Well partial orderings and their strength measured in terms of their maximal order types

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1 Summary of a talk given at the workshop on Computability, Reverse Mathematics and Combinatorics 2008

A *well partial order* (wpo) is a partial order $\langle X, \leq_X \rangle$ such that for all infinite sequences $\{x_i\}_{i=0}^{\infty}$ of elements in X there exist natural numbers i, j such that $i < j$ and $x_i \leq_X x_j$. There are lots of examples for wpo's known, for example well orders or wpo's resulting from Higman's Lemma and Kruskal's theorem. In the context of reverse mathematics it is of interest to know which subsystems of second order arithmetic are able to prove a given poset to be a wpo. A crucial invariant for this enterprise is the *maximal order type* of a wpo. According to a classical theorem of de Jongh and Parikh [2] there exists for any given wpo $\langle X, \leq_X \rangle$ a linear (or total) ordering $\leq^+$ on X such that $\leq_X \subseteq \leq^+$ and

$$otype(\leq^+) = o(X) := \sup\{otype(\leq') + 1 : \leq_X \subseteq \leq' \subset X \times X \text{ and } \leq' \text{ is total}\}.$$

If one puts $L_X(x) := \{y \in X : \neg x \leq_X y\}$ then one obtains the following formula to compute $o(X)$:

$$o(X) = \sup\{o(L_X(x) + 1 : x \in X\}.$$

If we have given two posets X_0 and X_1 we can define induced partial orders on the disjoint union $X_0 \oplus X_1$ and the cartesian product $X_0 \otimes X_1$ in the natural way. Moreover the set X^* of finite sequences of elements over X can be partially ordered using the natural pointwise ordering induced on subsequences (Higman ordering). With $\oplus$ and $\otimes$ we denote the (commutative) natural sum and the (commutative) natural product of ordinals. The following results are well known (by de Jongh and Parikh [2] or Schmidt [5]).

Theorem 1. *1. If X_0 and X_1 are wpo's then $X_0 \oplus X_1$ and $X_0 \otimes X_1$ are wpo's, $o(X_0 \oplus X_1) = o(X_0) \oplus o(X_1)$ and $o(X_0 \otimes X_1) = o(X_0) \otimes o(X_1)$.*

2. If X is a wpo then X^* is a wpo and

$$o(X^*) = \begin{cases} \omega^{\omega^{o(X)-1}} & \text{if } X \text{ is finite} \\ \omega^{\omega^{o(X)+1}} & \text{if } o(X) = \epsilon + n \text{ where } \epsilon \text{ is an epsilon number} \\ & \text{and } n \text{ is finite} \\ \omega^{\omega^{o(X)}} & \text{otherwise.} \end{cases}$$

To describe the ordinals resulting from Kruskal's theorem [3] we have to introduce a certain ordinal function (also known as *collapsing function*). Let Ω denote the first uncountable ordinal and $\varepsilon_{\Omega+1}$ the first epsilon number above Ω. First note that any ordinal $\alpha < \varepsilon_{\Omega+1}$ can be described uniquely in terms of its Cantor normal form:

$$\alpha = \Omega^{\alpha_1}\beta + \cdots + \Omega^{\alpha_n} \cdot \beta_n$$

where $\alpha_1 > \ldots > \alpha_n$ and $0 < \beta_1, \ldots, \beta_n < \Omega$. In this situation we define the countable subterms $K\alpha$ of α recursively via

$$K\alpha := K\alpha_1 \cup \ldots \cup K\alpha_n \cup \{\beta_1, \ldots, \beta_n\}$$

where $K0 := 0$. Let $AP = \{\omega^\delta : \delta \in ON\}$. We can then put

$$\vartheta\alpha := \min\{\beta \in AP : \beta \geq \max K\alpha \wedge \forall\gamma < \alpha(K\gamma < \beta \rightarrow \vartheta\gamma < \beta\}. \qquad (1)$$

One easily verifies $\vartheta < \Omega$ by induction on α using a cardinality argument. It is easy to verify that then $\varepsilon_0 = \vartheta\Omega$ and $\Gamma_0 = \vartheta\Omega^2$. (If we would have demanded that $\vartheta\alpha := \min\{\beta \geq \max K\alpha \wedge \forall\gamma < \alpha(K\gamma < \beta \rightarrow \vartheta\gamma < \beta\}$ then this would have changed the values of $\vartheta\alpha$ slightly but inessentially. This modified version would fit more nicely into the following general formula when relative small order types are involved.)

Let T^n the set of finite planar trees having outdegree exactly n (n-ary trees) and let $T^{<\omega}$ be the set of finite planar trees. These tree classes are wpo under homeomorphic embedding (which preserves g.l.b). The following result has been proved by Diana Schmidt [5] and for $n = 2$ by de Jongh and Parikh (unpublished).

Theorem 2. *1. $o(T^2) = \varepsilon_0$ for $n = 2$.*

2. $o(T^n) = \vartheta\Omega^n$ for $n \geq 3$.

3. $o(T^{<\omega}) = \vartheta\Omega^\omega$.

(The same results also hold for non planar trees.) Diana Schmidt obtained many more sharp results for labeled trees. For example if F is a countable wpo and $T(F)$ denotes the set of planar labeled trees with labels from F then $o(T(F)) = \vartheta(\Omega^\omega \cdot o(F))$. (For uncountable F one has to replace Ω by the successor cardinal of the cardinality of F in the definition of ϑ and the formula $o(T(F)) = \vartheta(\Omega^\omega \cdot o(F))$ then again is true.)

H. Friedman proved a long time ago (private communication) that the assertion $\forall F(F \text{ wpo } \Rightarrow T(F) \text{ wpo })$ yields ATR_0 over RCA_0. Now let $T(F)'$ be the set of F-labeled trees together with the Montalban embedding relation (which does not request preservation of g.l.b) In joint work with H. Friedman and A. Montalban we could show that $\forall F(F \text{ wpo } \Rightarrow T(F)' \text{ wpo }$ is equivalent to ATR_0 over RCA_0. (Two reasons for this result are that $o(F) < \Gamma_0$ implies $o(T(F)') < \Gamma_0$ and that Γ_0 is the proof theoretic ordinal of ATR_0.)

To investigate the maximal order types of the Friedman style (using a gap condition) Kruskal theorems one has to extend the domain of ϑ to intrinsically larger domains but this is rather easy. In a first step one defines a function $\vartheta_1 : \varepsilon_{\Omega_2+1} \to [\Omega, \Omega_2[$ in the same way as $\vartheta : \varepsilon_{\Omega+1} \to \Omega$ was defined previously. Here Ω_2 denotes the second uncountable ordinal and ε_{Ω_2+1} the next epsilon number above Ω_2. On the segment determined by $\vartheta_1\varepsilon_{\Omega_2+1}$ on can define the countable coefficient sets $K\alpha$ similarly as before using $K\vartheta_1\alpha = K\alpha$. Using this one can then define $\vartheta : \vartheta_1\varepsilon_{\Omega_2+1} \to \Omega$ by (1). This process can be iterated through the uncountable number classes to provide a function $\vartheta : \vartheta_1 \ldots \vartheta_n\varepsilon_{\Omega_{n+1}} \to \Omega$ giving an end-extension of the previously defined versions of ϑ. The limiting value of $\vartheta\vartheta_1 \ldots \vartheta_n\varepsilon_{\Omega_{n+1}}$ as $n \to \infty$ is known to be the ordinal related to the union of Friedman's assertions FKT_n which rely on an embeddability relation satisfying a gap condition. (Of course the iteration can be continued even further but this will not be needed here.)

Our long term goal is to classify the strengths of the assertions FKT_n for fixed n and moreover of variations thereof. For this purpose we recently developed (in a joint research project with Michael Rathjen) a very satisfying and general formula which predicts in all natural cases (at which we had a look at) good upper bounds for the maximal order type of a tree-based wpo under consideration.

To explain this formula informally let us consider a given explicit operator W which maps a wpo X to a wpo $W(X)$ so that the elements of $W(X)$ can be described as generalized terms in which the variables are replaced by constants for the elements of X. We assume that the ordering between elements of $W(X)$ is induced effectively by the ordering from X. (This resembles Feferman's notion of effective relative categoricity.) In concrete situations W may for example stand for an iterated application of basic constructions like disjoint union and cartesian product, the set of finite sequences construction, the multiset construction, or a tree constructor and the like. We assume that for W we have an explicit knowledge of $o(W(X))$ such that $o(W(X)) = o(W(o(X)))$ and such that this equality can be proved using an effective reification as in [4].

Using W we then build the set of W-constructor trees $T(W(Rec))$ as follows:

1. $\cdot \in T(W(Rec))$.
2. If (s_i) is a sequence of elements in $T(W(Rec))$ and $w((x_i))$ is a term from $W(X)$ then $\cdot(w((s_i))) \in T(W(Rec))$.

The embeddability relation $\trianglelefteq$ on $T(W(Rec))$ is defined recursively as follows:

1. $\cdot \trianglelefteq t$.

2. If $s \trianglelefteq t_i$ then $s \trianglelefteq \cdot(w((t_i)))$
3. If $w((s_i) \leq w'((t_j)$ mod W is induced recursively by $\trianglelefteq$ then $\cdot(w((s_i))) \trianglelefteq \cdot(w'((t_j)))$.

The general principle is now that

$$T(W(Rec)) \text{ is a wpo}$$

and

$$o(\langle T(W(Rec)), \trianglelefteq \rangle) \leq \vartheta o(W(\Omega)) \tag{2}$$

for $o(W(\Omega)) \in dom(\vartheta)$ with $o(W(\Omega)) \geq \Omega^3$. [Moreover the reverse inequality follows in many cases by direct inspection.] (For smaller values of $o(W(\Omega))$ one should use the slightly modified version of ϑ which does not necessarily enumerate additive principal numbers.) The formula (2) is true for several natural examples which appear as suborderings of Friedman's FKT^n. We believe that the formula will be the key property in finally analyzing Friedman's FKT^n and we have already obtained far reaching applications.

In general this formula can be proved along the following general outline. (This outline applies to all cases which we considered so far.)

Proof outline for (2). The inequality is proved by induction on $o(W(\Omega))$. Let $t = w((t_j)) \in T(W(Rec))$. We claim $o(L_{T(W(Rec))}(t)) < \vartheta o(W(\Omega))$ and may assume by induction hypothesis that

$$o(L_{T(W(Rec))}(t_j)) < \vartheta o(W(\Omega)).$$

If now $s \in L_{T(W(Rec))}(t)$ then there will be natural quasi-embedding putting s into a well partial order $W'(Rec, (t_i))$ such that

$$o(W'(\Omega, (t_i))) < o(W(\Omega))$$

and such that

$$K(o(\Omega, (t_i)))) \subseteq K(o(W(\Omega)) \cup \bigcup Kt_j).$$

This step uses the assumption that the maximal order type resulting from W can be computed by an effective reification a la [4] or [6]. Therefore the definition of ϑ can be used to show

$$\vartheta(o(W'(\Omega, (t_i))))) < \vartheta(o(W(\Omega)))).$$

By induction hypothesis

$$o(L_{T(W(Rec))}(t)) \leq o(T(W'(\Omega, (t_i)))) \leq \vartheta(T(W'(\Omega, (t_i))))$$

and we are done. □

This proof outline can be used to prove (rigorously) the main results of the Habilitationsschrift of Dianaschmidt in a short and uniform way, but there already have been lots of more applications (which exceed he realm of the usual Kruskal theorem).

Examples 1. *1. If $W(X) = X^*$ then $o(\langle T(W(Rec)), \trianglelefteq\rangle) = \vartheta o(\Omega^\omega)$ (since $\omega^{\omega^{\Omega+1}} = \Omega^\omega$).*

2. If $W(X) = \bigotimes_{i<n} X$ then $o(\langle T(W(Rec)), \trianglelefteq\rangle) = \vartheta o(\Omega^n)$ (since $\bigotimes_{i<n} \Omega = \Omega^n$).

3. If $W(X) = (X^*)^*$ then $o(\langle T(W(Rec)), \trianglelefteq\rangle) = \vartheta o(\Omega^{\Omega^{\Omega^\omega}})$.

Further examples arise from the *multiset construction.* Let $M(x)$ be the set of finite multisets over X ordered by $m \ll m' \iff (\forall x \in m \setminus m \cap m')(\exists y \in m' \setminus m \cap m')[x < y]$. Further let $B(X)$ be the set of binary trees labeled with elements from X ordered under homeomorphic embeddability.

Examples 2. *1. If $W(X) = M(X)$ then $o(\langle T(W(Rec)), \trianglelefteq\rangle) = \vartheta o(\Omega)$ (since $\omega^{\Omega+1} = \Omega$).*

2. If $W(X) = M(X \otimes X)$ then $o(\langle T(W(Rec)), \trianglelefteq\rangle) = \vartheta o(\Omega^\Omega)$ (since $\omega^{\Omega\otimes\Omega} = \Omega^\Omega$).

3. If $W(X) = B(X)$ then $o(\langle T(W(Rec)), \trianglelefteq\rangle) = \vartheta o(\varepsilon_{\Omega+1})$. (since $o(B(\Omega)) = \varepsilon_{\Omega+1}$.

Let $M'(X)$ be the set of finite multisets over X ordered by

$$m \leq^\diamond m' \iff (\exists f : m\backslash m\cap m' \hookrightarrow m'\backslash m\cap m')(\forall x \in m\backslash m\cap m')[x \leq f(x) \bmod X].$$

During the workshop the following result (which sharpens a bound provided by Aschenbrenner and Pong [1]) has been obtained (after a fruitful discussion with A. Montalban). If $o(X) = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \geq \alpha_1 \geq \ldots \geq \alpha_n$ then

$$ord(M'(X)) := \begin{cases} \omega^{\omega^{\alpha_1}+\cdots+\omega^{\alpha_n}} & \text{if } \alpha_1 \text{ is not an epsilon number,} \\ \omega^{\omega^{\alpha_1+1}+\cdots+\omega^{\alpha_n}} & \text{if } \alpha_1 \text{ is an epsilon number.} \end{cases}$$

This result and the formula (2) lead to the correct maximal order types for wpo's resulting from non planar trees.

The formula (2) is also reflected in the proof strenth of subsystems of second order arithmetic. To prove that $T(W(Rec))$ is a wpo one typically applies a minimal bad sequence argument. Roughly speaking this means that *Rec* can be considered as a wpo. Using a small amount of extra strength (extra comprehension or induction) one then can prove $W(Rec)$ is a wpo hence $T(W(Rec))$ is also a wpo. In the joint project with Michael Rathjen we calibrate precisely the proof-theoretic strength of various versions and extensions of Kruskal's theorem in terms of systems of second order arithmetic. Among other things the following results have been obtained so far:

Theorem 3. *1. $(\Pi^1_1 - CA)^- + \mathrm{ACA}_0 \nvdash T(B(Rec))$ is a wpo .*

2. $(\Pi^1_1 - CA)^- + \mathrm{ACA}_0 \vdash T((Rec)^{*\cdots *})$ is a wpo .

3. $(\Pi^1_1 - CA)^- + \mathrm{RCA}_0 \nvdash T((Rec)^*)$ is a wpo .

4. $(\Pi^1_1 - CA)^- + \mathrm{RCA}_0 \vdash T(\bigotimes_{i<n}(Rec))$ *is a wpo for every* $n \geq 1$.
5. $(\Pi^1_1 - CA)^- + \mathrm{RCA}_0 \nvdash T(M'(Rec))$ *is a wpo .*
6. $(\Pi^1_1 - CA)^- + \mathrm{RCA}_0 \vdash T(M(Rec))$ *is a wpo .*

Problems 1. 1. *Is it true that* $(\Pi^1_1 - CA)^- + \mathrm{RCA}^*_0 \nvdash WO(\varphi\omega 0)$*? (We already proved that* $(\Pi^1_1 - CA)^- + \mathrm{RCA}^*_0 \vdash WO(\alpha)$ *for all* $\alpha < \varphi\omega 0$*.)*

2. *If the answer for assertion 1 is no: Is it true that* $(\Pi^1_1 - CA)^- + (\Delta^0_0 - CA) \nvdash WO(\varphi\omega 0)$*? This problem is interesting since, if the answer would be no, then* $(\Pi^1_1 - CA)^-$*, which is commonly considered to be as the prototype of an impredicative comprehension, will have a predicative interpretation in a weak context.*

3. *How far does the general formula 2lead? Are there natural situations in which it fails?*

4. *Assume that* W *as before is a natural operatar mapping countable wpo's to countable wpo's and assume that* $o(W(\Omega) \geq \Omega^3$*. Does* $\mathrm{RCA}_0 + \forall X(WPO(X) \rightarrow WPO(T(W(X)))$ *have proof-theoretic ordinal* $\vartheta o(W(\Omega))$*?*

5. *Assume that* W *as before is a natural operatar mapping countable wpo's to countable wpo's and assume that* $o(W(\Omega) \geq \Omega^3$*. Is it always true that* $\mathrm{RCA}_0 \vdash WPO(T(W(Rec))) \leftrightarrow WO(\vartheta(o(W(\Omega))))$*?*

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