Banff International Research Station

for Mathematical Innovation and Discovery

Free Probability, Extensions, and Applications
January 13 - 18, 2008

MEALS

*Breakfast (Buffet): 7:00–9:00 am, Sally Borden Building, Monday–Friday
*Lunch (Buffet): 11:30 am–1:30 pm, Sally Borden Building, Monday–Friday
*Dinner (Buffet): 5:30–7:30 pm, Sally Borden Building, Sunday–Thursday
Coffee Breaks: As per daily schedule, 2nd floor lounge, Corbett Hall
***Please remember to scan your meal card at the host/hostess station in the dining room for each meal.**

MEETING ROOMS

All lectures will be held in Max Bell 159 (Max Bell Building accessible by bridge on 2nd floor of Corbett Hall). LCD projector, overhead projectors and blackboards are available for presentations. *Please note that the meeting space designated for BIRS is the lower level of Max Bell, Rooms 155–159. Please respect that all other space has been contracted to other Banff Centre guests, including any Food and Beverage in those areas.*

SCHEDULE

Monday

7:00–8:45	Breakfast
8:45–9:00	Introduction and Welcome to BIRS by BIRS Station Manager, Max Bell 159
9:00–10:00	Voiculescu
10:00–10:30	Coffee Break
10:30–11:30	Bercovici: Complex analytic methods in free probability theory
11:30–13:00	Lunch
13:00–14:00	Guided Tour of The Banff Centre; meet in the 2nd floor lounge, Corbett Hall
14:00–15:00	Verdu
15:00–15:30	Coffee Break
15:30–16:30	Letac: Beta distributions on matrices and applications
16:30–17:30	Mingo: Asymptotics of the Weingarten function
17:30–19:30	Dinner

Tuesday

7:00–9:00	Breakfast
9:00–10:00	Speicher: Survey on higher order freeness
10:00–10:30	Coffee Break
10:30–11:10	Moustakas: Synchronous minimum-mean-square-error signal-to-interference-ratio with synchronous interference: diagrams and replicas
11:20–12:00	Novak: Solution of nonhermitian spectral problem $H_1 + iH_2$, where H_1 and H_2 are arbitrary free hermitian ensembles
12:00–13:30	Lunch
14:00–15:00	Capitaine and Donati-Martin: Asymptotic spectrum of large deformed Wigner matrices
15:00–15:30	Coffee Break
15:30–16:10	Collins: Free Bessel laws
16:20–17:20	Hiai: Free pressure, free entropy and hypothesis testing
17:30–19:30	Dinner

Wednesday

7:00–9:00	Breakfast
9:00–10:00	Shlyakhtenko: Planar algebras, subfactors, random matrices and free probability
10:00–10:30	Coffee Break
10:30	Group Photo; meet on the front steps of Corbett Hall
10:30–11:30	Tulino: Large random matrices: applications to information theory and signal processing
11:30–12:10	Kargin: Lyapunov exponents of products of free operators
12:10–13:30	Lunch
	free afternoon
17:30–19:30	Dinner

Thursday

7:00–9:00	Breakfast
9:00–10:00	Silverstein: Spike models
10:00–10:30	Coffee Break
10:30–11:10	Dykema: Microstates in amalgamated free products
11:20–12:00	Kemp: Universal resolvent behaviour for $\mathscr{R}$-diagonal operators
12:00–13:30	Lunch
14:00–15:00	Edelman: On computational aspects of random matrices and free probability
15:00–15:30	Coffee Break
15:30–16:10	Anshelevich: Polynomial families and Boolean probability
16:20–17:00	Müller: Minimization of quadratic forms in wireless communications
17:00–18:00	Discussion Session
17:30–19:30	Dinner

Friday

7:00–9:00	Breakfast
9:00–10:00	Nica: Combinatorics of free probability and connections to the Boolean world
10:00–10:20	Coffee Break
10:20–11:00	Ryan: Channel capacity estimation using free probability theory
11:30–13:30	Lunch

** 5-day workshops are welcome to use the BIRS facilities (2nd Floor Lounge, Max Bell Meeting Rooms, Reading Room) until 3 pm on Friday, although participants are still required to checkout of the guest rooms by 12 noon. **

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ABSTRACTS
(in alphabetic order by speaker surname)

Speaker: **Anshelevich, Michael** (Texas A&M University)
Title: *Polynomial families and Boolean probability*
Abstract: The derivative operator generates polynomial families (Appell, Sheffer, Meixner) related to probability theory, while the difference quotient generates polynomials arising in free probability. A simpler one-variable version of the difference quotient also generates polynomial families, which turn out to be related to the third non-commutative probability theory, the Boolean theory. While the Boolean Appell polynomials are different, the Boolean versions of the Sheffer polynomials turn out to be the same as the free ones. In particular, the free Meixner distributions, such as the semicircle and the Marchenko-Pastur laws, also arise in the Boolean theory. Moreover, the Boolean-to-free version of the Bercovici-Pata bijection takes the free Meixner laws to themselves, and has a simples explicit form on them.
Everything above also holds for polynomials in many variables, but for the purposes of the presentation I will only describe the single-variable case.

Speaker: **Bercovici, Hari** (Indiana University)
Title: *Complex analytic methods in free probability theory*
Abstract: This will be a survey covering limit theorems in free probability, and regularity issues for free convolutions. We will discuss the prospects of extending these results to operator-valued variables, and to other variants of noncommutative probability theory.

Speaker: **Capitaine, Mireille** (CNRS, Universite Paul Sabatier)
Title: *Asymptotic spectrum of large deformed Wigner matrices, Part II*
Abstract: We investigate the asymptotic spectrum of complex or real Deformed Wigner matrices $(M_N)_N$ defined by $M_N = W_N/\sqrt{N} + A_N$ where W_N is a $N \times N$ Hermitian (resp. symmetric) Wigner matrix whose entries have a symmetric law satisfying a Poincaré inequality. The matrix A_N is Hermitian (resp. symmetric) and deterministic with all but finitely many eigenvalues equal to zero. We first show that, as soon as the first largest or last smallest eigenvalues of A_N are sufficiently far from zero, the corresponding eigenvalues of M_N almost surely exit the limiting semicircle compact support as the size N becomes large. The corresponding limits are universal in the sense that they only involve the variance of the entries of W_N. On the other hand, when A_N is diagonal with a sole simple non-null eigenvalue large enough, we prove that the fluctuations of the largest eigenvalue are not universal and vary with the particular distribution of the entries of W_N.

Speaker: **Collins, Benoit** (University of Ottawa)
Title: *Free Bessel laws*
Abstract: Motivated by the study of 'liberation' of compact groups, we introduce and study a remarkable family of real probability measures π_{st}, that we call free Bessel laws. This family has the property of having a semigroup type behavior for free additive convolution, a natural classical counterpart, and an interesting

matrix model. We also study analytic and combinatorial aspects of these probability distributions. This is joint work with Teo Banica, Serban Belinschi and Mireille Capitaine.

Speaker: **Donati-Martin, Catherine** (Universit Paris 6- CNRS)
Title: *Asymptotic spectrum of large deformed Wigner matrices, Part I*
Abstract: We investigate the asymptotic spectrum of complex or real Deformed Wigner matrices $(M_N)_N$ defined by $M_N = W_N/\sqrt{N} + A_N$ where W_N is a $N \times N$ Hermitian (resp. symmetric) Wigner matrix whose entries have a symmetric law satisfying a Poincaré inequality. The matrix A_N is Hermitian (resp. symmetric) and deterministic with all but finitely many eigenvalues equal to zero. We first show that, as soon as the first largest or last smallest eigenvalues of A_N are sufficiently far from zero, the corresponding eigenvalues of M_N almost surely exit the limiting semicircle compact support as the size N becomes large. The corresponding limits are universal in the sense that they only involve the variance of the entries of W_N. On the other hand, when A_N is diagonal with a sole simple non-null eigenvalue large enough, we prove that the fluctuations of the largest eigenvalue are not universal and vary with the particular distribution of the entries of W_N.

Speaker: **Dykema, Ken** (Texas A&M University)
Title: *Microstates in amalgamated free products*
Abstract: Under certain regularity assumptions, canonical generators in a free product of von Neumann algebras, with amalgamation over a hyperfinite subalgebra have the expected free entropy dimension.
(joint work with Nate Brown and Kenley Jung)

Speaker: **Hiai, Fumio** (Tohoku University)
Title: *Free pressure, free entropy and hypothesis testing*
Abstract: I discuss the free pressure and the free entropy for noncommutative self-adjoint multi-variables in the matricial microstate approach from the point of view of hypothesis testing, which may be considered as a sort of large deviation type problem. In particular, the free analog of Stein's lemma is considered and the asymptotic error bound is compared with the η-version of free entropy χ. This consideration provides a certain sufficient condition for a tracial distribution μ to satisfy the equality $\eta(\mu) = \chi(\mu)$. Moreover, in the single variable case, I show the complete free analogies of Stein's lemma and other results in hypothesis testing.

Speaker: **Kargin, Vladislav** (Courant Institute)
Title: *Lyapunov exponents of products of free operators*
Abstract: Lyapunov exponents of a dynamical system are a useful tool to gauge the stability and complexity of the system. In this talk I define and discuss Lyapunov exponents for a sequence of free linear operators. The definition is based on the concept of the extended Fuglede-Kadison determinant. I will explain how the Lyapunov exponents can be computed using the S-transform. I will also discuss the relations with the results on the products of large random matrices."

Speaker: **Kemp, Todd** (MIT)
Title: *Universal resolvent behaviour for $\mathscr{R}$-diagonal operators*
Abstract: Voiculescu's circular operator c, an important non-normal operator in free probability, is the large-N limit of the $N \times N$ Ginibre ensemble of matrices with independent normal entries of variance $\frac{1}{2N}$. The Haar unitary operator is the limit of Haar unitary random matrices. The free cumulants of both operators have symmetry properties that led Nica and Speicher to introduce $\mathscr{R}$-diagonal operators. They are characterized by a very strong rotational-invariance property.
In recent work with Uffe Haagerup and Roland Speicher, we have shown that $\mathscr{R}$-diagonal operators have universal resolvent blow-up. That is, if a is $\mathscr{R}$-diagonal (and not Haar unitary), the analytic function

$$\rho_a(z) = \frac{1}{z-a}$$

blows up as z approaches spec a at the asymptotic rate dist$(z, \operatorname{spec} a)^{-3/2}$; moreover, there is a universal constant depending only on the second and fourth moments of a. These results are proved using both the combinatorial techniques of Nica and Speicher and the complex analytic techniques discussed on Monday by Bercovici.

Speaker: **Letac, Gerard** (Universite Paul Sabatier, Toulouse, France)
Title: *Beta distributions on matrices and applications*
Abstract: The beta distributions on real symmetric matrices -or complex or quaternionic hermitian- are distributions of suitable quotients Z of Wishart variables. Complex versions occur in signal theory. We explain several ways of computing their moments, and we use the simplest, the 'Psi' method, for computing expectations of some quadratic forms of Z and bilinear forms of Z and Z^{-1}. We apply in particular these tools for proving the Thomae formulae for the multivariate hypergeometric function ${}_3F_2$ and for studying the Bryc-Bozejko problem ("when are $U = S^{-1}X^2S^{-1}$ and $S = X + Y$ independent if X and Y are iid symmetric matrices?").

Speaker: **Mingo, James** (Queen's University)
Title: *Asymptotics of the Weingarten Function*
Abstract: The expectation of products of entries of a Haar distributed random N x N unitary matrix may be expressed simply in terms of the Weingarten function on the symmetric group, much in the same way that Wick's formula gives the expectation of a product of Gaussian random variables.

The Weingarten function can be written as a power series is 1/N. The coefficient of the term of leading order was shown by B. Collins to the Moebius function of the lattice of non-crossing partitions. I will show that the coefficient of the sub-leading term can be computed from the second order Moebius function. A similar scheme will be explained for higher orders. This is joint work with Roland Speicher.

Speaker: **Moustakas, Aris** (National & Capodistrian University of Athens)
Title: *Synchronous Minimum-Mean-Square-Error Signal-to-Interference-Ratio with synchronous interference: Diagrams and Replicas*
Abstract: Using the diagrammatic method developed in physics, I will discuss a particular application of the asymptotic eigenvalue spectrum of the random matrix (H*U*J*U'*H'+G*V*P*V'*G'+...) where H, G are aribtrary fixed random matrices, J, P are arbitrary fixed positive definite hermitian matrices and U, V are independent random Haar-unitary or independent complex Gaussian matrices (the dots indicate generalizations by adding the same form of matrices). I will also discuss some related results obtained using the replica method.

Speaker: **Müller, Ralf** (Norwegian University of Science and Technology)
Title: *Minimization of quadratic forms in wireless communications*
Abstract: The problem of minimizing $\langle x|J|x\rangle$ over a certain set of vectors $\langle x|$ is considered where J is a random matrix with given R-transform. This problem has applications in transmitter processing of wireless communications. Generalizations of that problem that are still open, but also of interest are discussed as well.

Speaker: **Nowak, Maciej** (Jagiellonian University)
Title: *Solution of nonhermitian spectral problem $H_1 + iH_2$, where H_1 and H_2 are arbitrary free hermitian ensembles*
Abstract: Using quaternion valued extension of R-transformation we provide the construction and operational algorithm for inferring spectral properties of $H_1 + iH_2$, where H_i $(i = 1, 2)$ are mutually free hermitian ensembles. We exemplify the construction for H_i = GUE, Free Cauchy, CUE.., recovering in Gaussian case properties of Ginibre-Girko ensemble and in other cases providing results, which, up to our knowledge, are not known in physical and mathematical literature on the subject.

Speaker: **Ryan, Oyvind** (University of Oslo)
Title: *Channel capacity estimation using free probability theory*
Abstract: The aim of this talk is to find a simple application of free probability, where one obtains useful results when comparing to existing methods. The chosen example is channel capacity estimation in MIMO systems. We explain which free probability constructs are needed in this setting, and explain how they can be implemented in practice. We also show results of simulations where the free probability approach is compared with other methods. For some random matrix models the free probability approach comes short. We show one possible way of working around this by using exact formulas for the expectations of mixed moments of deterministic and Gaussian random matrices.

Speaker: **Shlyakhtenko, Dimitri** (UCLA)
Title: *Planar algebras, subfactors, random matrices and free probability*
Abstract: Planar algebras are interesting combinatorial objects arising from subfactors; they involve planar drawings. Very similar diagrams occur in random matrix theory. We give a construction of a natural trace on a planar algebra which can also be obtained via a random matrix model. This trace is used to give a new proof of Popa's theorem on realization of subfactors.
This is joint work with A. Guionnet and V.F.R. Jones.

Speaker: **Speicher, Roland** (Queen's University)
Title: *Survey on higher order freeness*
Abstract: I will give a survey on the questions, ideas, and results behind the concept of higher (in particular, second) order freeness.

Speaker: **Tulino, Antonia** (Universit degli Studi di Napoli Federico II)
Title: *Large random matrices: applications to information theory and signal processing*
Abstract: Of late, random matrices have attracted great interest in the engineering community because of their applications to the communications and information theory on the fundamental limits of wireless communication noisy vector channels. The purpose of this talk is to illustrate this synergy between random matrix theory and information theory through several classes of channels that arise in wireless communications. These channels are characterized by random matrices that admit various statistical descriptions depending on the actual application.
Throughout the talk, we apply the various findings in random matrix theory to the fundamental limits of wireless communication with focus on several classes of vector channels that arise in wireless communications: Code-division multiple-access (CDMA), Multi-carrier code-division multiple access (MC-CDMA). Channels with multiple receive and transmit antennas, incorporating features such as antenna correlation, polarization, and line-of-sight components.
For each of these channels, we analyze two performance measures of engineering interest: the average mutual information (highest data rate that can be conveyed reliably per unit bandwidth) and minimum mean-square error (smallest mean-square error that can be incurred estimating the channel input based on its noisy received observations), which are determined by the distribution of the singular values of the channel matrix.

Eigenvalues of Large
Dimensional Random Matrices

Jack W. Silverstein
Department of Mathematics
North Carolina State University

Abstract:

The talk will outline recent work on spectral properties of random matrices. One topic to be covered concerns the properties of individual eigenvalues of a class of matrices of sample covariance type. It is defined as $B_n = (1/N)T_n^{1/2}X_nX_n^*T_n^{1/2}$ where $X_n = (X_{ij})$ is $n \times N$ with i.i.d. complex standardized entries, and $T_n^{1/2}$ is a Hermitian square root of the nonnegative definite Hermitian matrix T_n. This matrix can be viewed as the sample covariance matrix of N i.i.d. samples of the n dimensional random vector $T_n^{1/2}(X_n)_{\cdot 1}$. It is known that if $n/N \to c > 0$ and the empirical distribution function (e.d.f.) of the eigenvalues of T_n converge as $n \to \infty$, then the e.d.f. of the eigenvalues of B_n converges a.s. to a nonrandom limit. This result is relevant in situations in multivariate analysis where the vector dimension is large, but the number of samples to adequately approximate the population matrix (required in standard statistical procedures) cannot be attained.

Consider a finite number of eigenvalues of T_n which are outside the support of its limiting spectral e.d.f. This is referred to as a "spiked population model". Results are obtained for the limiting behavior of those eigenvalues of B_n which correspond to the "spiked" eigenvalues of T_n. An application is given to the detection problem in array signal processing: determining the number of sources (presumed large) impinging on a bank of sensors in a noise filled environment (joint work with Jinho Baik at University of Michigan, and with Raj Rao at MIT).

Another class of matrices of the form $C_n = (1/N)(R_n + \sigma X_n)(R_n + \sigma X_n)^*$ where X_n is as in B_n, $\sigma > 0$, and R_n is $n \times N$ random, independent of X_n with the spectral e.d.f. of $(1/N)R_nR_n$ converging to a nonrandom limit. These matrices model situations, such as in array signal processing, where information is contained in the sampling of the vectors $R_{\cdot 1} \cdots R_{\cdot N}$, but the received vector is contaminated by additive noise (the columns of σX_n). The e.d.f. of the eigenvalues of C_n also converges a.s. as $n \to \infty$ (with $n/N \to c > 0$). Properites of the limiting distribution will be outlined. (joint work with Brent Dozier).

A third class to be discussed generalizes B_n. It is of the form $D_n = (1/N)T_n^{1/2}X_nS_nX_n^*T_n^{1/2}$ where S_n is $N \times N$ nonnegative definite Hermitian, and appears in the modeling of MIMO (multiple-input-multiple-out) systems in wireless communications (joint work with Debashis Paul at UC Davis).