The Rate of Convergence of LERW to SLE(2)

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1 Overview of the field

In 2000, O. Schramm [4] introduced a one-parameter family of random growth processes in two dimensions which he called the *stochastic Loewner evolution* (now also called the *Schramm-Loewner evolution*) or SLE. In the past several years, SLE techniques have been successfully applied to analyze a variety of two-dimensional statistical mechanics models including percolation, the Ising model, the Q-state Potts model, uniform spanning trees, loop-erased random walk, and self-avoiding walk. Furthermore, SLE has provided a mathematically rigorous framework for establishing predictions made by conformal field theory (CFT), and much current research is being done to further strengthen and explain the links between SLE and CFT.

The importance of proving predictions about such models made by conformal field theory in a rigorous mathematical sense was acknowledged when W. Werner was awarded a Fields medal in 2006 for "his contributions to the development of stochastic Loewner evolution, the geometry of two-dimensional Brownian motion, and conformal field theory." Although there is knowledge of the scaling limit in the aforementioned models, there is essentially nothing known about the rates of convergence of any of these discrete models to SLE. In fact, this mathematically important open problem was communicated by Schramm in his plenary lecture at the International Congress of Mathematicians in Madrid in 2006: "Obtain reasonable estimates for the speed of convergence of the discrete processes which are known to converge to SLE." (See [5], page 532.)

Therefore, the objective of our *Research In Teams* meeting was to study the rate of convergence of loop-erased random walk to SLE(2) (i.e., SLE with parameter 2). In our opinion, this was the most promising case and the first one that should be considered. Loop-erased random walk has been extensively studied, and there are a number of tools available for analyzing them including a detailed proof of convergence to radial SLE(2) by G. Lawler, O. Schramm, and W. Werner [3].

In order to determine a reasonable rate of convergence of loop-erased random walk to SLE(2), it was necessary to first understand Lawler, Schramm, and Werner's original proof of convergence [3]. The following is a description of the two convergence theorems they established for loop-erased random walk. Suppose that D is a simply connected proper subset of the complex plane $\mathbb{C}$ containing 0, and let $\delta > 0$. Denote by μ_δ the law of the loop-erasure of simple random walk on $\delta\mathbb{Z}^2$ started at 0 and stopped at ∂D, and denote by ν the law of the image of the radial SLE(2) path under a normalized conformal transformation from the unit disk $\mathbb{D}$ onto D fixing 0. Define the metric ρ on the space of unparametrized curves in $\mathbb{C}$ by setting

$$\rho(\beta, \gamma) = \inf \sup_{0 \le t \le 1} |\hat{\beta}(t) - \hat{\gamma}(t)|$$

where the infimum is taken over all possible parametrizations $\hat{\beta}$, $\hat{\gamma}$ in $[0, 1]$ of β, γ, respectively.

The main theorem is a precise statement about what it means for loop-erased random walk to converge in the scaling limit to radial SLE(2).

Theorem 1.1 (LERW scaling limit). *The measures μ_δ converge weakly to ν as $\delta \to 0$ with respect to the metric ρ on the space of curves.*

An important step in their proof was to establish that the Loewner driving process for the discrete process (loop-erased random walk) converged to a scaled Brownian motion (which is the driving process for SLE).

A *grid domain* is a simply connected subset of the complex plane with a boundary that is contained in the edge set of $\mathbb{Z}^2$.

Theorem 3.7 (Driving process convergence). *For every $T > 0$ and $\epsilon > 0$, there is an $r = r(\epsilon, T) > 0$ such that, for all grid domains $D \ni 0$ with $\mathrm{inrad}(D) > r$, there is a coupling of γ with Brownian motion $B(t)$ starting at a uniform random point in $[0, 2\pi]$ such that*

$$\mathbf{P}[\sup\{|\vartheta(t) - B(2t)| : 0 \leq t \leq T\} > \epsilon] < \epsilon.$$

2 Scientific progress made

The primary goal we had for our week at BIRS was to understand the original paper [3] by Lawler, Schramm, and Werner. We knew *a priori* that it would not be possible to establish a rate of convergence without a clear understanding of the convergence itself. After several days of studying their proof, it became clear to us that our first step would be to establish a reasonable rate for the "driving process convergence" (which is Theorem 3.7 in [3]). Indeed, by the end of our week at BIRS, we had an outline of the proof of such a rate.

The original proof of Theorem 3.7 in [3] of the driving process convergence required several preliminary results that together form the proof. These included their Proposition 2.2 ("hitting probability"), Proposition 3.4 ("the key estimate"), and Lemma 3.8 ("Skorohod embedding").

Therefore, in order for us to establish a reasonable rate for the "driving process convergence," the first thing that we had to do was understand in detail these three preliminary results. We eventually discovered how a modification of a result due to Kozdron and Lawler (Proposition 3.10 in [2]) could be used to give a rate in Proposition 2.2 ("hitting probability"). We were then able to use this rate to re-establish Proposition 3.4 ("the key estimate") with a rate of convergence. The final step was to study in detail the Skorohod embedding scheme for martingales. Having done this, we were then able to determine a specific rate for the driving process convergence. We are currently working on a manuscript [1] that writes this proof out carefully with complete details. Once that is complete, we will begin working on establishing a rate in Theorem 1.1 ("LERW scaling limit") of [3].

References

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