The Quasi Order of Graphs on an Ordinal

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4:55-5:45; 18.10.2007

A large part of the talk is from a paper with Jean Larson and Péter Komjáth.

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This partial quasiorder $(\mathcal{G}_{\aleph_0}; \preceq)$ is atomic with exactly two atoms.

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More generally let G be a graph.

Let $\mathcal{G}(\mathrm{G})$ be the set of all functions $f : E(\mathrm{G}) \to \{0,1\}$ and

$(\mathcal{G}(\mathrm{G}); \preceq)$ the partial quasiorder on $\mathcal{G}(\mathrm{G})$ for which $f \preceq g$ if there exists an embedding $h : \mathrm{G} \to \mathrm{G}$ with $f\{x,y\} = g\{h(x), h(y)\}$ for all $x, y \in V(\mathrm{G})$.

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For example, let R be the Rado graph. Then $(\mathcal{G}(\mathrm{R}); \preceq)$ is atomic with four atoms. Similar for the countable triangle free homogeneous graph.

Or, let $(S;\leq)$ be a total order.

Let $\mathcal{G}_{(S;\leq)}$ be the set of all functions $\mathrm{F} : [S]^2 \rightarrow \{0,1\}$.

Let $(\mathcal{G}_{(S;\leq)};\preceq)$ be the partial quasiorder for which $F \preceq G$ if there exists a strictly order preserving function h of S to S so that $\mathrm{F}\{x,y\} = \mathrm{G}\{h(x),h(y)\}$ for all $x,y \in S$.

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Then $(\mathcal{G}_{\mathbb{Q}}; \preceq)$ is atomic with exactly four atoms. We have seen that $(\mathcal{G}_\omega; \preceq)$ is atomic with two atoms.

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Problem: Let μ be an ordinal. Is the partial quasiorder $(\mathcal{G}_\mu; \preceq)$ atomic? If so, how many atoms? If not, what is the coinitiality of $(\mathcal{G}_\mu; \preceq)$?

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For $\mathrm{F}, \mathrm{G} \in \mathcal{G}(n)$ we write $F \preceq G$ if there is a strictly order preserving function $h : \mathcal{W}(n) \to \mathcal{W}(n)$ so that $\{x, y\} \in E(\mathrm{F})$ if and only if $\{h(x), h(y)\} \in E(\mathrm{G})$.

Given two elements of $\mathcal{W}(n)$, they exhibit a *pattern*:
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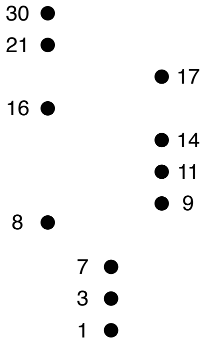
n=7

<1,3,7,8,16,21,30>
<1,3,7,9,11,14,17>



k=4

{0,1,1,1,0,1,0,0}

If $1 \leq k \leq n$ then a *k-pattern* is a function

$u : \{1, 2, \ldots, 2k\} \to \{0, 1\}$ with

$u(1) = 0$, $|u^{-1}(0)| = k$, $|u^{-1}(1)| = k$.

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Note: The number of k-patterns is

$$\binom{2k-1}{k-1}.$$

The number of all regular patterns is

$$T_n = \sum_{k=1}^{n} \binom{2k-1}{k-1}.$$

The pair $(\mathbf{a}_1, \mathbf{b}_1)$ of elements of $\mathcal{W}(n)$ is equivalent to the pair $(\mathbf{a}_2, \mathbf{b_2})$, written $(\mathbf{a}_1, \mathbf{b}_1) \equiv (\mathbf{a}_2, \mathbf{b_2})$, if the pattern of $(\mathbf{a}_1, \mathbf{b}_1)$ is equal to the pattern of $(\mathbf{a}_2, \mathbf{b_2})$.

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A graph $\mathrm{G} \in \mathcal{G}(n)$ is *pattern uniform* if $E(\mathrm{G})$ is the union of $\equiv$ equivalence classes.

Lemma

For every graph $\mathrm{G} \in \mathcal{G}(n)$ *there exists a pattern uniform graph* A *with* $\mathrm{A} \preceq \mathrm{G}$.

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Proof.

Note that if u is a k-pattern and $\mathbf{c} \in (W)(n+k)$ then there is a unique pair $(\mathbf{a}, \mathbf{b})$ of elements in $\mathcal{W}(n)$ whose pattern is u and for which the entries of $\mathbf{c}$ are the entries of $\mathbf{a}$ union the entries of $\mathbf{b}$.

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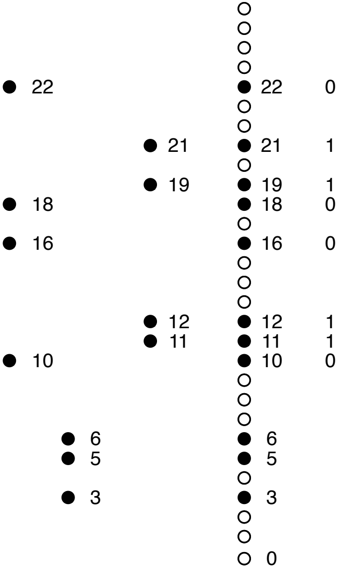
22 22 0
21 21 1
19 19 1
18 18 0
16 16 0
12 12 1
11 11 1
10 10 0
6 6
5 5
3 3
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If $\{\mathbf{a},\mathbf{b}\}$ is an edge of G then $l(\mathbf{c}) = 1$ and otherwise $l(\mathbf{c}) = 0$.

Apply Ramsey's theorem. Let $f(u) = 1$ if there is an infinite subset S of ω so that for every $\mathbf{c} \in \mathcal{W}(n) \cap S^{n+k}$ we have $l(\mathbf{c}) = 1$) and $f(u) = 0$ otherwise. Repeat for all k-patterns and all $1 \leq k \leq n$.

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We obtain an infinite subset S of ω and a function f of all patterns into $\{0, 1\}$ so that for all $\mathbf{a}, \mathbf{b} \in \mathcal{W}(n) \cap S^n$ the pair $\{\mathbf{a}, \mathbf{b}\}$ is an edge of G if and only if $f(u) = 1$ for the pattern u of $(\mathbf{a}, \mathbf{b})$.

□

Note that every order preserving map of $\mathcal{W}(n)$ into $\mathcal{W}(n)$ preserves patterns. Hence:

Lemma

Let A and B be pattern regular graphs in $\mathcal{G}(n)$ with $\mathrm{A} \preceq \mathrm{B}$. Then $\mathrm{B} \preceq \mathrm{A}$.

Combining both Lemmas we obtain:

Theorem

In the partial quasiorder of graphs G with $V(\mathrm{G}) = \omega^n$ ordered under order preserving embeddings there is an atom below every graph and the number of atoms is

$$2^{T_n} \quad \textit{with} \quad T_n = \sum_{k=1}^{n} \binom{2k-1}{k-1}.$$

Theorem

Assume that $\alpha < \omega^\omega$. In the partial quasiorder of graphs G with $V(\mathrm{G}) = \alpha$ ordered under order preserving embeddings there is an atom below every graph. If α has the Hausdorff normal form $\alpha = \omega^{n_0} + \cdots + \omega^{n_k}$ with $n_0 \geq \cdots \geq n_k$ then the number of atoms is 2^{a+b} where

$$a = \sum_{i=0}^{k} T_{n_i}$$

and

$$b = \sum_{i<j} \binom{n_i + n_j}{n_i}.$$

Authors

Introduction

The atoms of $\mathcal{G}_{\omega^n}$

The case ω^ω

Ultrametric spaces

Theorem

In the partial quasiorder of graphs G *with* $V(\mathrm{G}) = \omega^\omega$ *ordered under order preserving embeddings there is an atom below every graph. This quasiorder of graphs has* $2^{\aleph_0}$ *atoms.*

An ultrametric space is a metric space satisfying the inequality

$$\mathrm{d}(x, z) \leq \max\{\mathrm{d}(x, y), \mathrm{d}(y, z)\}$$

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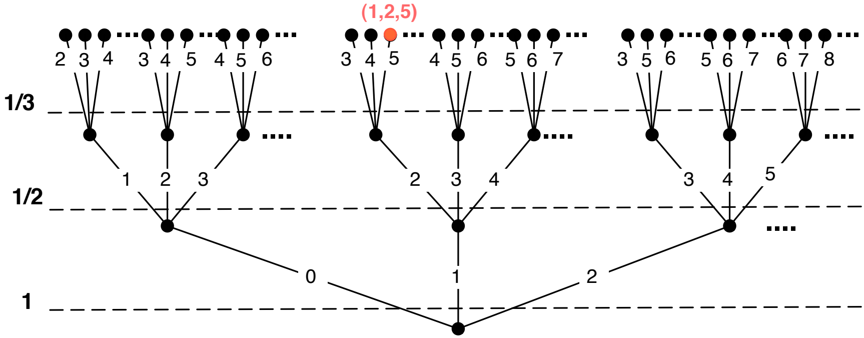
A metric space is *homogeneous* if every isometry of finite subspaces extends to an automorphism of the metric space.

Let $n \in \omega$ and $\mathbf{a}, \mathbf{b} \in \mathcal{W}(n)$ with $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$.

If $\mathbf{a} \neq \mathbf{b}$ let $\mathrm{d}(\mathbf{a}, \mathbf{b}) = 1/i$ where i is the smallest index for which $a_i \neq b_i$.

If $\mathbf{a} = \mathbf{b}$ let $\mathrm{d}(\mathbf{a}, \mathbf{b}) = 0$.

$\mathcal{W}(3)$:



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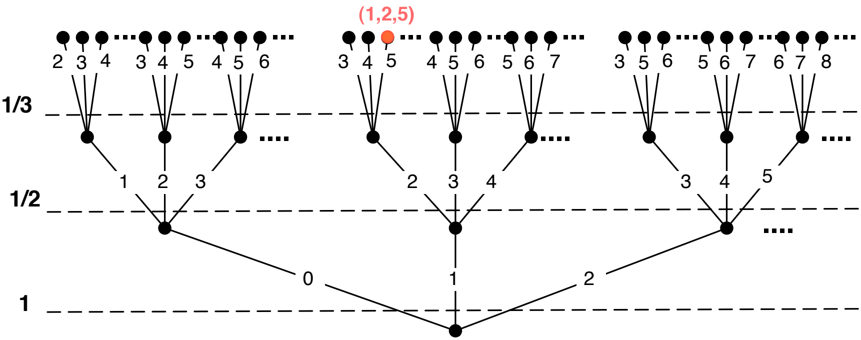

$(\mathcal{W}(n), \mathrm{d})$ is a countable homogeneous ultrametric space, its big Ramsey degree has been determined by Lionel Nguyen Van Thé.

Let V be a countable set of non negative real numbers which is dually well ordered. Using the idea of Fraïssé limits one can show that there exists a homogeneous ultrametric space $\mathbb{U}lt_V$ whose set of finite subspaces is the set of all finite ultrametric spaces whose set of distances is a subset of V.

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The *Nerv* of an ultrametric space is the set of closed balls of the ultrametric space, whose diameter is attained.

Theorem

Let M *be a denumerable ultrametric space. The following properties are equivalent:*

(i) M *is isometric to some* $\mathbb{U}lt_V$*, where* V *is dually well-ordered;*

(ii) M *is point-homogeneous,* $P := (Nerv(\mathrm{M}), \supseteq)$ *is well founded and the degree of every non maximal element is infinite;*

(iii) M *is homogeneous and indivisible;*