Nonstandard Arithmetic, Reverse Mathematics, and Recursive Comprehension

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First order reasoning about hyperintegers proves things about sets of integers. The advantage is that hyperintegers have more structure than sets of integers.

1. The languages $L_1, L_2, *L_1$

2. Basic Nonstandard Arithmetic (BNA)

3. The Standard Part Principle (STP)

4. Nonstandard Counterparts

5. Weak Koenig Lemma

6. Arithmetical Comprehension

7. Arithmetical Transfinite Recursion

8. $\Pi^1_1$-Comprehension

9. Recursive Comprehension
Reminder: The 5 basic theories in $L_2$

$$RCA_0 =\ I\Sigma^0_1 + \Sigma^0_1 \text{ Induction} + \Delta^0_1 \text{ Comprehension}.$$

$$WKL_0 = RCA_0 + \text{ the Weak Koenig Lemma}$$
(every infinite binary tree has an infinite branch).

$$ACA_0 = RCA_0 + \text{ Arithmetical Comprehension}$$
(for each $k$, every $\Sigma^0_k$ formula defines a set).

$$ATR_0 = RCA_0 + \Sigma^1_1 \text{ Separation}$$
(any two disjoint $\Sigma^1_1$ properties are separated by a set).

$$\Pi^1_1$-CA$_0 = RCA_0 + \Pi^1_1 \text{ Comprehension}$$
(any $\Pi^1_1$ property defines a set).
Some References


1. The languages $L_1, L_2, *L_1$

First Order Arithmetic $L_1$:
Sort $N$ with variables $m, n, q, r, \ldots$
Vocabulary $0, 1, +, -, \cdot, <$
(where $m - n = \max(m - n, 0)$)
Terms $s(\vec{n})$ of sort $N$

2nd Order Arithmetic $L_2$:
$L_1 \subseteq L_2$. Models: $(\mathcal{N}, \mathcal{P}), \mathcal{P} \subseteq \mathcal{P}(\mathcal{N})$
Sort $P$ with variables $X, Y, Z, \ldots$
Relation $\in$ of sort $N \times P$

Nonstandard Arithmetic $*L_1$:
$L_1 \subseteq *L_1$. Models $(\mathcal{N}, *\mathcal{N}), \mathcal{N} \subseteq *\mathcal{N}$
Sort $*N$ with variables $x, y, z, \ldots$
Vocabulary $0, 1, +, -, \cdot, <$ in both sorts
Terms $t(\vec{u})$ of sort $*N$
where $\vec{u}$ has variables of both sorts.

Combined language: $L_2 \cup *L_1$
Models: $(\mathcal{N}, \mathcal{P}, *\mathcal{N})$
2. Basic Nonstandard Arithmetic (BNA)

A weak theory in $^\ast L_1$
which says $\mathcal{N} \equiv \forall \ast \mathcal{N}$ and $\mathcal{N} \subset_{end} \ast \mathcal{N}$

Axioms of $I\Sigma_1$ in sort $N$:
Recursive rules for $0, 1, +, -, \cdot, <$,
$\Sigma^0_1$ Induction in $L_1$

$\forall$-Transfer: $\forall \vec{m} \varphi(\vec{m}) \leftrightarrow \forall \vec{x} \varphi(\vec{x})$,
$\forall \vec{m} \varphi(\vec{m})$ a universal sentence in $L_1$

Proper Initial Segment Axioms:

\[
\forall n \exists x (x = n)
\]

\[
\forall n \forall x [x < n \rightarrow \exists m x = m]
\]

\[
\exists y \forall n [n < y]
\]
3. The Standard Part Principle (STP)

The bridge between $*L_1$ and $L_2$.

STP is a sentence in $L_2 \cup *L_1$ meaning:
“Every hyperinteger codes a set, and every set is coded by a hyperinteger”

$(p_n|x)$ means “The $n$-th prime divides $x$”,

$st(x) = \varphi(\cdot, \vec{u})$ denotes $\forall m [(p_m|x) \iff \varphi(m, \vec{u})]$
“$x$ codes the class $\{m : \varphi(m, \vec{u})\}$”

$st(x) = X$ denotes $\forall m [(p_m|x) \iff m \in X]$
“$x$ codes $X$”, “$X$ is the standard part of $x$”

STP: $\forall x \exists X \ st(x) = X \land \forall X \exists x \ st(x) = X$
4. Nonstandard Counterparts

A theory $T'$ in $L_2 \cup \ast L_1$ is conservative over a theory $T$ in $L_2$ if every sentence of $L_2$ provable from $T'$ is provable from $T$.

$T'$ is a nonstandard counterpart of $T$ if $T'$ implies and is conservative over $T$.

We will give nonstandard counterparts of each of the five basic theories $\text{RCA}_0$, $\text{WKL}_0$, $\text{ACA}_0$, $\text{ATR}_0$, and $\Pi^1_1$-$\text{CA}_0$ of second order arithmetic.

Each of these counterparts will be of the form $U + \text{STP}$ where $U$ is a theory in $\ast L_1$. 
5. Weak Koenig Lemma

**Theorem.** The theory
\[ \ast WKL_0 = \text{BNA} + \text{Int-IND} + \Sigma^S_1-\text{IND} + \text{STP} \]
is a nonstandard counterpart of \( WKL_0 \).
So is \( \ast WKL_0 + \) Transfer for FO sentences.

\( \Delta^S_0 \) formula in \( *L_1 \): Built from
atomic formulas, connectives, and
bdd quantifiers \((\forall m < s(\vec{n}))\), \((\forall x < t(\vec{u}))\).
\( \Sigma^S_1 \) means \( \exists m \varphi \) where \( \varphi \) is \( \Delta^S_0 \). And so on.

**Internal Induction (Int-IND):**
\[ [\varphi(0, \vec{u}) \land \forall x [\varphi(x, \vec{u}) \rightarrow \varphi(x + 1, \vec{u})]] \rightarrow \forall x \varphi(x, \vec{u}) \]
where \( \varphi(x, \vec{u}) \) is \( \Delta^S_0 \).

**\( \Sigma^S_1 \) Induction (\( \Sigma^S_1 \)-IND):**
\[ [\psi(0, \vec{u}) \land \forall n [\psi(n, \vec{u}) \rightarrow \psi(n + 1, \vec{u})]] \rightarrow \forall n \psi(n, \vec{u}) \]
where \( \psi(n, \vec{u}) \) is \( \Sigma^S_1 \).

Transfer for FO sentences says \( \mathcal{N} \equiv \ast \mathcal{N} \).

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5. Weak Koenig Lemma (Continued)

Proof that $^*\text{WKL}_0$ implies $\text{WKL}_0$ uses the **Overspill Lemma**:
For each $\Delta^S_0$ formula $\varphi(x, \vec{u})$, 
$$\forall n \varphi(n, \vec{u}) \to \exists x [\varphi(x, \vec{u}) \land \forall m m < x]$$

Proof that $^*\text{WKL}_0$ is conservative over $\text{WKL}_0$ uses Tanaka’s result that
Every countable nonstandard model of $\text{WKL}_0$ has an isomorphic proper end extension.
6. Arithmetical Comprehension

**Theorem.** *The theory* \( {^*}\text{WKL}_0 + S\text{-ACA} \) *is a nonstandard counterpart of ACA}_0. *So is* \( {^*}\text{WKL}_0 + S\text{-ACA} + \text{FOT} \).

\( S\)-Arithmetical Comprehension (\( S\text{-ACA} \)):
(Each \( S\)-arithmetical class is coded by an \( x \))

\[
\exists x \text{ st}(x) = \varphi(\cdot, \vec{u})
\]

where \( \varphi(m, \vec{u}) \in \bigcup_k \Sigma^S_k \).

FOT is Transfer for all first order formulas (\( \mathcal{N} \prec {^*}\mathcal{N} \)).

Using results of Enayat, one can get even stronger nonstandard counterparts of ACA}_0.
7. Arithmetical Transfinite Recursion

**Theorem.** The theory $^{\ast}WKL_0 + \Sigma^*_1$-SEP is a nonstandard counterpart of $\text{ATR}_0$. So is $^{\ast}WKL_0 + \Sigma^*_1$-SEP + FOT.

$\Sigma^*_1$ formula: $\exists x \varphi(x, \vec{u})$ where $\varphi(m, \vec{u}) \in \bigcup_k \Sigma^S_k$.

$\Sigma^*_1$-Separation ($\Sigma^*_1$-SEP):
(Two disjoint $\Sigma^*_1$ classes can be separated by an $x$)

$\forall m[\psi(m, \vec{u}) \rightarrow \neg \theta(m, \vec{u})] \rightarrow$

$\exists x[\psi(\cdot, \vec{u}) \subseteq st(x) \land st(x) \subseteq \neg \theta(\cdot, \vec{u})]$

where $\psi(m, \vec{u}), \theta(m, \vec{u})$ are $\Sigma^*_1$. 
8. $\Pi_1^1$ Comprehension

**Theorem.** The theory $^*\text{WKL}_0 + \Pi_1^*-\text{CA}$ is a nonstandard counterpart to $\Pi_1^1\text{-CA}_0$. So is $^*\text{WKL}_0 + \Pi_1^*-\text{CA} + \text{FOT}$.

$\Pi_1^*$ formula: $\forall x \varphi(x, \vec{u})$ where $\varphi(m, \vec{u}) \in \bigcup_k \Sigma^S_k$.

$\Pi_1^*$ Comprehension ($\Pi_1^*-\text{CA}$):
(Each $\Pi_1^*$ class is coded by an $x$)

$$\exists x \, st(x) = \varphi(\cdot, \vec{u})$$

where $\varphi(m, \vec{u})$ is $\Pi_1^*$.

The 1984 paper of Henson, Kaufmann, and Keisler gave nonstandard counterparts of some theories which are stronger than $\Pi_1^1\text{-CA}$. 
9. Recursive Comprehension

The theory $^\ast\text{RCA}_0 = ^\ast\text{WKL}_0 - \text{STP} + \text{weak STP}$ is a nonstandard counterpart of RCA$_0$. Here is another, with full STP:

**Theorem.** The theory $^\ast\text{RCA}_0' = \text{BNA} + \text{Special } \Sigma^S_1\text{-IND} + \text{Special } \Delta^S_1\text{-CA} + \text{STP}$ is a nonstandard counterpart of RCA$_0$.

Special $\Delta^S_0$ formulas: Built from atomic formulas, connectives, $(s(\vec{n})|t(\vec{u}))$, bdd quantifiers $(\forall m < s(\vec{n}))$ of sort $N$. Special $\Sigma^S_1$ means $\exists m \varphi$, $\varphi$ special $\Delta^S_0$.

Special $\Delta^S_1$ Comprehension (Special $\Delta^S_1\text{-CA}$): (Each special $\Delta^S_1$ class is coded by an $x$)

$$\forall m[\varphi(m, \vec{u}) \leftrightarrow \neg \psi(m, \vec{u})] \rightarrow \exists x \text{ st}(x) = \varphi(\cdot, \vec{u})$$

where $\varphi(m, \vec{u}), \psi(m, \vec{u})$ are special $\Sigma^S_1$. 

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9. Recursive Comprehension (Continued)

But each of the following is a nonstandard counterpart of WKL₀:

* \( \text{RCA}_0' + \) Overspill Lemma

* \( \text{RCA}_0' + \) Internal Induction

* \( \text{RCA}_0' + \) Transfer for \( \Pi^0_1 \) sentences

* \( \text{RCA}_0' + \Delta^S_0 \) Comprehension

The analogue of \( \text{RCA}_0' \) with a symbol for each primitive recursive function.
9. Recursive Comprehension (Open Questions)

Is \( \text{BNA} + \Sigma^S_1\text{-IND} + \text{STP} \) conservative over \( \text{RCA}_0 \)?

Is \( ^*\text{RCA}_0 ' + \text{Transfer} \) for universal formulas (rather than sentences) conservative over \( \text{RCA}_0 \)?

Note: The above two theories do not imply the Weak Koenig Lemma.

Is the analogue of \( ^*\text{RCA}_0 ' \) with an added symbol for exponentiation conservative over \( \text{RCA}_0 \)?

What happens if one uses a different method of coding sets by hyperintegers?