The Survival Game

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- $\overline{v}_k = v_1, \ldots, v_k$

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- Otherwise Chooser wins when $|S_i| < t$.

Key Technical Result

Theorem (HK-Konjevod)

For all positive integers p, s, t, with $s \leq p$, Presenter has a winning strategy in the (p, s, t)-survival game.

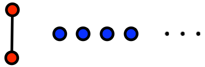
Example

Theorem (Grytczuk, Hałuszczak and HK)

For all positive integers p, t Presenter can win the $(p, 2, t)$-survival game.

Proof

$\exists x \exists y E(x, y)$



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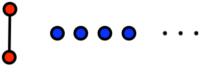

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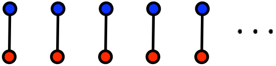

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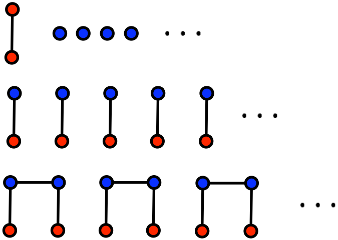

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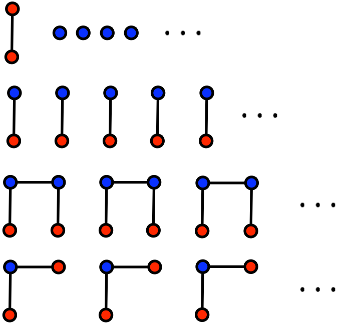

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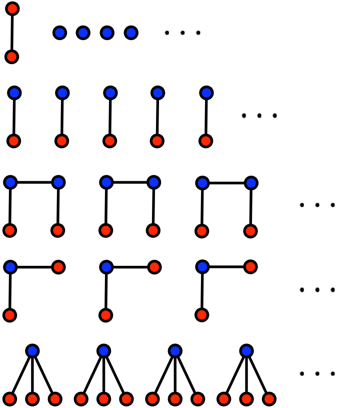

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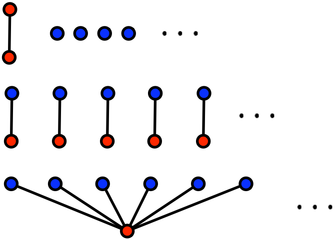

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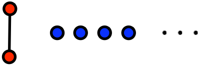

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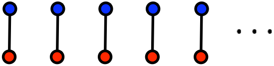

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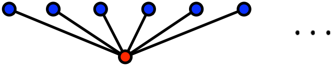

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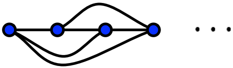

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- For notational convenience, let $\overline{v}_0 = \lambda = v_0$.

Definition of Satisfaction for Basic Formulas

Let H be a partitioned s-graph and $\overline{v}_h \subseteq U \cup W$.

- A basic formula has the form:

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  for some $w \in W$ with $w \gtrdot v_h$ $\;H \models \overline{Q\xi}E(\overline{v}_h, w, \overline{\xi})$.

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- A sentence φ is f-satisfiable if for any n, Presenter has a strategy starting from $f(n)$ vertices, so that some H_i contains a subgraph (V, E) that can be partitioned as $\{U, W\}$ so that $(U, W, E) \models \varphi$ and $|U| = n$.

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- Base Step: $\varphi_0 = \exists\xi_1 \ldots \exists\xi_s E(\bar{\xi})$ is f- satisfiable, where $f(n) = n + p$.

Induction Step

Lemma

If $\varphi = \forall \overline{\xi}_\ell \exists \xi_{\ell+1} \psi$ *is f-satisfiable then* $\varphi^+ = \exists \overline{\xi}_\ell \forall \xi_{\ell+1} \psi$ *is F-satisfiable, where F is defined recursively by*

$$F(0) = s$$
$$F(j+1) = f(F(j)),\ \text{if } j \geq 0.$$

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- (y_i) is strictly increasing, since $x_\ell \lessdot y_i$ in H_i and $y_{i+1} \in U_i$.

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- Let $\overline{x}_\ell \in U_n$. Then $H_i \models \exists\xi_{\ell+1}\overline{Q\xi}E(\overline{x}_\ell, \xi_{\ell+1}, \ldots, \xi_s)$.
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- (y_i) is strictly increasing, since $x_\ell \lessdot y_i$ in H_i and $y_{i+1} \in U_i$.
- Let H^+ be induced by

$$U^+ := \{y_i : i = 0, \ldots, n-1\} \text{ and } W^+ := \bigcup W_i - U^+.$$

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Substructure Lemma

Lemma

Suppose $H = (U, W, E)$ *and* $H' = (U', W', E')$ *are partitioned s-graphs and* $\overline{v}_h \subseteq (U \cup W) \cap (U' \cup W')$.

$$\textit{If } H \models \overline{Q\xi} E(\overline{v}_h, \overline{\xi}) \textit{ then } H' \models \overline{Q\xi} E(\overline{v}_h, \overline{\xi}),$$

provided the following conditions are all satisfied:

1. *If* $\overline{y}_s \in E$ *then* $\overline{y}_s \in E'$ *for all* $\overline{y}_s \subseteq (U \cup W) \cap (U' \cup W')$.
2. $U' - \{v : v \leq v_h\} \subseteq U$.
3. $W - \{v : v \leq v_h\} \subseteq W'$.

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Comment

The winning strategy for Presenter requires more than $A(2^s - 1, t)$ starting vertices, where A is the Ackermann function.

Main Theorem

Theorem (HK and Konjevod)

For all $c, s, t \in \mathbb{N}$, the on-line coloring Ramsey number satisfies the trivial lower bound

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Theorem (HK and Konjevod)

For all $c, s, t \in \mathbb{N}$ and on-line s-edge coloring algorithms A there exists a k-colorable s-graph G such that if A colors G with c colors then G contains a monochromatic K_s^t, where $k = \chi(K_s^t)$.