An interaction between reverse mathematics and computable analysis

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Outline

1. **Computable analysis**
2. **Reverse mathematics vs computable analysis**
3. **The WKL_0 level**
4. **The Hahn-Banach Theorem**

Computable analysis (Weihrauch's approach)

"Computable analysis uses the point of view of computability and complexity theory to study problems in the domain of analysis" (Brattka)

How can we compute with infinite objects?

Computation + Approximation

TTE machines

A TTE machine is a Turing machine with an input tape,
an output tape, and one or more working tapes.

The most important restriction is that the head of the output tape moves to the right after writing, and never moves left.

This means that we cannot correct the output:
once a digit is written it will not change
so that at each stage of the computation the partial output is reliable.

TTE provides a realistic model of computation.

Computable functions

$F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is computable if there exists a program such that when we start the computation with $p \in \mathrm{dom}(F)$ on the input tape the computation never terminates,
and in the end $F(p)$ is written on the output tape.

Every computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous.

Representations

A representation σ_X of a set X is a surjective $\sigma_X : \subseteq \mathbb{N}^{\mathbb{N}} \to X$.
The pair (X, σ_X) is a represented space.

If $x \in X$ a σ_X-name for x is any $p \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma_X(p) = x$.

representations in computable analysis = codings in reverse mathematics

Representing separable metric spaces

An effective metric space is a triple (X, d, a) where:

- (X, d) is a separable metric space;
- $a : \mathbb{N} \to X$ is a dense sequence in X.

When $d : \mathrm{ran}(a) \times \mathrm{ran}(a) \to \mathbb{R}$ is computable,
(X, d, a) is a computable metric space.

The Cauchy representation δ_X of the effective metric space (X, d, a):

- $p \in \mathrm{dom}(\delta_X)$ iff for all i and all $j \geq i$, $d(a(p(i)), a(p(j))) \leq 2^{-i}$;
- $\delta_X(p) = x$ if and only if $\lim a(p(n)) = x$.

Representing closed sets

For X an effective metric space $\mathcal{A}_+(X)$ and $\mathcal{A}_-(X)$ are the hyperspace of closed subsets of X with representations ψ_+^X and ψ_-^X:

- $\psi_+^X(p) = A$ if and only if $\forall i\, p_i \in \mathrm{dom}(\delta_X)$ and $A = \overline{\{\, \delta_X(p_i) \mid i \in \mathbb{N}\,\}}$;
- $\psi_-^X(p) = A$ if and only if $X \setminus A = \bigcup B_{p(i)}^X$ ($\{B_n^X\}$ enumerates all rational open balls in X).

$$\mathcal{A}_+(X) = \text{separably closed sets}$$
$$\mathcal{A}_-(X) = \text{closed sets}$$

Computable functions between represented spaces

If (X, σ_X) and (Y, σ_Y) are represented spaces and $f : \subseteq X \to Y$, we say that $F : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a (σ_X, σ_Y)-realizer of f when $\forall p \in \mathrm{dom}(f \circ \sigma_X)\, f(\sigma_X(p)) = \sigma_Y(F(p))$.

The function f is (σ_X, σ_Y)-computable if it has a computable (σ_X, σ_Y)-realizer.

If X and Y are effective metric spaces, every (δ_X, δ_Y)-computable function is continuous.

Multi-valued functions and computable reducibility

If (X, σ_X) and (Y, σ_Y) are represented spaces and $f :\subseteq X \rightrightarrows Y$ is a multi-valued function,
we say that $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a (σ_X, σ_Y)-realizer of f when $\forall p \in \operatorname{dom}(f \circ \sigma_X)\, \sigma_Y(F(p)) \in f(\sigma_X(p))$.

We do not require $\sigma_X(p) = \sigma_X(p') \implies \sigma_Y(F(p)) = \sigma_Y(F(p'))$.

The multi-valued function f is (σ_X, σ_Y)-computable if it has a computable (σ_X, σ_Y)-realizer.

Let (X, σ_X), (Y, σ_Y), (Z, σ_Z), (W, σ_W) be represented spaces.
Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be multi-valued functions.

$f \leqslant_c g$ if there exist computable $h :\subseteq X \rightrightarrows Z$ and $k :\subseteq X \times W \rightrightarrows Y$ such that $\forall x \in \operatorname{dom}(f)\, k(x, (g \circ h)(x)) \subseteq f(x)$.

Σ^0_2-completeness

Let $C_1 : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be defined by

$$C_1(p)(n) = \begin{cases} 0 & \text{if } \exists m\, p(\langle n, m\rangle) \neq 0; \\ 1 & \text{otherwise.} \end{cases}$$

If $f \leqslant_c C_1$ we say that f is $\mathbf{\Sigma}^0_2$-computable.
If $f \cong_c C_1$ we say that f is $\mathbf{\Sigma}^0_2$-complete.

Theorem (von Stein, Mylatz)

The differential operator $' : \subseteq C[0,1] \to C[0,1]$, $f \mapsto f'$, *is* $\mathbf{\Sigma}^0_2$*-complete.*

Theorem (Brattka/Gherardi)

The function $\mathcal{A}_+(X) \to \mathcal{A}_+(X)$, $A \mapsto \overline{X \setminus A}$, *is* $\mathbf{\Sigma}^0_2$*-complete for every computable metric space* X *which is complete and perfect.*
The function $\mathcal{A}_-(X) \to \mathcal{A}_+(X)$, $A \mapsto \overline{X \setminus A}$, *is computable for every computable metric space* X.

Between computable and Σ^0_2-complete

Let X be a computable separable Banach space, and define $\mathbf{HB}_X$ to be the multi-valued function mapping a closed linear subspace $A \subseteq X$ and a bounded linear functional $f : A \to R$ with $\|f\| = 1$ to the set of all bounded linear functionals $g : X \to R$ which extend f and are such that $\|g\| = 1$.

Theorem (Brattka)

For many computable separable Banach spaces X, $\mathbf{HB}_X$ is incomputable.
For every computable separable Banach space X, $\mathbf{HB}_X <_c C_1$.

How incomputable is the Hahn-Banach theorem?

From reverse mathematics to computable analysis

Many mathematical statements expressed in $\mathcal{L}_2$ have the form

$$\forall X (\psi(X) \implies \exists Y \, \varphi(X, Y)).$$

If this is true, we define the multi-valued function $f : \subseteq \mathcal{P}(\mathbb{N}) \rightrightarrows \mathcal{P}(\mathbb{N})$ such that $\mathrm{dom}(f) = \{ X \in \mathcal{P}(\mathbb{N}) \mid \psi(X) \}$ and $f(X) = \{ Y \mid \varphi(X, Y) \}$.

We can also unravel the coding used in $\mathcal{L}_2$,
so that the domain and the range of f are appropriate represented spaces.

f can be studied with the tools of computable analysis.

From computable analysis to reverse mathematics

Reversing the procedure, to study from the viewpoint of computable analysis a multi-valued function $f : \subseteq X \rightrightarrows Y$,
we can look at the reverse mathematics of the statement

$$\forall x (x \in \operatorname{dom}(f) \implies \exists y \in Y \, y \in f(x)).$$

If f is computable we expect a statement provable in RCA_0.

From C_1 we obtain a statement equivalent to ACA_0.

Successes of the correspondence

- Let $\mathbf{Range} : \subseteq \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the function that maps any one-to-one function to the characteristic function of its range. Then $\mathbf{Range} \cong_c C_1$.
- Let $\mathbf{Sup} : [0,1]^{\mathbb{N}} \to [0,1]$ be the function that maps any sequence in $[0,1]^{\mathbb{N}}$ to its least upper bound. Then $\mathbf{Sup} \cong_c C_1$.

Failures of the correspondence 1

Let $\mathbf{Sel} :\subseteq \mathcal{A}_-(2^{\mathbb{N}}) \rightrightarrows 2^{\mathbb{N}}$ be the multi-valued function which selects a point from nonempty closed subsets of $2^{\mathbb{N}}$: $\mathbf{Sel}(A) = A$,
where A is a closed set on the lhs and a set of points on the rhs.

The statement corresponding to $\mathbf{Sel}$ is

$$\forall A \in \mathcal{A}_-(2^{\mathbb{N}})(A \neq \emptyset \implies \exists x\, x \in A)$$

which is a tautology and hence provable in RCA_0.

On the other hand, $\mathbf{Sel}$ is incomputable (in fact $\mathbf{Sel} \cong_c \mathbf{Sep}$).

Failures of the correspondence 2

The Heine-Borel compactness of the interval $[0,1]$ is equivalent to WKL_0.

Theorem (Weihrauch)

The function which maps each open covering of $[0,1]$ consisting of intervals with rational endpoints to a finite subcovering is computable.

Proof.

There exists a computable enumeration $(\mathfrak{C}_n)$ of all finite open coverings of $[0,1]$ consisting of intervals with rational endpoints.
If (U_k) is an open covering of $[0,1]$ with intervals with rational endpoints, search for $j, n \in \mathbb{N}$ such that every interval in $\mathfrak{C}_n$ is U_k for some $k \leq j$.
Then $\{\, U_k \mid k \leq j \,\}$ is the desired finite subcovering. □

RCA_0 proves that $\{\mathfrak{C}_n\}$ exists and can define the algorithm, but fails to prove its termination.

Sep

Let $\mathbf{Sep} : \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \rightrightarrows 2^\mathbb{N}$ be defined by
$\mathrm{dom}(\mathbf{Sep}) = \{ (p, q) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \mid \forall n \, \forall m \, p(n) \neq q(m) \}$,

$$\mathbf{Sep}(p, q) = \left\{ r \in 2^\mathbb{N} \mid \forall n (r(p(n)) = 0 \land r(q(n)) = 1) \right\}.$$

Theorem

- $\mathbf{Sep}$ *is not computable;*
- $\mathbf{Sep} <_c C_1$.

Let (X, σ_X), (Y, σ_Y) be represented spaces and $f : \subseteq X \rightrightarrows Y$.
f is **Sep-computable** if $f \leqslant_c \mathbf{Sep}$.
f is **Sep-complete** if $f \cong_c \mathbf{Sep}$.

Iterating Sep-computable functions

Theorem

Let $f :\subseteq X \rightrightarrows Y$ *and* $g :\subseteq Y \rightrightarrows Z$ *be* $\mathbf{Sep}$*-computable multi-valued functions between represented spaces.*
Then $g \circ f :\subseteq X \rightrightarrows Z$ *is* $\mathbf{Sep}$*-computable.*

The Hahn-Banach multi-valued function

We define the notion of effective Banach space,
and the space $\mathcal{BAN}$ of all effective Banach spaces with its representation.

We introduce the space $\mathcal{PF}$ of partial linear bounded functionals on effective Banach spaces with its representation.
An element of $\mathcal{PF}$ is of the form $f_{(X,A,r)}$: a linear functional f defined on $A \in \mathcal{A}_+(X)$ where $X \in \mathcal{BAN}$ with $\|f\| = r$.

Let $\mathbf{HB} : \subseteq \mathcal{PF} \rightrightarrows \mathcal{PF}$ be the multi-valued function with $\mathrm{dom}(\mathbf{HB}) = \{f_{(X,A,1)} \in \mathcal{PF}\}$ defined by

$$\mathbf{HB}(f_{(X,A,1)}) = \left\{\, g_{(X,X,1)} \mid g \upharpoonright A = f \,\right\}.$$

This is the Hahn-Banach multi-valued function:
it is the global version of Brattka's $\mathbf{HB}_X$.

$\mathbf{HB}$ is $\mathbf{Sep}$-complete

Theorem

$\mathbf{HB}$ *is* $\mathbf{Sep}$*-complete.*

The proof of $\mathbf{Sep} \leqslant_c \mathbf{HB}$ uses the Bishop/Metakides/Nerode/Shore argument used also in reverse mathematics.

In the proof of $\mathbf{HB} \leqslant_c \mathbf{Sep}$ the only $\mathbf{Sep}$-computable function we use is a selector of the solution.