Hamilton Circles in Locally Finite Graphs

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1. Introduction

- **The Four Color Theorem**: Every planar map is 4-face colorable.

- **Theorem** (Whitney 1931). Every 4-connected planar triangulation contains a Hamilton cycle.

- **Theorem** (Tutte 1956). 4-Connected planar graphs contain Hamilton cycles.

2. Spanning Rays

- An infinite graph G is k-*indivisible*, where k is a positive integer, if for any finite $X \subseteq V(G)$, $G - X$ has at most $k-1$ infinite components.

- For locally finite graphs, a graph is k-indivisible iff it has at most $k-1$ ends.

- **Conjecture** (Nash-Williams, 1971). A 4-connected infinite planar graph contains a spanning ray iff it is 2-indivisible.

  —established by Dean, Thomas and Y. (1997)

- **Conjecture** (Nash-Williams, 1971). A 4-connected infinite planar graph contains a spanning double ray iff it is 3-indivisible.

  —established by Y. (1999-2004).

- **Conjecture** (Bruhn, 2005?). Every locally finite 4-connected planar graph admits a Hamilton circle.

- True for 6-connected graphs with finitely many ends (Bruhn and Y. 2005).

3. 2-Indivisible Plane Graphs

- A *dividing* cycle C in an infinite plane graph G is a cycle such that each closed region bounded by C contains infinitely many vertices of G.

- If the cycle C is not dividing, then we can define $I(C)$, the maximal subgraph of G contained in the closed region bounded by C which contain only finitely many vertices of G.

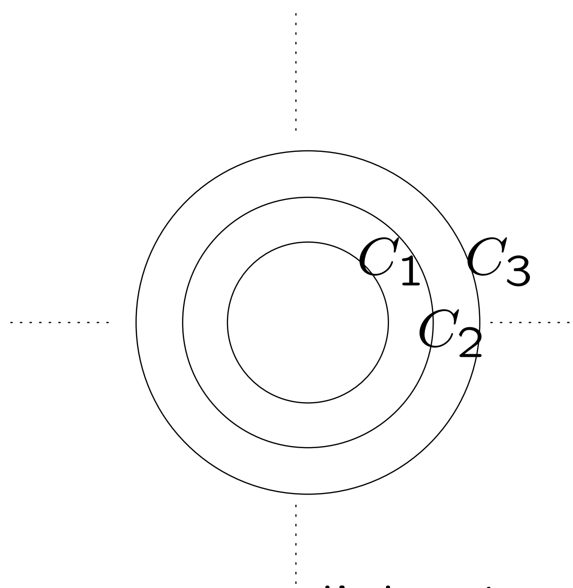
- If an infinite plane graph is 2-indivisible, then it contains no dividing cycles.

- **Theorem** (Dean, Thomas and Y. 1997). Let G be a 2-indivisible *locally finite* infinite plane graph, with an appropriate connectivity condition. Then there exist cycles $C_1, C_2, \ldots$ such that either

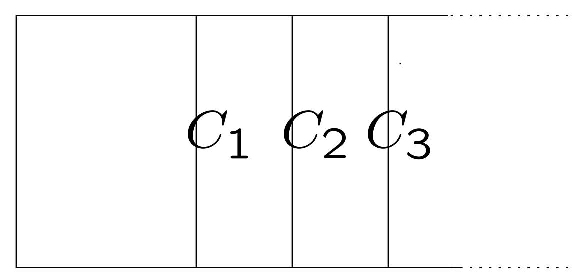
(1) $C_i \cap C_j = \emptyset$ for $i \neq j$, $I(C_i) \subseteq I(C_{i+1})$, and $G = \bigcup I(C_i)$, or

(2) $I(C_i) \subseteq I(C_{i+1})$, $C_i \cap C_{i+1}$ is subpath of $C_{i+1} \cap C_{i+2}$ with no common endvertex, and $G = \bigcup I(C_i)$.

- $(C_1, C_2, \ldots)$ is a *radial net* if (1) is satisfied, and *ladder net* otherwise.



radial net



ladder net

4. 3-Indivisible graphs

- Let $\gamma(G)$ denote the maximum number of vertex disjoint dividing cycles in G.

- 3-Indivisible infinite plane graphs can be divided into three classes:

  – those with $\gamma(G) = 0$,

  – those with $\gamma(G) = \infty$, and

  – those with $0 < \gamma(G) < \infty$.

5. Graphs with $\gamma(G) = 0$

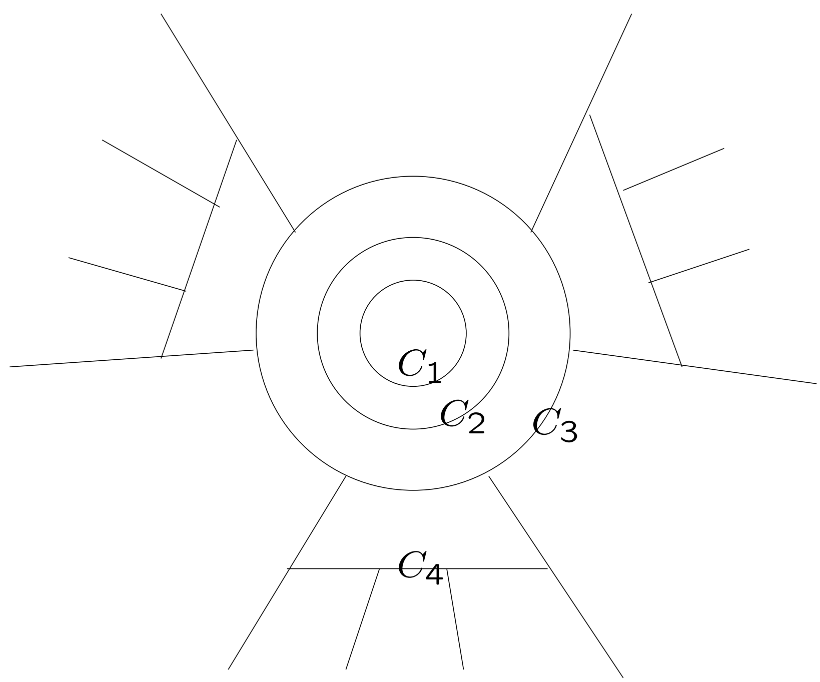
- Structure: If G is an infinite plane graph with $\gamma(G) = 0$ (suitably connected), then there is a sequence of cycles $(C_1, C_2, \ldots)$, called a *net*, in G such that

  - $I(C_i) \subseteq I(C_{i+1})$,

  - each component of $C_i \cap C_{i+1}$ is subpath of some component of $C_{i+1} \cap C_{i+2}$, with no common endvertex,

  - $G = \bigcup I(C_i)$.

- Nice embdedding of G.


C_1
C_2
C_3
C_4

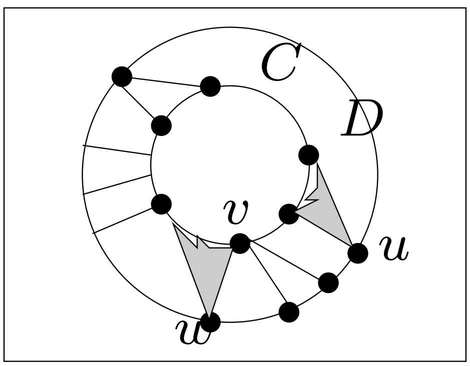
- Let G be a locally finite plane graph with $\gamma(G) = 0$.

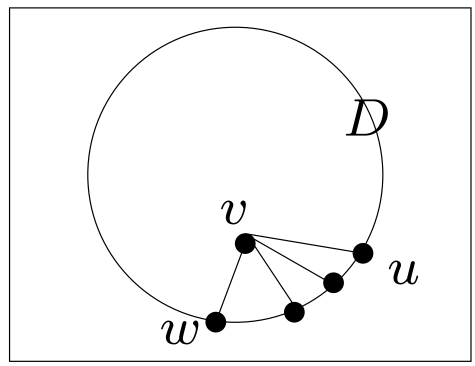
- We may assume that G is nicely embedded.

- Let C be a facial cycle of G, and e be an edge of C.

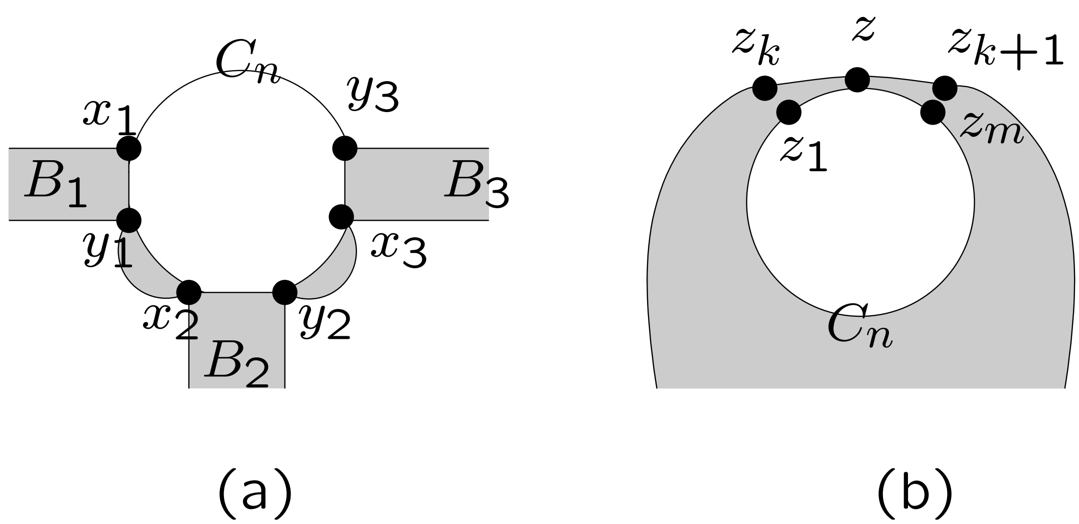
- Will show that G contains a collection of double rays so that the closure of their union is a Hamilton circle.

6. First reduction

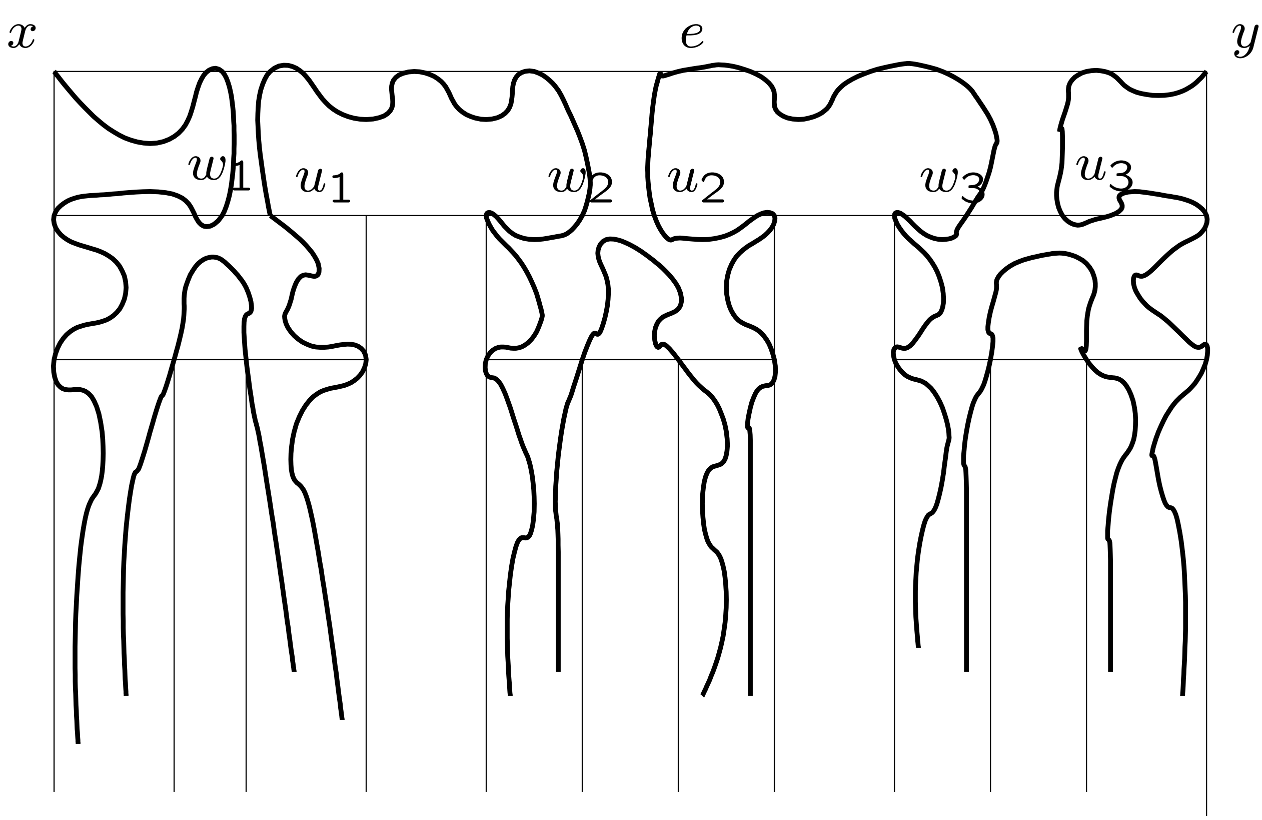


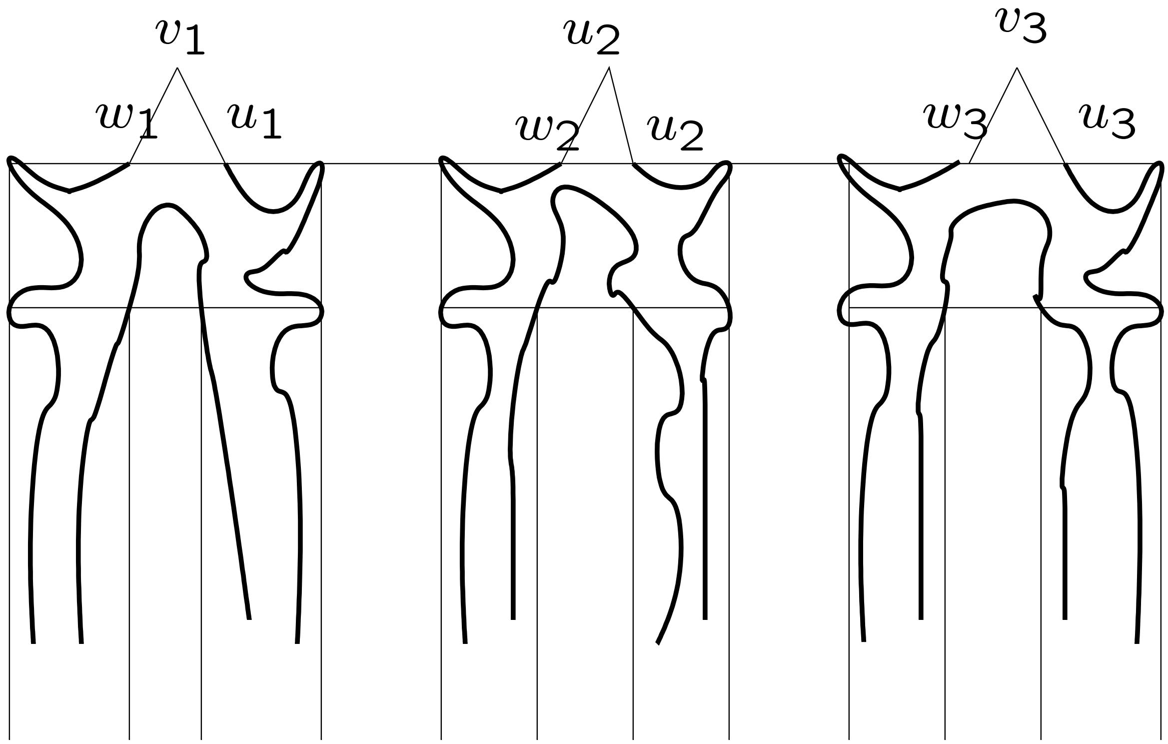


We may assume that G has at least two ends.



C_n
x_1
y_3
B_1
B_3
y_1
x_3
x_2
y_2
B_2
(a)
z_k
z
z_{k+1}
z_1
z_m
C_n
(b)


x
e
y
w_1
u_1
w_2
u_2
w_3
u_3


v_1
w_1
u_1
u_2
w_2
u_2
v_3
w_3
u_3