New Topological Contexts for Galois Theory and Algebraic Geometry

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1 Overview of the Field

There were two main themes present in the workshop. One is probably best described by the term *arithmetic of structured ring spectra* and the other might be called *arithmetically defined cohomology theories*.

Cohomology theories are classical subjects in topology and of stable homotopy theory. Different cohomology theories detect different aspects of topological spaces. Particularly nice cohomology theories are the ones that carry a multiplicative structure, for example singular cohomology with coefficients in a ring has cup products, cobordism theories have products related to cartesian products of manifolds, and K-theories have products related to tensor producs of vector bundles. Such structures make it possible to use algebraic methods in order to investigate generalized cohomology groups of spaces. Cohomology theories can be represented by spectra. These are the objects of study in the stable homotopy category. Multiplicative structures on a cohomology theory give rise to so-called homotopy ring structures on the associated spectrum.

Since the middle of the 90's several symmetric monoidal categories of spectra were developed that model the stable homotopy category, for instance symmetric spectra [10] and S-modules [8]. From now on we will call such strict versions of homotopy ring spectra *structured ring spectra* or just *ring spectra*.

In good cases it is possible to improve homotopy ring structures on spectra to actual monoid structures in these strict models; sometimes this is done using classical approaches [13], but there are other means such as direct constructions, obstruction theories or the application of functors to ring spectra that have ring spectra as an outcome.

With these strict models the analogy to classical algebra is a sound one that can be exploited: it is possible to study the algebraic features of ring spectra such as Galois and Hopf-Galois theory of structured ring spectra as defined by Rognes, and to investigate arithmetic properties by studying algebraic K-theory of ring spectra.

Classical rings give rise to Eilenberg-MacLane spectra, the representing objects of singular cohomology, and this process embeds classical algebra and arithmetic into the world of ring spectra. However, homotopy theory provides a much richer variety of *rings* than algebra: the initial ring spectrum is the sphere spectrum and for instance there are many Galois extensions sitting between the sphere spectrum and the field with p elements for a prime p.

In order to understand algebraic K-theory of ring spectra it is often useful to investigate topological Hochschild homology [7] or topological cyclic homology of ring spectra. These homology theories are topological analogues of Hochschild and cyclic homology for algebras. For commutative ring spectra topological André-Quillen homology [4] is an important homology theory. Determining algebraic K-theory is usually a demanding task, but substantial progress has been made with respect to concrete calculations [3, 2, 1] and with respect to statements about structural properties (see for instance [6, 16]).

The construction of some important examples of structured ring spectra is inspired by arithmetic and algebraic geometric ideas, and relies on machinery from derived algebraic geometry newly developed by Lurie, Toën and Vessozi [11, 17, 18]. Most prominently, Hopkins' spectrum of topological modular forms, TMF [9], and Behrens-Lawson's topological automorphic forms, TAF [5].

Following the foundational developments during the last decade, the subject area seems to be moving into a period in which applications in homotopy theory as well as algebraic geometry and other fields can be seen.

2 The workshop and talks

The BIRS workshop on **New Topological Contexts for Galois Theory and Algebraic Geometry** took place at a crucial moment in the development of the subject. With many of the leading experts on various aspects of the subject attending the conference, the participants and in particular younger researchers had an excellent opportunity to communicate and exchange their ideas.

The programme was organised around a series of invited talks prepared by small groups of participants or indivduals, and these were intended to provide to give the audience an up-to-date perspective in these (often specialised) topics. The rest of the talks were selected from those offered by participants.

The invited talks included two on Hopf Galois theory [14], three on ∞-categories, one on topological André-Quillen cohomology [4], two on algebraic K-theory for S-algebras, two on topological logarithmic structures, as well as talks on étale K-theory, topological automorphic forms, Hopf Galois extensions in monoidal model categories, realisability of Landweber exact theories, and ended with a timely survey/historical talk by May on E_∞ ring spaces and spectra. The other talks covered a range of topics all well-related to the subject of the workshop.

A poster board was available during the workshop, and a number of posters appeared, with a session during which their authors were available to discuss them.

The last morning consisted of a problem session in which individuals proposed problems and gave short background talks.

The following talks were given at the workshop. We list the speakers in alphabetical order; the abstracts were written by the speakers.

Speakers: **Clark Barwick & David Gepner** (Universities of Oslo, Sheffield)
Title: *∞-categories and applications, I & II*
Abstracts:
I Infinity Categories and Applications – Models for the ∞-category of ∞-categories.
∞-categories provide an excellent foundation for abstract homotopy theory, offering several advantages over the more traditional approach via Quillen model categories. After discussing some of their merits, we turn to definitions and models for higher categories and specifically $(\infty, 1)$-categories, including simplicial categories, Segal categories, complete Segal spaces, and quasicategories. Finally, we focus on quasicategories, which are in some sense the smallest and most streamlined of the known models.

II Post-Prandial Infinity-Categories – Third Half.
In this talk, I describe the relationship between model categories and infinity-categories. By the coherent nerve construction discussed in part I, one can convert any model category into an infinity category. But when one performs infinity categorical constructions on coherent nerves of model categories, how does one understand the result? The answer comes in the form of the Strictification Theorems, which give models for infinity-categories constructed by homotopy limits and internal homs. Further, there are multi-infinity-category versions of this result as well, which allow for the strictification of weak algebras over weak operads. By combining these results with the theory of operator categories, one can prove a number of interesting results, including the assertion that the space of E_n structures on the infinity-category of left modules over an E_1 ring spectrum A is canonically equivalent to the space of E_{n+1} structures on A itself.

Speakers: **Christian Ausoni & Christian Schlichtkrull** (Universities of Bonn, Bergen)
Title: *Algebraic K-theory and traces for structured ring spectra, I and II*

The aim of this series of lectures is to give an overview of the trace invariants associated to the algebraic K-theory $K(A)$ of a connective structured ring spectrum A. We begin with the definition of $K(A)$ based

on matrices, and we explain how it is related to Waldhausen's A-theory of spaces by means of spherical group-rings. We then review the Bökstedt trace map to topological Hochschild homology, and its refinement given by the cyclotomic trace to topological cyclic homology. We also present some general results, like the theorems on relative K-theory of Dundas and McCarthy. In the second part of this lecture, we turn to more specific examples of computations of $K(A)$, first with rational coefficients, and then with finite coefficients. Finally, we mention an extension of the Lichtenbaum-Quillen conjectures to this context, due to John Rognes.

Speaker: **Maria Basterra** (University of New Hampshire Durham)
Title: *Topological André-Quillen cohomology and related topics*

In his seminal work *Homotopical Algebra*, Quillen defined cohomology in a general model category in terms of the derived functors of *abelianization*. In this talk I will give an overview of my work with Mike Mandell investigating Quillen cohomology in the category of commutative S-algebras. We present several perspectives and some of the results obtained from the different points of view.

Speaker: **Andrew Blumberg** (Stanford University)
Title: *Localization for THH and TC of schemes*

Abstract: In this talk I will give a construction of the topological Hochschild homology and topological cyclic homology of a scheme in terms of the 'spectral derived category' of the scheme. I will discuss the application of this construction to show that there is a localization sequence associated to the inclusion of an open subscheme, naturally connected via the cyclotomic trace to the localization sequence of Thomason-Trobaugh in K-theory. This is joint work with Michael Mandell.

Speaker: **Ethan Devinatz** (University of Washington)
Title: *Towards the finiteness of the homotopy groups of the $K(n)$-local sphere*

Abstract: I will replace the notion of finiteness by a related notion of 'essentially finite rank' which is relevant for certain homotopy or homology groups of finite $K(n-2)$-acyclic spectra which are annihilated by p. Using the Devinatz-Hopkins continuous homotopy fixed point spectra, I will outline a program for proving that, if X is such a spectrum, then the homotopy groups of its $K(n)$-localization are of essentially finite rank, and I will indicate what progress I have made. One consequence of my work is the result that if z is an element of the p-Sylow subgroup of S_n and is non-torsion in the quotient of this group by its center, then the only units in $(E_n)_*$ fixed by z are the units in the Witt vectors.

Speaker: **Paul Goerss** (Northwestern University)
Title: *Realizing families of Landweber exact homology theories*

Abstract: In this overview, I would like to revisit and explore the following realization problem: given a continuous family of Landweber exact homology theories, when can it be lifted to a diagram of commutative ring spectra? This is not always possible, but work of Hopkins, Miller, Lurie, Behrens, and Lawson has given us a number of important examples where realization is possible, and I would like to meditate some on these in order to see what makes them work. Because of the relative simplicity and because it is easy to make explicit the role of p-divisible groups, I will emphasize the example of elliptic homology theories (i.e., topological modular forms) and the Hopkins-Miller realization result.

Speaker: **Kathryn Hess** (EPF Lausanne)
Title: *Hopf-Galois extensions in monoidal model categories*

Abstract: Rognes introduced the notion of a Hopf-Galois extension of structured ring spectra, of which the unit map from the sphere spectrum to MU is an important example. In this talk I will outline a theory of homotopic Hopf-Galois extensions in a monoidal category with compatible model category structure that generalizes the case of structured ring spectra. I will give examples of homotopic Hopf-Galois extensions in other categories. This is joint work with Cédric Bujard.

Speaker: **Mark Hovey** (Wesleyan University)
Title: *Semisimple ring spectra*

Abstract: In this talk, we present a definition and partial classification of semisimple ring spectra. This can be looked at as part of the general program of extending ring theory to structured ring spectra. It is also very closely related to the following purely algebraic question: for which graded rings R can the category of projective R-modules be triangulated, where the suspension is the shift? This is joint work with Keir Lockridge.

Speaker: **Rick Jardine** (University of Western Ontario)
Title: *Étale K-theory: a modern view*

Abstract: This lecture shows how to construct the étale K-theory of a scheme S and investigate its properties, with modern tools. There is a rather simply defined version of the K-theory presheaf of spectra K on the big site for S that is constructed with Waldhausen's techniques from big site vector bundles, and which is a presheaf of symmetric spectra. Let n be a number which is relatively prime to the residue characteristics of S: then the mod n étale K-theory presheaf of spectra is constructed from the presheaf of spectra K by taking a stably fibrant model FK/n for the presheaf K/n with respect to the etale topology. I shall display descent spectral sequences for mod n étale K-theory, including a spectral sequence of Galois descent type for the mod n étale K-theory of S which starts with Galois cohomology of the Grothendieck fundamental group of S, with coefficients in étale K-groups. We shall also discuss some standard comparisons with other flavours of K-theory.

Speaker: **Tyler Lawson** (University of Minnesota)
Title: *Topological Automorphic Forms*

Abstract: In this talk, I will discuss joint work with Mark Behrens on families of Landweber exact homology theories arising from moduli of higher-dimensional abelian varieties with extra structure. I will explain how these moduli give rise to p-divisible groups and a result of Jacob Lurie that allows realization of them. In particular, I will discuss the case of abelian surfaces with complex multiplication and how these can be related to the case of elliptic homology theories.

Speaker: **Peter May** (University of Chicago)
Title: *E_∞ ring spaces and E_∞ ring spectra*

Abstract: E_∞ ring spectra were defined in 1972, but the term has since acquired several alternative meanings. The same is true of several other terms. The new formulations and constructions are not always known to be equivalent to the old ones and even when they are, the notion of 'equivalence' needs discussion: Quillen equivalent categories can be quite seriously inequivalent. Part of the confusion stems from a gap in the modern resurgence of interest in E_∞ structures. E_∞ ring spaces were also defined in 1972 and and have never been redefined. They were central to the early applications and they tie in to modern applications. We give an informal open ended discussion of the relationships between the original notions and various new ones, explaining what is and what is not known.

Speaker: **Susan Montgomery** (University of Southern California)
Title: *Hopf Galois theory: I & II*

Abstract: The first talk will consist of definitions, examples, and a few basic results about Hopf Galois extensions. In the second lecture, I will try to discuss topics raised by the audience at the first talk.

Speaker: **John Rognes** (University of Oslo)
Title: *Topological logarithmic structures, I and II*

Abstract: A logarithmic structure on a commutative S-algebra B is a suitable map $M \to \Omega^\infty B$ of E_∞ spaces (with zero). It specifies a topological algebro-geometric object, intermediate between $\mathrm{Spec}(B)$ and $\mathrm{Spec}(B[M^{-1}])$. We report on work in progress on how to define logarithmic versions of TAQ and THH. In the case $B = HA$, where A is the valuation ring of a p-adic number field K and M is freely generated by the uniformizer, this agrees with the relative $THH(A|K)$ of Hesselholt and Madsen. In the case $B = ku$, the hope is that this framework provides a setting where the 'fraction field' of complex K-theory makes sense.

Speaker: **Steffen Sagave** (University of Oslo)
Title: *DG-algebras and derived A_∞-algebras*

Abstract: Let A be a differential graded algebra over a commutative ring k. We show that the homology of A admits a k-projective resolution E coming with a family of higher multiplications. This E is an instance of a 'minimal derived A_∞-algebra'. The main result is that minimal derived A_∞-algebras provide an alternative description of quasi-isomorphism types of dgas.

3 Mathematical outcomes

The Workshop provided a timely event at which a large number of the mathematicians interested in these topics were able to interact. In particular the following conclusions can be drawn from teh activities of the Workshop.

It is clear that the rapidly developing technical subject area of ∞-categories will have a major impact in the near future, with work of Lurie [12], Barwick, Gepner, Töen and Vezzosi being central.

The continuing development of Galois theoretic ideas within the framework structured ring spectra and also more general monoidal model categories, leads to the need to make greater use of existing algebraic Galois theoretic machinery.

Algebraic K-theory, with its links to Hochschild and cyclic homology will continue to provide a major stimulus, with versions for structured ring specra currently being the subject of a great deal of computational and conceptual activity. Topological logarithmic structures and suitable (co)homology theories are being developed and these provide further enrichment of the subject and interactions with existing algebraic and number theoretic areas.

The development of topological modular forms and the more recent topological automorphic forms, particularly as examples of the applicability of Lurie's methods in derived algebraic geometry, point to increasingly rich sources of interaction with algebraic geometry and interesting new objects to study in algebraic topology. One motivation for the latter is its connections (both established and speculative) with the dominant chromatic viewpoint on stable homotopy.

4 Outcome of the Meeting

The workshop ran at almost 100% capacity (we had a few very late withdrawals and managed to fill all but one place because of the great demand to attend). The programme was a mixture of invited talks and short sequences of talks, as well as talks chosen from those offered by participants.

It was notable that a great number of the speakers were young mathematicians and the standard of presentations was high. We have had many positive comments about the workshop from participants since it occurred. The facilities at BIRS provide a marvellous environment for a research meeting.

There will be a proceedings volume of the conference and we have encouraged all of the invited speakers to produce written versions of their talks. These papers should provide an up-to-date introduction and survey of the topics of the workshop and fill an existing gap in the literature. We have been promised a substantial number of articles which we expect to amount to more than 300 pages.

References

[1] V. Angeltveit, M. Hill, T. Lawson, Topological Hochschild homology of ℓ and ko, arXiv:0710.4368

[2] C. Ausoni, On the algebraic K-theory of the complex K-theory spectrum, preprint

[3] C. Ausoni and J. Rognes, Algebraic K-theory of topological K-theory, *Acta Math.* **188**, (2002), 1–39

[4] M. Basterra, André-Quillen cohomology of commutative S-algebras, *J. Pure Appl. Algebra* **144** (1999), 111–143.

[5] M. Behrens and T. Lawson, Topological automorphic forms, to appear in *Memoirs of the Amer. Math. Soc.*, arxiv math/0702719

[6] A. Blumberg, M. Mandell, The localization sequence for the algebraic K-theory of topological K-theory, to appear in Acta Mathematica, arxiv math/0606513.

[7] M. Bökstedt, Topological Hochschild homology, Preprint (1985), Bielefeld.

[8] A. D. Elmendorf, I. Kriz, M. A. Mandell, J. P. May, *Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole*, Mathematical Surveys and Monographs **47**, American Mathematical Society, Providence, RI (1997) xii+249 pp.

[9] M. J. Hopkins, Algebraic topology and modular forms. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, 291–317, Higher Ed. Press, Beijing, 2002.

[10] M. Hovey, B. Shipley, J. Smith, Symmetric spectra, *J. Amer. Math. Soc.* **13** (2000), 149–208.

[11] J. Lurie, A survey of elliptic cohomology, `http://www-math.mit.edu/ lurie/papers/survey.pdf`.

[12] J. Lurie, Higher Topos Theory, arxiv math.CT/0608040.

[13] J. P. May, E_∞ ring spaces and E_∞ ring spectra. With contributions by F. Quinn, N. Ray, and J. Tornehave, *Lecture Notes in Mathematics*, **577**, Springer-Verlag, Berlin-New York (1977), 268 pp.

[14] S. Montgomery, Hopf algebras and their actions on rings. *CBMS Regional Conference Series in Mathematics*, **82**, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, (1993), xiv+238 pp.

[15] J. Rognes, Galois extensions of structured ring spectra, *Memoirs of the Amer. Math. Soc.*, vol. 192, **898** (2008), 1–97.

[16] C. Schlichtkrull, Units of ring spectra and their traces in algebraic K-theory, *Geom. Topol.*, **8** (2004), 645–673.

[17] B. Toën and G. Vezzosi, Homotopical Algebraic Geometry I: Topos theory, *Adv. in Math.* **193** (2005), 257–372.

[18] B. Toën and G. Vezzosi, Homotopical Algebraic Geometry II: Geometric stacks and applications, *Memoirs of the Amer. Math. Soc.* vol. 193, **902** (2008), 224 pp.