Geodesic topological cycles in locally finite graphs

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joint work with A. Georgakopoulos

Finite graphs

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	$\iff$	No 'shortcut'

Finite graphs

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$\iff$ No 'shortcut'

Theorem

The cycle space of a finite graph is generated by geodesic circuits.

Proof by picture.

Our plan:

- Extend theorem to locally finite graphs

Infinite graphs

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Infinite graphs

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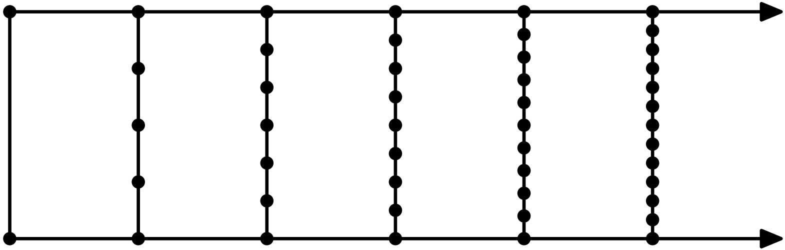
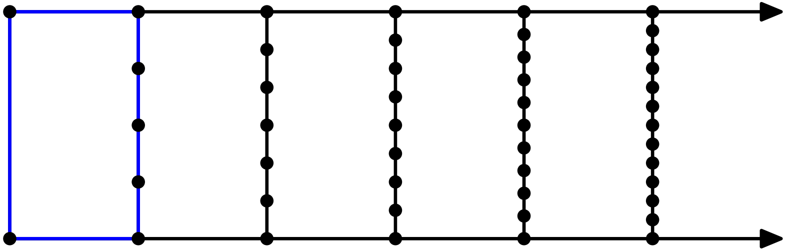
Theorem is false for infinite graphs.

A counterexample



A counterexample



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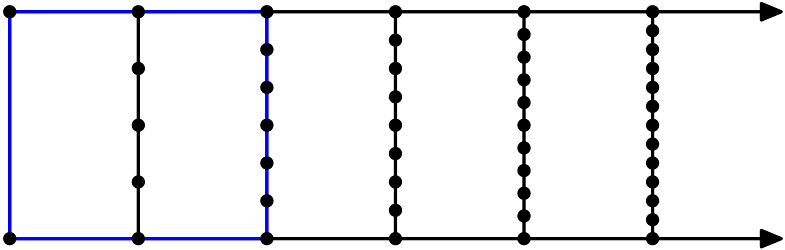

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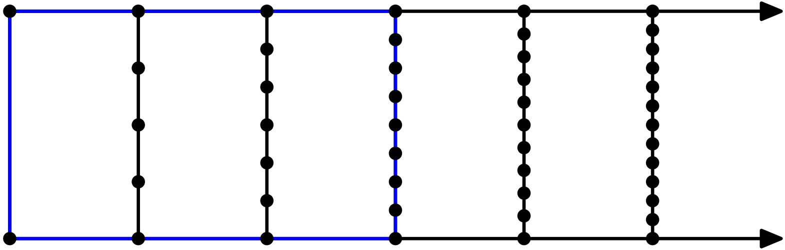

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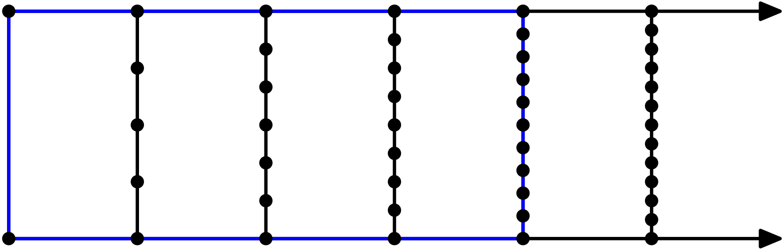

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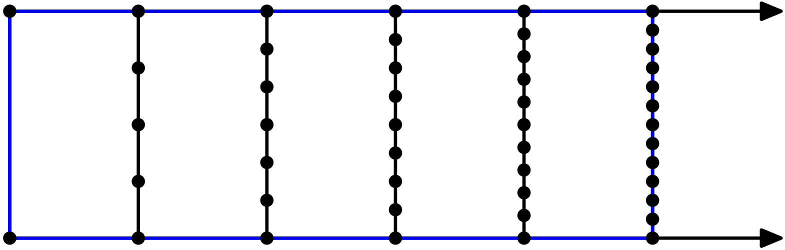

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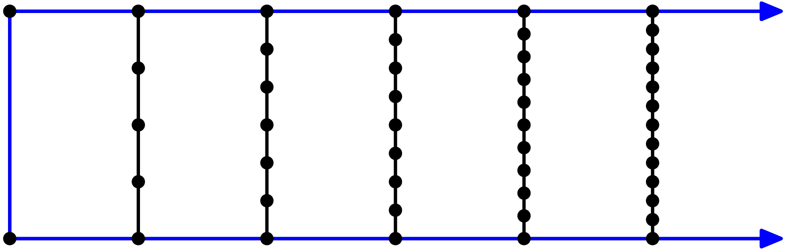

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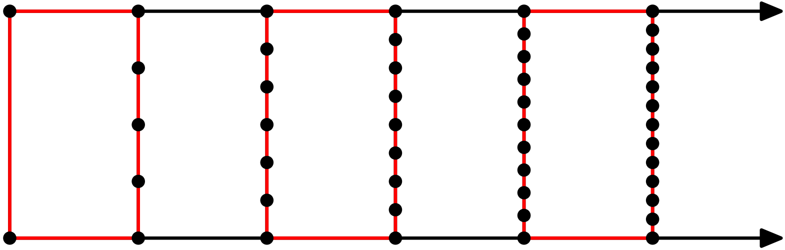

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Infinite circles can have finite length.

ℓ-Geodesic circles may be infinite, but always of finite length.

The new theorem

Theorem

Given suitable edge-lengths $\ell(e)$*, the* ℓ*-geodesic circuits generate* $\mathcal{C}(G)$*.*

The proof

Problems:

The proof

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- Generating circuits is not enough.

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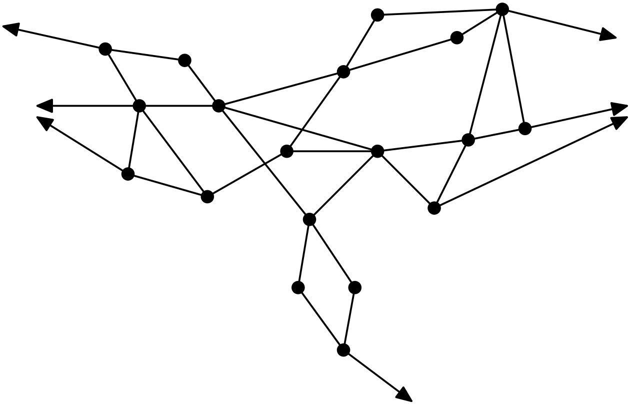
Problems:

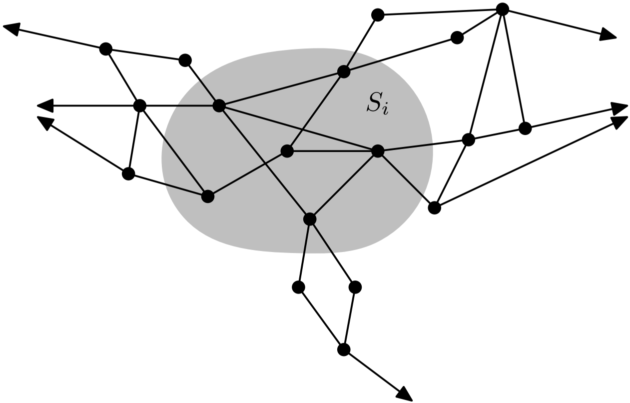
- Generating circuits is not enough.
- Construct geodesic circuits.
- Construct a *thin* family of geodesic circuits.

Constructing geodesic circuits



Constructing geodesic circuits

Finite sets $S_0 \subset S_1 \subset S_2 \subset \cdots$ with $\bigcup S_i = V(G)$.



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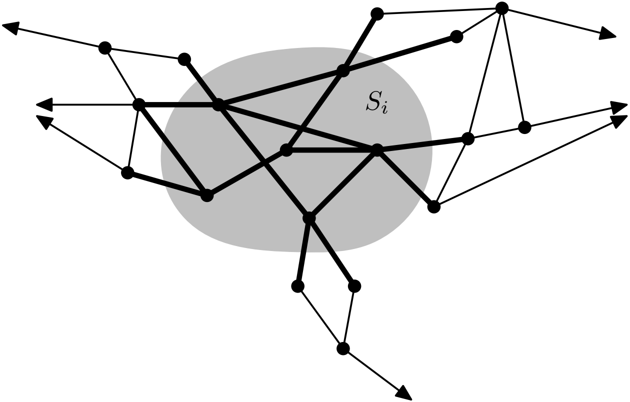

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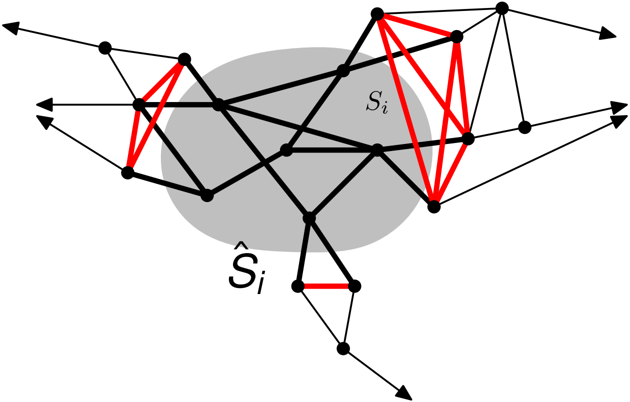

Constructing geodesic circuits

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ℓ_i-Geodesic circuit in each $\hat{S}_i$ $\longrightarrow$ ℓ-Geodesic circuit in $|G|$

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Using compactness will not yield a thin family.

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The family is thin.

Final remark

Go to Angelos' workshop!