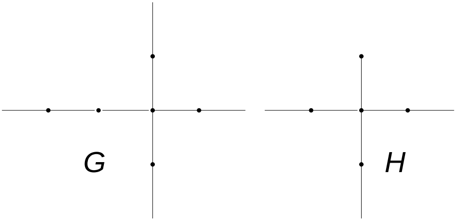
in G each vertex has even degree, but there is **no two-way infinite Euler trail**,
in H there is exactly one vertex with odd degree but there is **no one-way infinite Euler trail**.

Observation

The plain generalization fails for infinite graphs:

in G each vertex has even degree, but there is **no two-way infinite Euler trail**,

in H there is exactly one vertex with odd degree but there is **no one-way infinite Euler trail**.

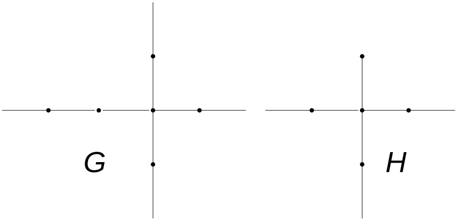


Euler Theorem

Observation

The plain generalization fails for infinite graphs:

in G each vertex has even degree, but there is **no two-way infinite Euler trail**,
in H there is exactly one vertex with odd degree but there is **no one-way infinite Euler trail**.



Euler Theorem

Theorem (Erdős, P.; Grünwald, T.; Vázsonyi, E., 1938)

A graph $G = (V, E)$ has a ***one-way infinite Euler trail with end-vertex*** $v \in V$ *iff (o1)-(o4) below hold:*

1. *G is connected, $|E(G)| = \aleph_0$,*
2. *$d_G(v)$ is odd or infinite,*
3. *$d_G(v')$ is even or infinite for each $v' \in V(G) \setminus \{v\}$,*
4. *$G \setminus E'$ has one infinite component for each finite $E' \subset E$.*

write $owit(G, v)$ iff (1)-(4) above hold for G and v.

Lemma

Assume that G is a graph, $v \in V(G)$, $e \in E(G)$ and $owit(G, v)$ holds. Then there is there is a trail T with endpoints v and v' such that $e \in E(T)$ and $owit(G \setminus T, v')$ holds.

Euler Theorem

Theorem (Erdős, P.; Grünwald, T.; Vázsonyi, E., 1938)

A graph $G = (V, E)$ has a ***one-way infinite Euler trail with end-vertex*** $v \in V$ *iff (o1)-(o4) below hold:*

1. G *is connected,* $|E(G)| = \aleph_0$,
2. $d_G(v)$ *is odd or infinite,*
3. $d_G(v')$ *is even or infinite for each* $v' \in V(G) \setminus \{v\}$,
4. $G \setminus E'$ *has one infinite component for each finite* $E' \subset E$.

write $owit(G, v)$ iff (1)-(4) above hold for G and v.

Lemma

Assume that G is a graph, $v \in V(G)$, $e \in E(G)$ and $owit(G, v)$ holds. Then there is there is a trail T with endpoints v and v' such that $e \in E(T)$ and $owit(G \setminus T, v')$ holds.

Euler Theorem

Theorem (Erdős, P.; Grünwald, T.; Vázsonyi, E., 1938)

A graph $G = (V, E)$ has a ***one-way infinite Euler trail with end-vertex*** $v \in V$ *iff (o1)-(o4) below hold:*

1. *G is connected, $|E(G)| = \aleph_0$,*
2. *$d_G(v)$ is odd or infinite,*
3. *$d_G(v')$ is even or infinite for each $v' \in V(G) \setminus \{v\}$,*
4. *$G \setminus E'$ has one infinite component for each finite $E' \subset E$.*

write $owit(G, v)$ iff (1)-(4) above hold for G and v.

Lemma

Assume that G is a graph, $v \in V(G)$, $e \in E(G)$ and $owit(G, v)$ holds. Then there is there is a trail T with endpoints v and v' such that $e \in E(T)$ and $owit(G \setminus T, v')$ holds.

Euler Theorem

Theorem (Erdős, P.; Grünwald, T.; Vázsonyi, E., 1938)

A graph $G = (V, E)$ has a ***one-way infinite Euler trail with end-vertex*** $v \in V$ *iff (o1)-(o4) below hold:*

1. G *is connected,* $|E(G)| = \aleph_0$,
2. $d_G(v)$ *is odd or infinite,*
3. $d_G(v')$ *is even or infinite for each* $v' \in V(G) \setminus \{v\}$,
4. $G \setminus E'$ *has one infinite component for each finite* $E' \subset E$.

write $owit(G, v)$ iff (1)-(4) above hold for G and v.

Lemma

Assume that G is a graph, $v \in V(G)$, $e \in E(G)$ and $owit(G, v)$ holds. Then there is there is a trail T with endpoints v and v' such that $e \in E(T)$ and $owit(G \setminus T, v')$ holds.

Euler Theorem

Theorem (Erdős, P.; Grünwald, T.; Vázsonyi, E., 1938)

A graph $G = (V, E)$ has a ***one-way infinite Euler trail with end-vertex*** $v \in V$ *iff (o1)-(o4) below hold:*

1. G *is connected,* $|E(G)| = \aleph_0$,
2. $d_G(v)$ *is odd or infinite,*
3. $d_G(v')$ *is even or infinite for each* $v' \in V(G) \setminus \{v\}$,
4. $G \setminus E'$ *has one infinite component for each finite* $E' \subset E$.

write $owit(G, v)$ iff (1)-(4) above hold for G and v.

Lemma

Assume that G is a graph, $v \in V(G)$, $e \in E(G)$ and $owit(G, v)$ holds. Then there is there is a trail T with endpoints v and v' such that $e \in E(T)$ and $owit(G \setminus T, v')$ holds.

Euler Theorem

Theorem (Erdős, P.; Grünwald, T.; Vázsonyi, E., 1938)

A graph $G = (V, E)$ has a ***one-way infinite Euler trail with end-vertex*** $v \in V$ *iff (o1)-(o4) below hold:*

1. G *is connected,* $|E(G)| = \aleph_0$,
2. $d_G(v)$ *is odd or infinite,*
3. $d_G(v')$ *is even or infinite for each* $v' \in V(G) \setminus \{v\}$,
4. $G \setminus E'$ *has one infinite component for each finite* $E' \subset E$.

write $owit(G, v)$ iff (1)-(4) above hold for G and v.

Lemma

Assume that G is a graph, $v \in V(G)$, $e \in E(G)$ and $owit(G, v)$ holds. Then there is there is a trail T with endpoints v and v' such that $e \in E(T)$ and $owit(G \setminus T, v')$ holds.

Euler Theorem

Theorem (Erdős, P.; Grünwald, T.; Vázsonyi, E., 1938)

A graph $G = (V, E)$ has a ***one-way infinite Euler trail with end-vertex*** $v \in V$ *iff (o1)-(o4) below hold:*

1. G *is connected,* $|E(G)| = \aleph_0$,
2. $d_G(v)$ *is odd or infinite,*
3. $d_G(v')$ *is even or infinite for each* $v' \in V(G) \setminus \{v\}$,
4. $G \setminus E'$ *has one infinite component for each finite* $E' \subset E$.

write $owit(G, v)$ iff (1)-(4) above hold for G and v.

Lemma

Assume that G is a graph, $v \in V(G)$, $e \in E(G)$ and $owit(G, v)$ holds.
Then there is there is a trail T with endpoints v and v' such that $e \in E(T)$ and $owit(G \setminus T, v')$ holds.

Euler Theorem

Theorem (Erdős, P.; Grünwald, T.; Vázsonyi, E., 1938)

*A graph $G = (V, E)$ has a **one-way infinite Euler trail with end-vertex** $v \in V$ iff (o1)-(o4) below hold:*

1. *G is connected, $|E(G)| = \aleph_0$,*
2. *$d_G(v)$ is odd or infinite,*
3. *$d_G(v')$ is even or infinite for each $v' \in V(G) \setminus \{v\}$,*
4. *$G \setminus E'$ has one infinite component for each finite $E' \subset E$.*

write $owit(G, v)$ iff (1)-(4) above hold for G and v.

Lemma

Assume that G is a graph, $v \in V(G)$, $e \in E(G)$ and $owit(G, v)$ holds. Then there is there is a trail T with endpoints v and v' such that $e \in E(T)$ and $owit(G \setminus T, v')$ holds.

Euler Theorem

Theorem (Erdős, P.; Grünwald, T.; Vázsonyi, E., 1938)

A graph $G = (V, E)$ has a ***one-way infinite Euler trail with end-vertex*** $v \in V$ *iff (o1)-(o4) below hold:*

1. G *is connected,* $|E(G)| = \aleph_0$*,*
2. $d_G(v)$ *is odd or infinite,*
3. $d_G(v')$ *is even or infinite for each* $v' \in V(G) \setminus \{v\}$*,*
4. $G \setminus E'$ *has one infinite component for each finite* $E' \subset E$*.*

write $owit(G, v)$ iff (1)-(4) above hold for G and v.

Lemma

Assume that G is a graph, $v \in V(G)$, $e \in E(G)$ and $owit(G, v)$ holds. Then there is there is a trail T with endpoints v and v' such that $e \in E(T)$ and $owit(G \setminus T, v')$ holds.

Euler Theorem

Theorem (Erdős, P; Grünwald, T.; Vázsonyi, E., 1938)

A graph G has a two-way infinite Euler trail iff (t1)–(t4) below hold:

1. G is connected, $|E(G)| = \aleph_0$,
2. $d_G(v)$ is even or infinite for each $v' \in V(G)$
3. $G \setminus E'$ has at most two infinite component for each finite $E' \subset E$.
4. $G \setminus E'$ has one infinite component for a finite $E' \subset E$ provided that every degree is even in $\langle V, E' \rangle$.

Euler Theorem

Theorem (Erdős, P; Grünwald, T.; Vázsonyi, E., 1938)

A graph G has a two-way infinite Euler trail iff (t1)–(t4) below hold:

1. *G is connected,* $|E(G)| = \aleph_0$,
2. $d_G(v)$ *is even or infinite for each* $v' \in V(G)$
3. $G \setminus E'$ *has at most two infinite component for each finite* $E' \subset E$.
4. $G \setminus E'$ *has one infinite component for a finite* $E' \subset E$ *provided that every degree is even in* $\langle V, E' \rangle$.

Euler Theorem

Theorem (Erdős, P; Grünwald, T.; Vázsonyi, E., 1938)

A graph G has a two-way infinite Euler trail iff (t1)–(t4) below hold:

1. *G is connected,* $|E(G)| = \aleph_0$,
2. $d_G(v)$ *is even or infinite for each* $v' \in V(G)$
3. $G \setminus E'$ *has at most two infinite component for each finite* $E' \subset E$.
4. $G \setminus E'$ *has one infinite component for a finite* $E' \subset E$ *provided that every degree is even in* $\langle V, E' \rangle$.

Euler Theorem

Theorem (Erdős, P; Grünwald, T.; Vázsonyi, E., 1938)

A graph G has a two-way infinite Euler trail iff (t1)–(t4) below hold:

1. *G is connected,* $|E(G)| = \aleph_0$,
2. $d_G(v)$ *is even or infinite for each* $v' \in V(G)$
3. $G \setminus E'$ *has at most two infinite component for each finite* $E' \subset E$.
4. $G \setminus E'$ *has one infinite component for a finite* $E' \subset E$ *provided that every degree is even in* $\langle V, E' \rangle$.

Euler Theorem

Theorem (Erdős, P; Grünwald, T.; Vázsonyi, E., 1938)

A graph G has a two-way infinite Euler trail iff (t1)–(t4) below hold:

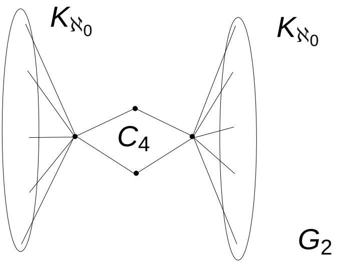
1. *G is connected,* $|E(G)| = \aleph_0$*,*
2. $d_G(v)$ *is even or infinite for each* $v' \in V(G)$
3. $G \setminus E'$ *has at most two infinite component for each finite* $E' \subset E$*.*
4. $G \setminus E'$ *has one infinite component for a finite* $E' \subset E$ *provided that every degree is even in* $\langle V, E' \rangle$*.*

Euler Theorem

Theorem (Erdős, P; Grünwald, T.; Vázsonyi, E., 1938)

A graph G has a two-way infinite Euler trail iff (t1)–(t4) below hold:

1. *G is connected, $|E(G)| = \aleph_0$,*
2. *$d_G(v)$ is even or infinite for each $v' \in V(G)$*
3. *$G \setminus E'$ has at most two infinite component for each finite $E' \subset E$.*
4. *$G \setminus E'$ has one infinite component for a finite $E' \subset E$ provided that every degree is even in $\langle V, E' \rangle$.*



G_2 satisfies (1)-(3) but it does not have a two-way infinite Euler trail.

Euler Theorem

write *twit*(G) iff (1)-(4) above hold for G.

$(*)$ For each finite trail T the graph $G \setminus T$ has **one infinite component**.

Lemma

Let G be a graph, $v \in V(G)$ and $e \in E(G)$. If $twit(G)$ and $(*)$ hold then there is a circuit T in G such that $v \in V(T)$, $e \in E(T)$ and $twit(G \setminus T)$.

If T witnesses that $(*)$ fails then there is a trail T' in G such that

1. the endpoints of T and T' are the same, v_1 and v_2,
2. $G \setminus T'$ has exactly two componets, G_1 and G_2
3. $owit(G_1, v_1)$ and $owit(G_2, v_2)$.

Euler Theorem

write *twit*(G) iff (1)-(4) above hold for G.

(∗) For each finite trail T the graph $G \setminus T$ has **one infinite component**.

Lemma

Let G be a graph, $v \in V(G)$ and $e \in E(G)$. If twit(G) and $(*)$ hold then there is a circuit T in G such that $v \in V(T)$, $e \in E(T)$ and twit$(G \setminus T)$.

If T witnesses that $(*)$ fails then there is a trail T' in G such that

1. the endpoints of T and T' are the same, v_1 and v_2,
2. $G \setminus T'$ has exactly two componets, G_1 and G_2
3. *owit*(G_1, v_1) and *owit*(G_2, v_2).

Euler Theorem

write *twit*(G) iff (1)-(4) above hold for G.

(∗) For each finite trail T the graph $G \setminus T$ has **one infinite component**.

Lemma

Let G be a graph, $v \in V(G)$ and $e \in E(G)$. If twit(G) and $(*)$ hold then there is a circuit T in G such that $v \in V(T)$, $e \in E(T)$ and twit$(G \setminus T)$.

If T witnesses that $(*)$ fails then there is a trail T' in G such that

1. the endpoints of T and T' are the same, v_1 and v_2,
2. $G \setminus T'$ has exactly two componets, G_1 and G_2
3. *owit*(G_1, v_1) and *owit*(G_2, v_2).

Euler Theorem

write *twit*(G) iff (1)-(4) above hold for G.

($*$) For each finite trail T the graph $G \setminus T$ has **one infinite component**.

Lemma

Let G be a graph, $v \in V(G)$ and $e \in E(G)$. If twit(G) and $(*)$ hold then there is a circuit T in G such that $v \in V(T)$, $e \in E(T)$ and twit$(G \setminus T)$.

If T witnesses that $(*)$ fails then there is a trail T' in G such that

1. the endpoints of T and T' are the same, v_1 and v_2,
2. $G \setminus T'$ has exactly two componets, G_1 and G_2
3. *owit*(G_1, v_1) and *owit*(G_2, v_2).

Euler Theorem

write *twit*(G) iff (1)-(4) above hold for G.

$(*)$ For each finite trail T the graph $G \setminus T$ has **one infinite component**.

Lemma

Let G be a graph, $v \in V(G)$ and $e \in E(G)$. If $twit(G)$ and $(*)$ hold then there is a circuit T in G such that $v \in V(T)$, $e \in E(T)$ and $twit(G \setminus T)$.

If T witnesses that $(*)$ fails then there is a trail T' in G such that

1. the endpoints of T and T' are the same, v_1 and v_2,
2. $G \setminus T'$ has exactly two componets, G_1 and G_2
3. $owit(G_1, v_1)$ and $owit(G_2, v_2)$.

Euler Theorem

write *twit*(G) iff (1)-(4) above hold for G.

(∗) For each finite trail T the graph $G \setminus T$ has **one infinite component**.

Lemma

Let G be a graph, $v \in V(G)$ and $e \in E(G)$. If twit(G) and $(*)$ hold then there is a circuit T in G such that $v \in V(T)$, $e \in E(T)$ and twit$(G \setminus T)$.

If T witnesses that $(*)$ fails then there is a trail T' in G such that

1. the endpoints of T and T' are the same, v_1 and v_2,
2. $G \setminus T'$ has exactly two componets, G_1 and G_2
3. *owit*(G_1, v_1) and *owit*(G_2, v_2).

Euler Theorem

write $twit(G)$ iff (1)-(4) above hold for G.

($*$) For each finite trail T the graph $G \setminus T$ has **one infinite component**.

Lemma

Let G be a graph, $v \in V(G)$ and $e \in E(G)$. If $twit(G)$ and $(*)$ hold then there is a circuit T in G such that $v \in V(T)$, $e \in E(T)$ and $twit(G \setminus T)$.

If T witnesses that $(*)$ fails then there is a trail T' in G such that

1. the endpoints of T and T' are the same, v_1 and v_2,
2. $G \setminus T'$ has exactly two componets, G_1 and G_2
3. $owit(G_1, v_1)$ and $owit(G_2, v_2)$.