Non-oscillatory very high order Residual Distribution schemes for steady hyperbolic conservation laws : preliminary results

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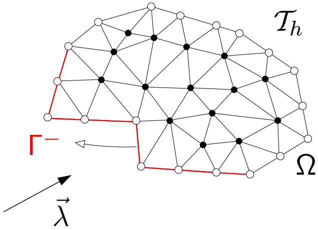
Outline

Framework for scalar $\mathcal{CL}$s

$$\begin{aligned} \nabla \cdot \mathcal{F}(u) &= 0 \qquad \text{in} \quad \Omega \\ u &= g \qquad \text{on} \quad \Gamma^- \\ \vec{\lambda}(u) &= \frac{\partial \mathcal{F}}{\partial u} \end{aligned} \qquad (1)$$



Some notation...

- Consider $\mathcal{T}_h$ triangulation of Ω (can do with quads...)
- Unknowns (Degrees of Freedom, DoF) : $u_i \approx u(M_i)$
- $M_i \in \mathcal{T}_h$ a given set of nodes (vertices +other dofs)
- Denote by u_h continuous piecewise polynomial interpolation (e.g. P^k Lagrange triangles) : $u_h = \sum_i \psi_i u_i$

Residual Distribution ($\mathcal{RD}$), up to 2nd order

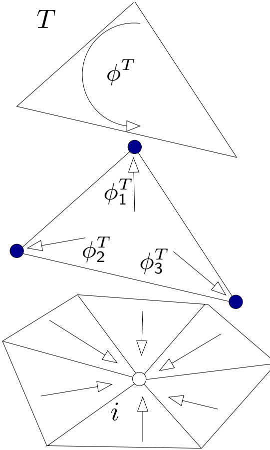
1. $\forall T \in \mathcal{T}_h$ compute : $\phi^T = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h(u_h)$

2. Distribution : $\phi^T = \sum_{i \in T} \phi_i^T$

   Distribution coeff.s : $\phi_i^T = \beta_i^T \phi^T$

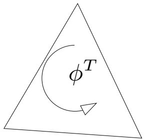
3. Compute nodal values : solve algebraic system

$$\sum_{T | i \in T} \phi_i^T = 0, \quad \forall\, i \in \mathcal{T}_h \qquad (2)$$



Principle for higher order

1. $\forall T \in \mathcal{T}_h$ compute : $\phi^T = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h(u_h)$



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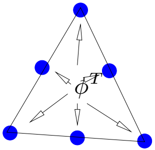
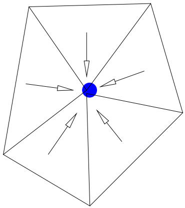


3. Compute nodal values : solve algebraic system

$$\sum_{T|i \in T} \phi_i^T = 0, \quad \forall\, i \in \mathcal{T}_h \qquad (3)$$



Design properties

Structural conditions, basic properties

Under which conditions on the ϕ_i^Ts we get

- Correct weak solutions (if convergent with h)
- Formal k^{th} order of accuracy
- Monotonicity (discrete max priciple)
- Convergence (with h, and with n !)

Condition 1 : conservation

Lax-Wendroff theorem (Abgrall & Barth, *SIAM J.Sci.Comp.* 24, 2002 ; Abgrall & Roe, *J.Sci.Comp.* 19, 2003)

(i)Technical assumptions, e.g. : continuity of ϕ_i^T, consistency of flux approximation ($\nabla \cdot \boldsymbol{\mathcal{F}}_h = 0$ and $\phi_i^T = 0$ if $u_h = c^t$).

(ii) If there is a $\boldsymbol{\mathcal{F}}_h$, continuous approximation of $\boldsymbol{\mathcal{F}}$ such that

$$\phi^T = \sum_{j \in T} \phi_j^T = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h \cdot \hat{n} \qquad (4)$$

then

If a bounded sequence u_h, solution of scheme (2), converges (with h) to $u \Longrightarrow u$ is a weak solution of the problem.

Condition 1 : conservation

Remark. Conservation : 2 underlying conditions

1. Existence of continuous flux approximation $\boldsymbol{\mathcal{F}}_h$ such that

$$\phi^T = \int_T \nabla \cdot \boldsymbol{\mathcal{F}}_h = \oint_{\partial T} \boldsymbol{\mathcal{F}}_h \cdot \hat{n}$$

for example $\boldsymbol{\mathcal{F}}_h = \boldsymbol{\mathcal{F}}(u_h)$, but also $\boldsymbol{\mathcal{F}}_h = \sum_i \psi_i \boldsymbol{\mathcal{F}}_i$!!

2. "Consistency" relation

$$\sum_{j \in T} \phi_j^T = \phi^T$$

Condition 2 : accuracy

Truncation error analysis (Abgrall, *J.Comp.Phys* 167, 2001 ; Ricchiuto *et al.*, *J.Comp.Phys* 222, 2007)

Error estimates built on variational formulation and stability analysis (coercivity) not available.

1. Given w_h discrete interpolation of nodal values of smooth exact solution w ;
2. Given φ a $C_0^1(\Omega)$ class function, and φ_h the discrete interpolation of $\{\varphi_i\}_{i\in\mathcal{T}_h}$, the nodal values of φ ;

Truncation error

$$\mathcal{E}(w_h) := \sum_{i\in\mathcal{T}_h} \varphi_i \Big(\sum_{T|\, i\in T} \phi_i^T(w_h) \Big)$$

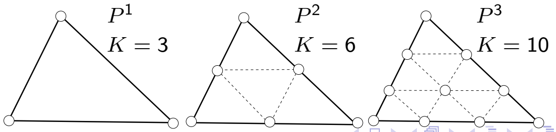
Condition 2 : accuracy

Guiding principle

Under which condition the $\mathcal{RD}$ scheme equivalent to the Galerkin scheme plus terms introducing and error (formally) within the one of the Galerkin approx.

$$\mathcal{E}(w_h) = \overbrace{\int_\Omega \varphi_h \nabla \cdot \boldsymbol{\mathcal{F}}_h}^{I \equiv \mathcal{E}^{\text{Galerkin}}} + \overbrace{\frac{1}{K} \sum_{T \in \mathcal{T}_h} \sum_{i,j \in T} (\varphi_i - \varphi_j)(\phi_i^T - \phi_i^{\text{Gal}})}^{II}$$

with ϕ_i^{Gal} elemental contribution of the standard (continuous) Galerkin discretization, and K the number of DoF per element.



Condition 2 : accuracy

- Final result

  If the (continuous) spatial approximations are $k+1^{\text{th}}$ order accurate (e.g. P^k Lagrange approximation), then one has the global estimate

$$|\mathcal{E}(w_h)| \leq C'(\mathcal{T}_h, w)\|\nabla\varphi\|_\infty \, h^{k+1}$$

provided that (in 2D) $\forall i \in T$ and $\forall T \in \mathcal{T}_h$

$$|\phi_i^T(w_h)| \leq C''(\mathcal{T}_h, w)h^{k+2} = \mathcal{O}(h^{k+2})$$

Condition 2 : accuracy

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Condition 2 : accuracy

Linearity (Accuracy) preserving schemes

The condition $\phi_i^T(w_h) = \mathcal{O}(h^{k+2})$ gives a design criterion. In particular, one ca show that for a regular solution

$$\phi^T(w_h) = \int_T \nabla\cdot\boldsymbol{\mathcal{F}}_h(w_h) \overbrace{=}^{\nabla\cdot\boldsymbol{\mathcal{F}}(w)=0} \int_T \nabla\cdot(F_h(w_h)-F(w)) =$$

$$\oint_{\partial T} (\boldsymbol{\mathcal{F}}_h(w_h) - \boldsymbol{\mathcal{F}}(w))\cdot\hat{n} = \mathcal{O}(\boldsymbol{\mathcal{F}}_h(w_h)-\boldsymbol{\mathcal{F}}(w)) \times \mathcal{O}(|\partial T|)$$

$$\overbrace{=}^{k+1^{\text{th}}\ \text{order approx.}} \mathcal{O}(h^{k+1})\times\mathcal{O}(h) = \mathcal{O}(h^{k+2})$$

Condition 2 : accuracy

Linearity (Accuracy) preserving schemes

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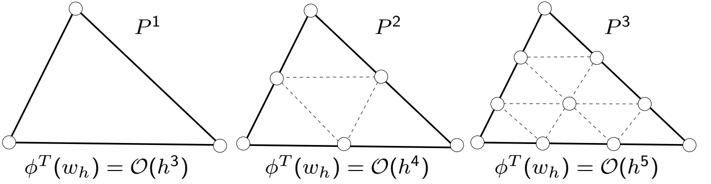
schemes for which

$$\phi_i^T = \beta_i^T \phi^T$$

with β_i^T uniformly bounded distribution coeff.s, are formally $k+1^{\text{th}}$ order accurate (for $k+1^{\text{th}}$ order spatial interpolation)



Condition 3 : monotonicity

Scalar advection and positivity theory

$$\vec{\lambda} \cdot \nabla u = 0, \qquad \vec{\lambda} = \text{const}$$

Construct schemes for which

$$\phi_i^T = \sum_{\substack{j \in T \\ j \neq i}} c_{ij}(u_i - u_j), \quad c_{ij} \geq 0$$

Theory of positive coefficient schemes ⇒ discrete max principle

(Spekreijse, *Math.Comp.* 49, 1987)

$$u_i^{n+1} = u_i^n - \omega_i \sum_{T|\, i \in T} \sum_{\substack{j \in T \\ j \neq i}} c_{ij}(u_i^n - u_j^n) \overset{c_{ij} \geq 0}{\underset{\omega_i \leq \omega_i^{\max}}{\Longrightarrow}} \min_j u_j^n \leq u_i^{n+1} \leq \max_j u_j^n$$

Examples of positive schemes

Positive schemes : the Rusanov scheme (Local Lax Friedrichs)

Centered linear first order distribution :

$$\phi_i^{\text{Rv}} = \frac{1}{K}\phi^T + \frac{\alpha}{K}\sum_{\substack{j\in T\\ j\neq i}}(u_i - u_j), \quad \alpha \geq \max_{j\in T}\left|\int_T \vec{\lambda}\cdot\nabla\psi_j\right|$$

- K number of DoF per element
- ψ_j Lagrange basis fcn. relative to node j
- The Rv scheme is cheap and has general formulation
- The Rv scheme is positive (energy stable in the P^1 case)

Nonlinear higher order schemes

Generalizations of the PSI of Struijs (Struijs, *PhD*, Delft U., 1994 ; Deconinck *et al.*, *Comp.Mech.* 11, 1993)

1. Starting point: a positive 1st order scheme (ϕ_i^{P})
2. Devise stretegy to construct a splitting (ϕ_i^*) such that $\phi_i^* = \alpha_i \phi_i^{\mathsf{P}},\ \alpha_i \geq 0$ and $\phi_i^* = \beta_i^* \phi^T$ with β_i^* bounded
3. The two conditions lead to the following construction.
   (a) If $\phi^T = 0$, set $\phi_i^* = 0\ \forall i \in T$
   (b) Otherwise, compute $\beta_i^p = \phi_i^p / \phi^T \ \ \forall i \in T$ and map them onto bounded coefficients verifying

$$\beta_i^* \beta_i^p \geq 0 \text{ (equivalent to } \alpha_i \geq 0\text{)} \quad \text{and} \quad \sum_{j \in T} \beta_j^* = 1$$

4. Mapping ?

Nonlinear higher order schemes

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4. For example take

$$\beta_i^* = \frac{\max(0, \beta_i^p)}{\sum_{j \in T} \max(0, \beta_j^p)}$$

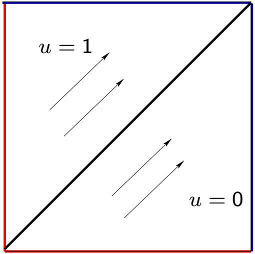
Limited Rv (LRv) scheme

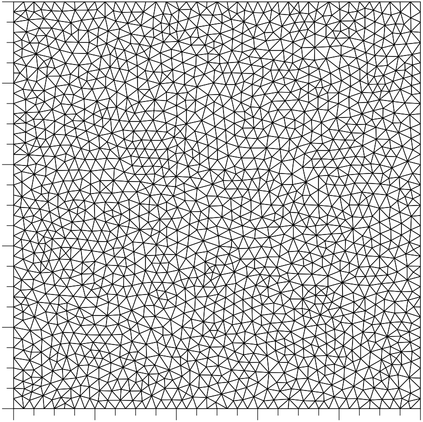
Summarizing

1. $\forall T \in \mathcal{T}_h$:
   (a) Compute ϕ^T (for ex. use P^k interpolation for flux)
   (b) Compute Rv distribution ϕ_i^{Rv}, $\forall i \in T$
   (c) Compute Rv distribution coeff.s and map them $\Rightarrow \phi_i^{\mathrm{Rv}*} = \beta_i^{\mathrm{Rv}*}\phi^T$, $\forall i \in T$
2. Evolve nodal values : $u_i^{n+1} = u_i^n - \omega_i \sum_{T|i\in T} \phi_i^{\mathrm{Rv}*}$

Apply the mapping to the Rv scheme$\Rightarrow$ Limited Rv scheme

Numerical example : advection





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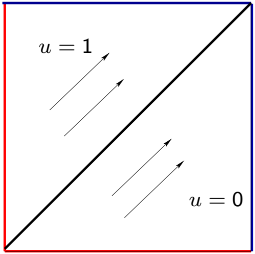
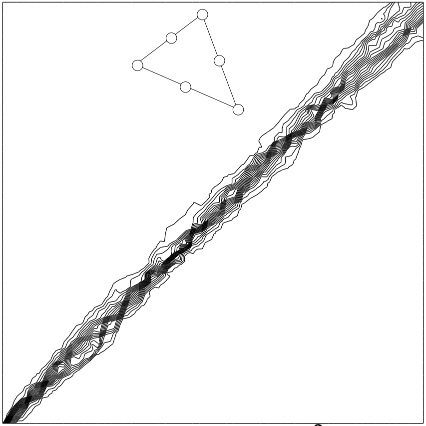
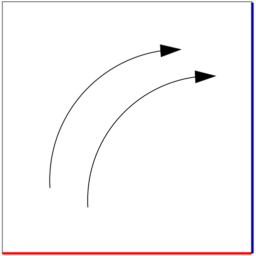




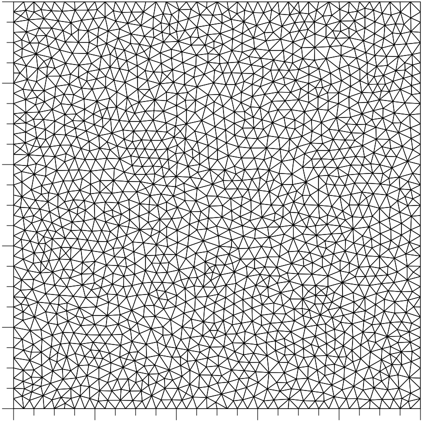
LRv scheme, P^2 interpolation

seems to work fine, no oscillation

Numerical example : rotation



$u_{\text{inlet}} = \cos^2(2\pi x)$
$0.25 \leq x \leq 0.75$



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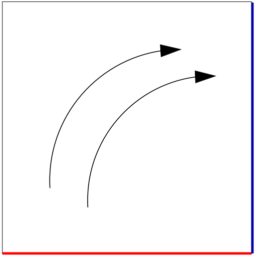

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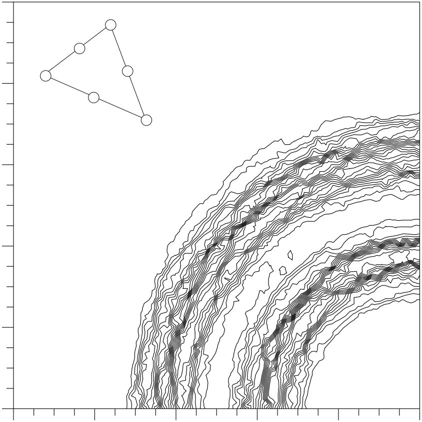
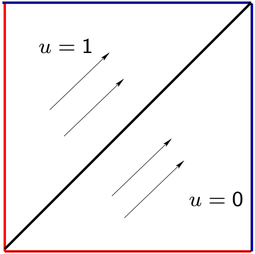
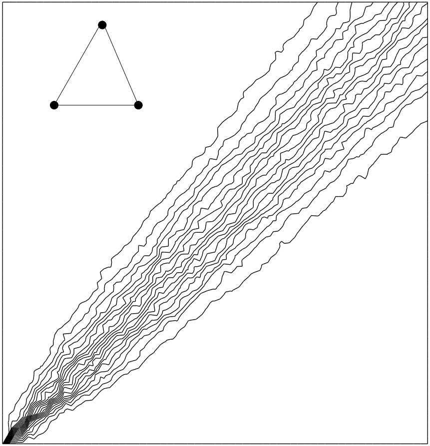


LRv scheme, P^2 interpolation

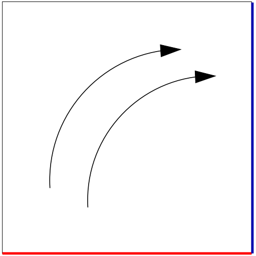
Numerical example : advection





LRv scheme, P^1 interpolation

Numerical example : rotation



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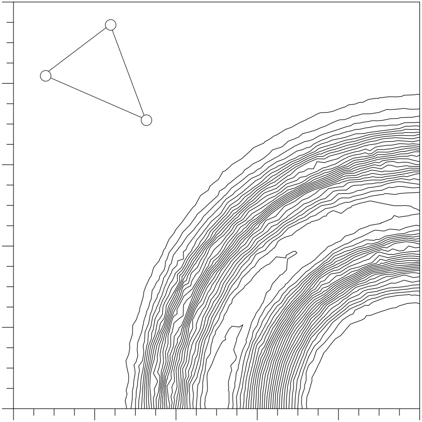
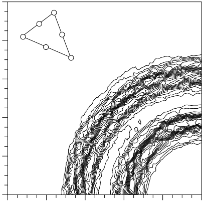


LRv scheme, P^1 interpolation

Upwinding & energy stability issues



Symptoms

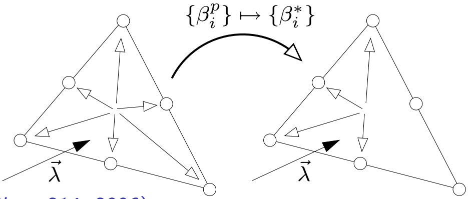
Smooth sol.s Lack of smoothness, staircase structure ;

Contacts (linear) : Monotone capturing. Spread over several cells, and then same as smooth parts ;

Shocks (nonlinear) : Monotone capturing. Kept in 1 or 2 cells, no staircases ;

Convergence Lack of iterative convergence (smooth sol.s)
$\Rightarrow$ Poor grid convergence (1^{st} order at most)

Upwinding & energy stability issues



Analysis (Abgrall, *J.Comp.Phys.* 214, 2006)

1. Positivity preserving mapping + central schemes : likely to get downwind discretizations, hence lack of stability (and consequently spurious modes, lack of convergence, etc..)
2. Possible cure : add upwind biasing/energy stabilizing term

$$\phi_i^{*S} = \beta_i^* \phi^T + \theta(u_h, \mathcal{T}_h, \vec{\lambda})\, h \int_T \vec{\lambda}\cdot\nabla\psi_i\, \vec{\lambda}\cdot\nabla u_h$$

with ψ_i Lagrange basis fcn. of node i

Upwinding & energy stability issues

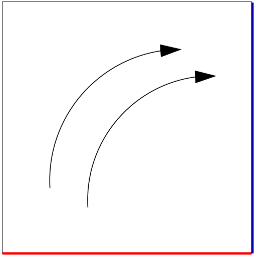
$$\phi_i^{*S} = \beta_i^* \phi^T + \theta(u_h, \mathcal{T}_h, \vec{\lambda}) \int_T \vec{\lambda} \cdot \nabla \psi_i \, \vec{\lambda} \cdot \nabla u_h$$

Requirements on $\theta(u_h, \mathcal{T}_h, \vec{\lambda})$

1. Correct scaling w.r.t. $\vec{\lambda}$ and mesh size : $\theta \propto h/\|\vec{\lambda}\|$;
2. Smooth sol.s : provide sufficient dissipation. Exact estimates can be derived asking the final discretization to be coercive. In practice, $\theta = h/\|\vec{\lambda}\|$ is more than enough ;
3. Discontinuous sol.s : since the basic scheme is positive, and no staircase effect is observed in discontinuities, we ask $\theta \propto h^2/\|\vec{\lambda}\|$ on discontinuous sol.s ;

$$\theta(u_h, \mathcal{T}_h, \vec{\lambda}) = \min\left(1, \frac{\|\vec{\lambda}\|_T \|u_h\|_T h^2}{|\phi^T|}\right) \frac{1}{\sum_{j \in T} |\vec{\lambda} \cdot \nabla \psi_j|_{P^1}}$$

Numerical example : rotation



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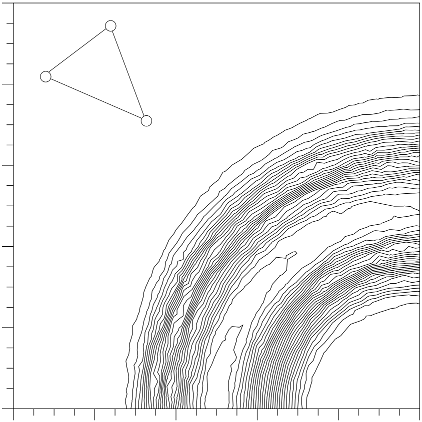
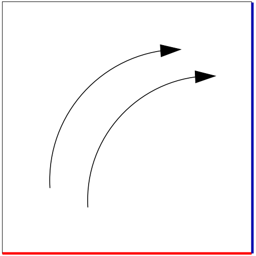


LRv scheme, P^1 interpolation

Numerical example : rotation



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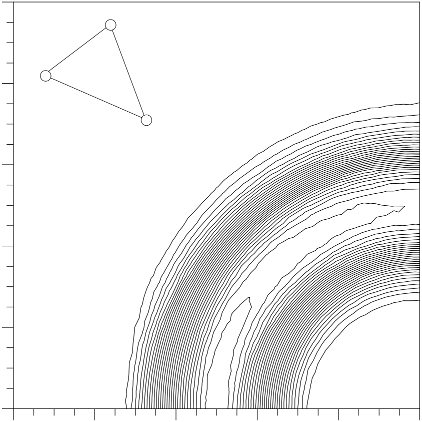
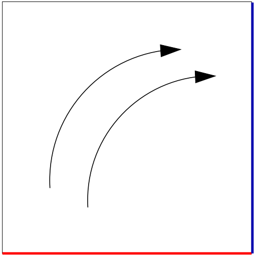


LRvS scheme, P^1 interpolation

Numerical example : rotation



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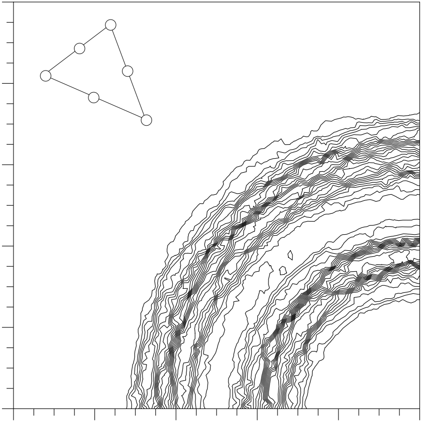
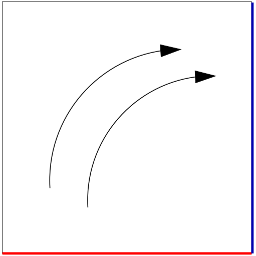


LRv scheme, P^2 interpolation

Numerical example : rotation



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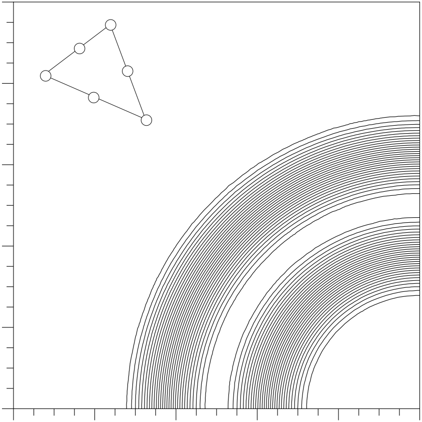
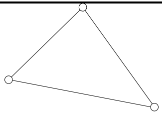
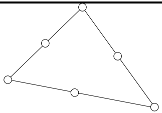
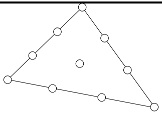
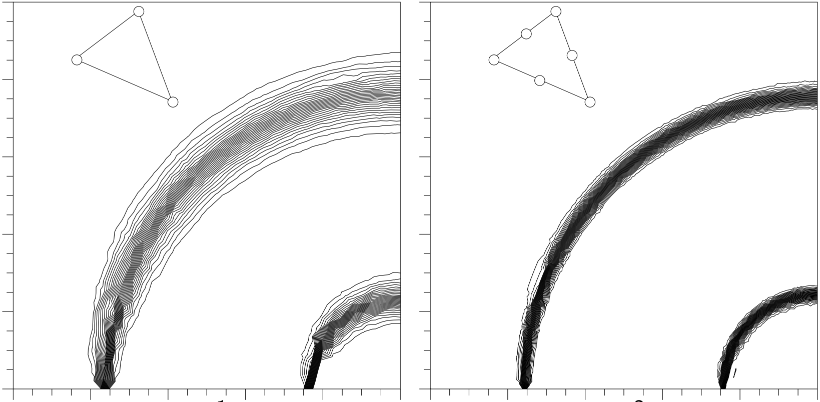


LRvS scheme, P^2 interpolation

Grid convergence

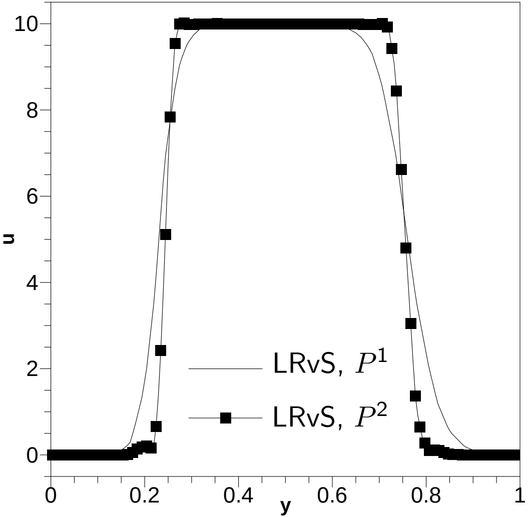
			
h	$\epsilon_{L^2}(P^1)$	$\epsilon_{L^2}(P^2)$	$\epsilon_{L^2}(P^3)$
$1/25$	0.50493E-02	0.32612E-04	0.12071E-05
$1/50$	0.14684E-02	0.48741E-05	0.90642E-07
$1/75$	0.74684E-03	0.13334E-05	0.16245E-07
$1/100$	0.41019E-03	0.66019E-06	0.53860E-08
	$\mathcal{O}^{\text{ls}}_{L^2}$ =**1.790**	$\mathcal{O}^{\text{ls}}_{L^2}$ =**2.848**	$\mathcal{O}^{\text{ls}}_{L^2}$ =**3.920**

Rotation of a top hat


LRvS scheme, P^1 interpolation | LRvS scheme, P^2 interpolation

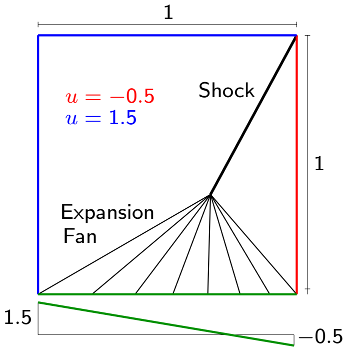
Contact in spread on same numer of DoF (fewer cells in P^2 case)

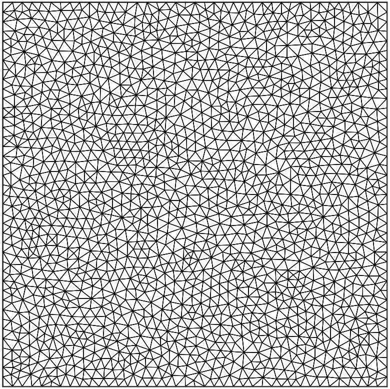
Rotation of a top hat : outlet profile



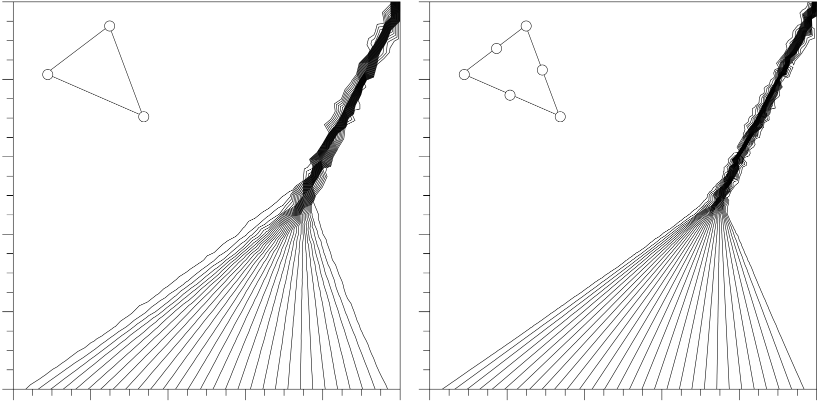
Numerical example : Burger's eq.n

$$\nabla \cdot \left(\frac{u^2}{2}, u\right) = 0$$





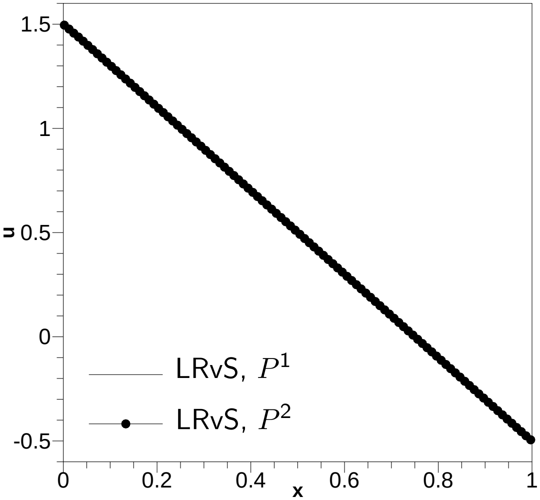
Numerical example : Burger's eq.n



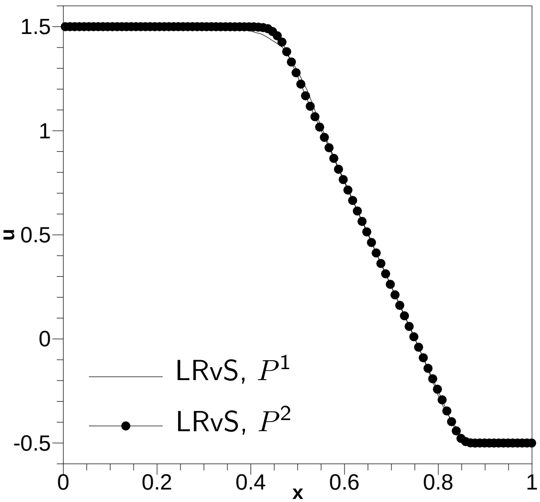
LRvS scheme, P^1 interpolation   LRvS scheme, P^2 interpolation

Shock captured in 1 or 2 cells (more DoF in P^2 case)

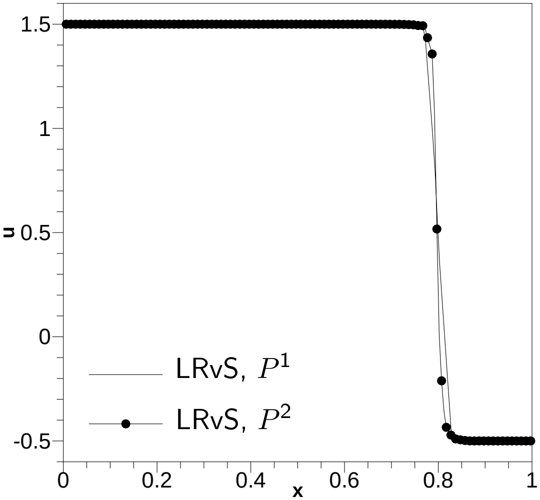
Burger's eq.n : cut at $y = 0.0$



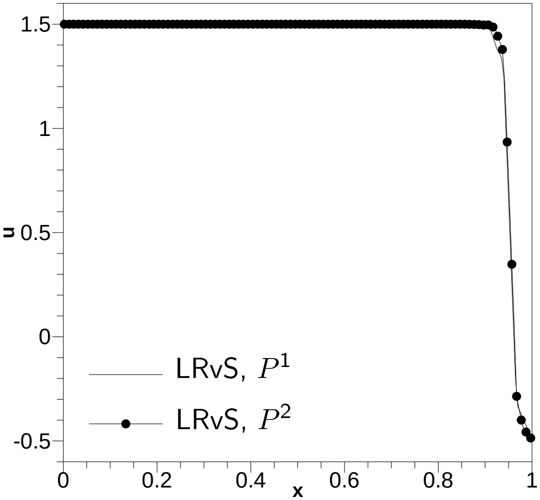
Burger's eq.n : cut at $y = 0.3$



Burger's eq.n : cut at $y = 0.6$



Burger's eq.n : cut at $y = 0.9$



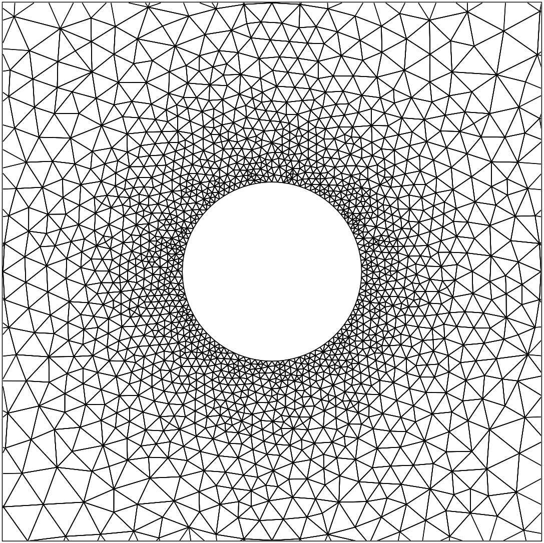
Extension to systems

$$\nabla \cdot \boldsymbol{\mathcal{F}}(\mathbf{u}) = 0$$

- Schemes formally identical to scalar case
- Nonlinear mapping on scalar residuals obtained by locally projecting on Eigenvector basis
- Stabilization : same as in the scalar case with matrix notation
- Solution monitor $\theta(\mathbf{u}_h)$ computed using energy or entropy component of cell residual

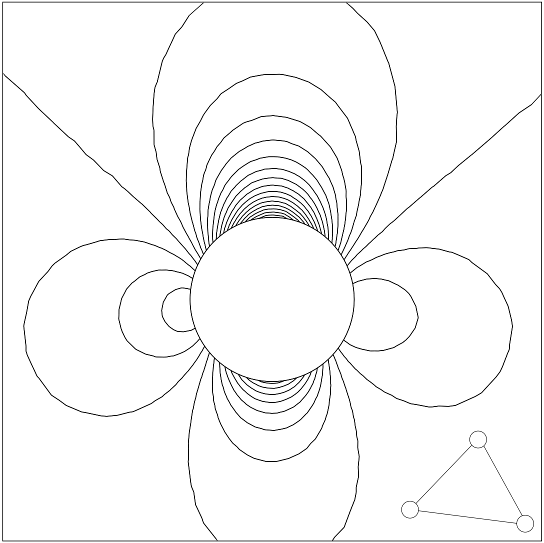
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
Mesh :
2719 nodes
5308 elements
100 nodes
on cylinder



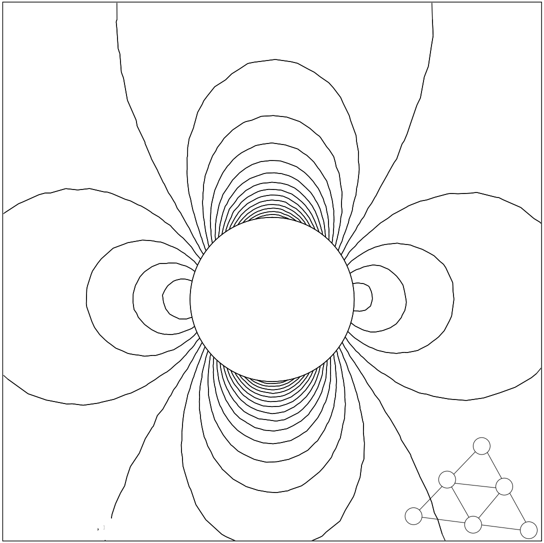
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LRvS scheme
P^1 elements :
pressure



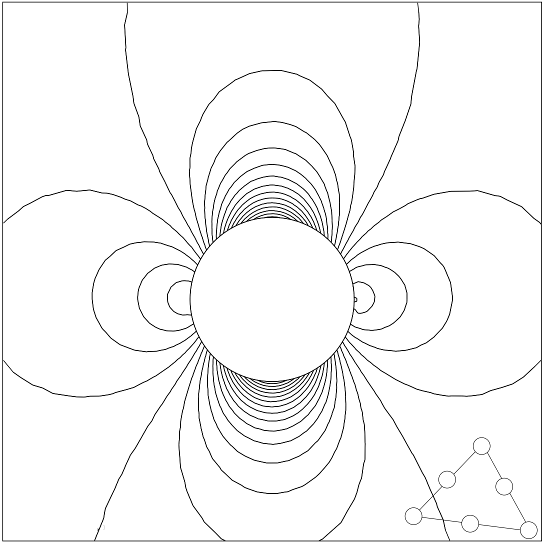
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LRvS scheme
P^1 elements
P^2 conformal
sub-triangulation :
pressure



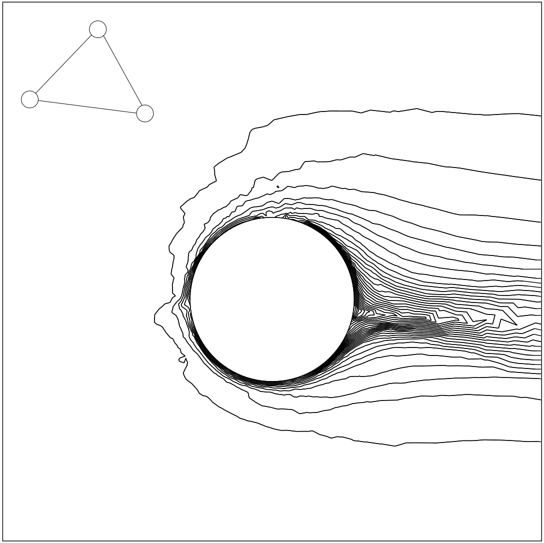
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LRvS scheme
P^2 elements :
pressure
(linear
boundary
representation)



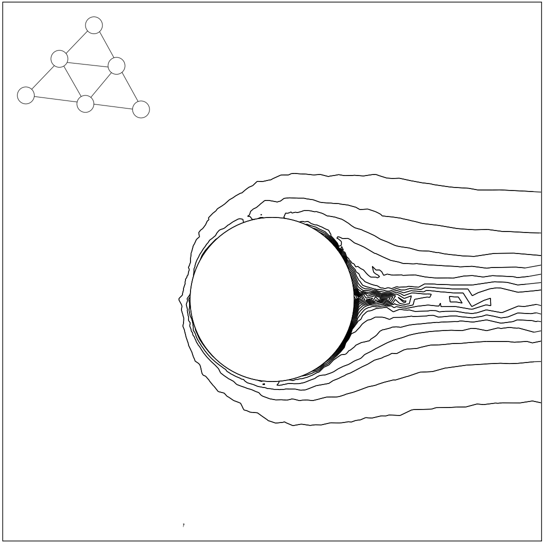
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LRvS scheme
P^1 elements :
entropy



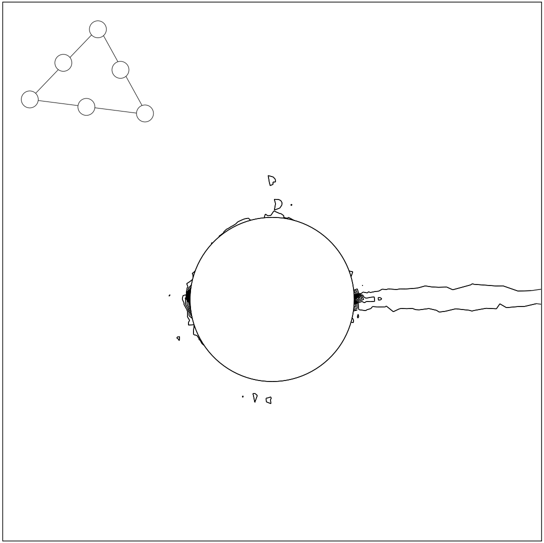
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LRvS scheme
P^1 elements
P^2 conformal
sub-triangulation :
entropy



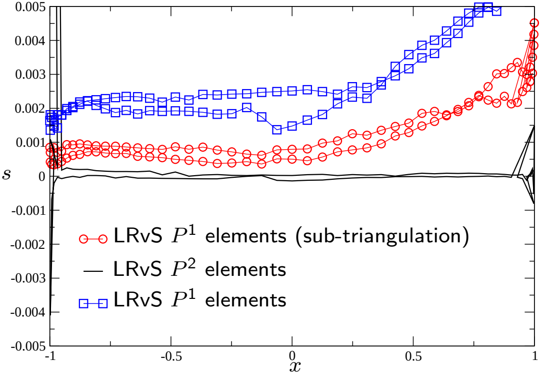
Euler eq.s : $Ma = 0.35$ cylinder flow

$Ma = 0.35$
flow on cylinder
LRvS scheme
P^2 elements :
entropy
(linear
boundary
representation)



$Ma = 0.35$ cylinder flow : entropy distribution

$Ma = 0.35$
flow on cylinder
LRvS scheme :
entropy
on the cylinder



Euler eq.s : self-similar sol.s

Consider families of self-similar unsteady solutions of the type

$$\mathbf{u}(t,x,y) = \mathbf{u}(\eta,\xi) \quad \text{with} \quad \eta = \frac{x}{t},\ \xi = \frac{x}{t}$$

The time dependent problem can be easily recast as :

$$\mathbf{u}_t + \nabla \cdot \boldsymbol{\mathcal{F}}(\mathbf{u}) = -\eta \mathbf{u}_\eta - \xi \mathbf{u}_\xi + \nabla_{\eta\xi} \cdot \boldsymbol{\mathcal{F}}(\mathbf{u})$$

For a fixed final time $t = t^*$, we solve the steady problem

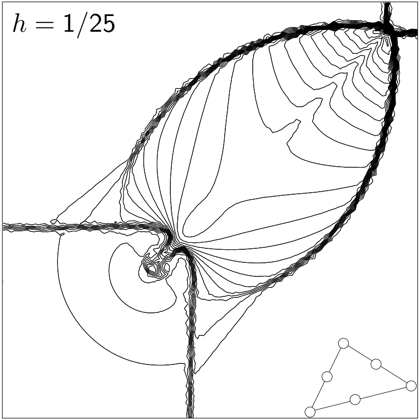
$$-x\mathbf{u}_x - y\mathbf{u}_y + \nabla \cdot \boldsymbol{\mathcal{F}}(\mathbf{u}) = 0$$

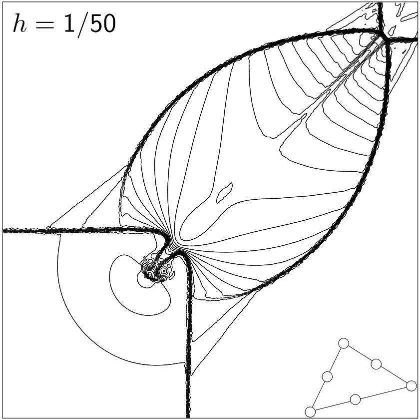
Extra terms included both in the element residual and in the stabilization term

$t = t^* \Longrightarrow$ linear scaling of computational domain

Euler eq.s : 2D RP

From (Kurganov & Tadmor, *Num.Meth. for Part.Diff.Eq.* 18, 2002)
2D Riemann Problem, configuration 12

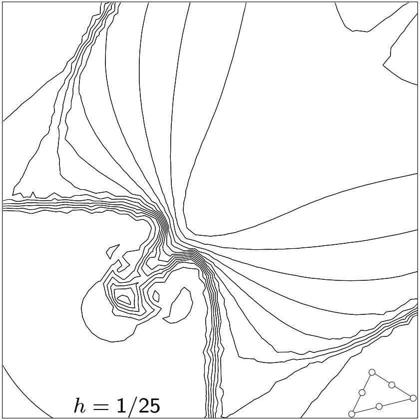


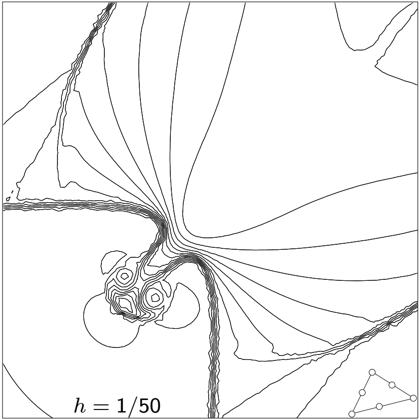


Density contours
LRvS scheme, P^2 elements

Euler eq.s : 2D RP

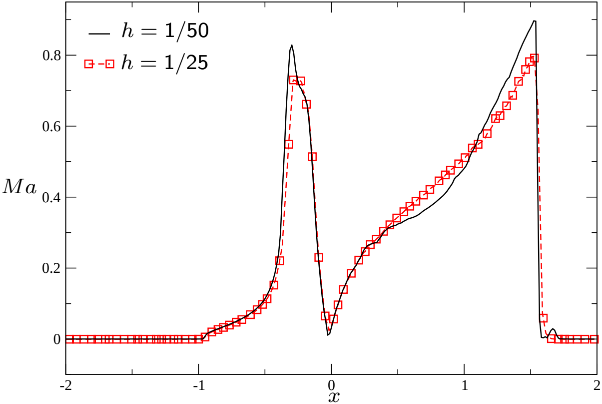
From (Kurganov & Tadmor, *Num.Meth. for Part.Diff.Eq.* 18, 2002)
2D Riemann Problem, configuration 12





Density contours
LRvS scheme, P^2 elements

Euler eq.s : 2D RP, data on $y = x$



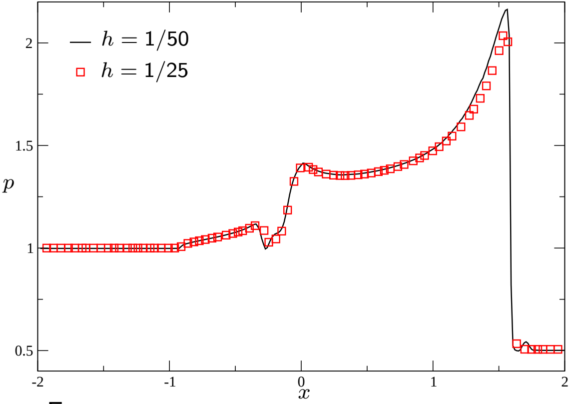
Mach number

From (Kurganov & Tadmor, *Num.Meth. for Part.Diff.Eq.* 18, 2002)

2D Riemann Problem, configuration 12

LRvS scheme, P^2 elements

Euler eq.s : 2D RP, data on $y = x$



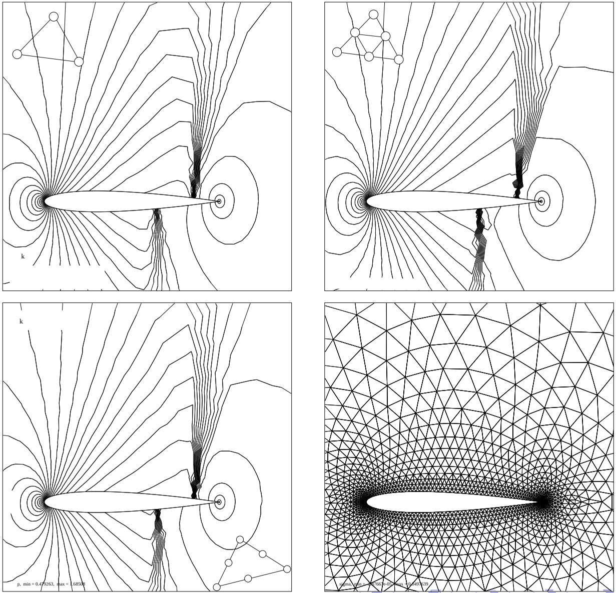
Pressure

From (Kurganov & Tadmor, *Num.Meth. for Part.Diff.Eq.* 18, 2002)

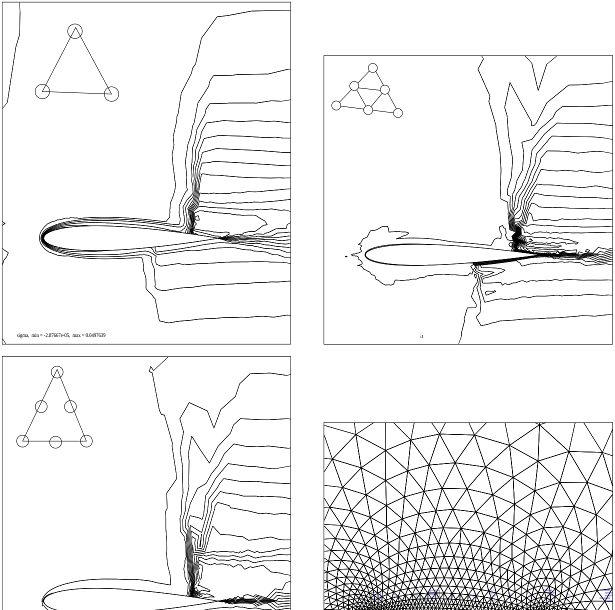
2D Riemann Problem, configuration 12

LRvS scheme, P^2 elements

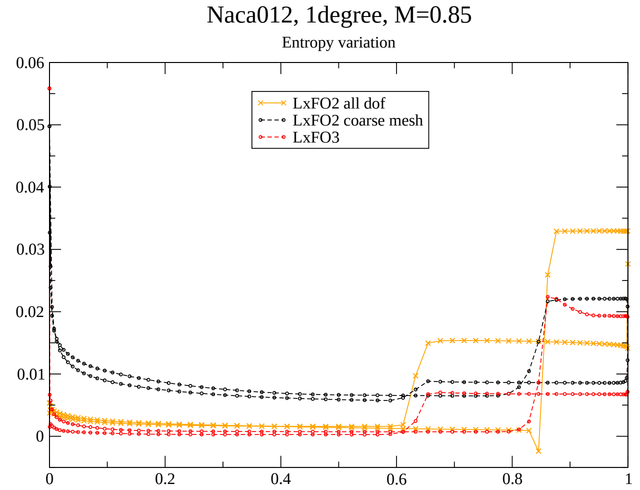
NACA 0012, M=0.85, 2° pressure



Entropy deviation



entropy deviation



Conclusions and perspectives

Conclusions

- Convergent higher order non-oscillatory $\mathcal{RD}$ schemes
- General procedure
- Efficient discretizations (fewer DoF and op.s w.r.t. DG)
- For systems less matrix algebra than with upwind schemes

Problems and perspectives

- Time-dependent
- Actual comparison with DG (error *vs* CPU)
- Viscous terms (in progress)