On Fenchel Duality and Some of Its Variants

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Preliminary notions and results

Consider

- X a separated locally convex space and its topological dual space X^* endowed with the weak* topology $\omega(X^*, X)$;
- for $C \subseteq X$ convex, $\text{core}(C)$, the algebraic interior of C. One has $x \in \text{core}(C)$ if and only if $\cup_{\lambda>0} \lambda(C - x) = X$;
- for $C \subseteq X$ convex, $\text{sqri}(C)$, the strong-quasi relative interior of C. One has $x \in \text{sqri}(C)$ if and only if $\cup_{\lambda>0} \lambda(C - x)$ is a closed linear subspace of X;
- for a given set $C \subseteq X$, the indicator function of C, $\delta_C : X \to \overline{\mathbb{R}}$, defined as $\delta_C(x) = 0$, if $x \in C$ and $\delta_C(x) = +\infty$, otherwise.

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For $f : X \to \overline{\mathbb{R}}$ we consider the following notions

- domain: $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$;
- f is proper: $f(x) > -\infty \ \forall x \in X$ and $\operatorname{dom} f \neq \emptyset$;
- epigraph: $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$;
- lower semicontinuous envelope of f: the function $\operatorname{cl}(f) : X \to \overline{\mathbb{R}}$ defined by $\operatorname{epi}(\operatorname{cl}(f)) = \operatorname{cl}(\operatorname{epi} f)$;
- conjugate function of f: $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup \{\langle x, x^* \rangle - f(x) : x \in X\}$;
- for $\varepsilon \geq 0$ and $\bar{x} \in X$ with $f(\bar{x}) \in \mathbb{R}$ the (convex) ε-subdifferential of f at $\bar{x}$: $\partial_\varepsilon f(\bar{x}) = \{x^* \in X^* : f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - \varepsilon \ \forall x \in X\}$; otherwise, $\partial_\varepsilon f(\bar{x}) = \emptyset$;
- the (convex) subdifferential of f at $\bar{x} \in X$: $\partial f(\bar{x}) := \partial_0 f(\bar{x})$.

- When $f, g : X \to \overline{\mathbb{R}}$ are proper functions, their infimal convolution is defined by $f \square g : X \to \overline{\mathbb{R}}$, $f \square g(x) = \inf\{f(x - y) + g(y) : y \in X\}$.
- We say that $f \square g$ is exact at $x \in X$ if there exists some $y \in X$ for which the infimum is attained.

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- Consider $f, g : X \to \overline{\mathbb{R}}$ two arbitrary proper convex functions and the following convex optimization problem

$$(P) \quad \inf_{x \in X} \{f(x) + g(x)\}.$$

- The Fenchel dual problem to (P) is

$$(D) \quad \sup_{z^* \in X^*} \{-f^*(-z^*) - g^*(z^*)\}.$$

We say that

- the pair f, g satisfy stable Fenchel duality if for all $x^* \in X^*$, there exists $z^* \in X^*$ such that $(f+g)^*(x^*) = f^*(x^* - z^*) + g^*(z^*)$
- the pair f, g satisfy the classical Fenchel duality if there exists $z^* \in X^*$ such that $(f+g)^*(0) = f^*(-z^*) + g^*(z^*)$
- the pair f, g is totally Fenchel unstable if f, g satisfy Fenchel duality but $y^*, z^* \in X^*$ and

$$(f+g)^*(y^* + z^*) = f^*(y^*) + g^*(z^*) \Longrightarrow y^* + z^* = 0.$$

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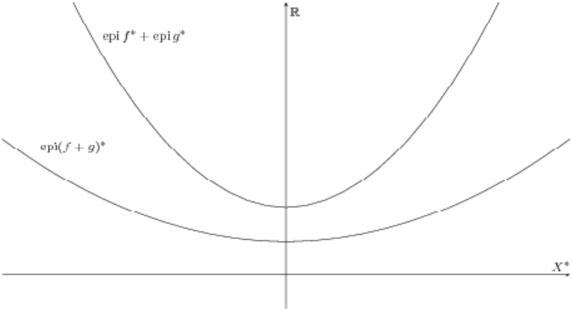
One always has

$$\operatorname{epi} f^* + \operatorname{epi} g^* \subseteq \operatorname{epi}(f+g)^*.$$



The pair f, g satisfy stable Fenchel duality if and only if

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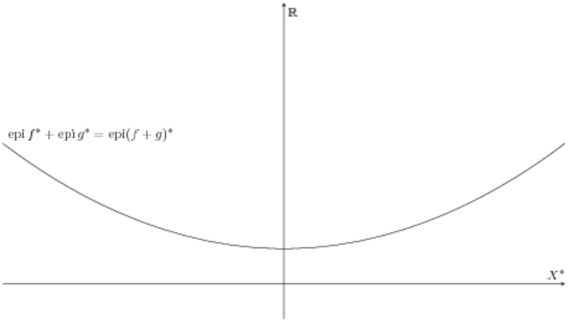

The pair f, g satisfy Fenchel duality if and only if

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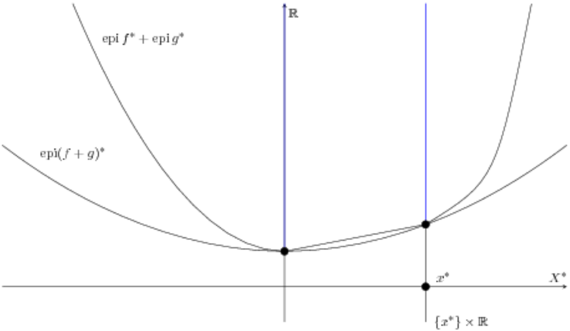
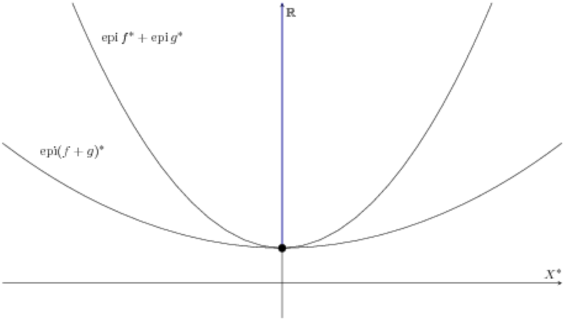


The pair f, g is totally Fenchel unstable if and only if

$$\operatorname{epi}(f+g)^* \cap (\{0\} \times \mathbb{R}) = (\operatorname{epi} f^* + \operatorname{epi} g^*) \cap (\{0\} \times \mathbb{R})$$

and there is no $x^* \in X^* \setminus \{0\}$ such that

$$\operatorname{epi}(f+g)^* \cap (\{x^*\} \times \mathbb{R}) = (\operatorname{epi} f^* + \operatorname{epi} g^*) \cap (\{x^*\} \times \mathbb{R}).$$



Regularity conditions for Fenchel duality

Assume that $f, g : X \to \overline{\mathbb{R}}$ are proper convex functions such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. In the literature there exist different classes of regularity conditions for stable Fenchel duality:

(i) f is continuous at $x' \in \operatorname{dom} f \cap \operatorname{dom} g$;

Interior point regularity conditions:

(ii) $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$;

(iii) $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$ (Rockafellar, 1974);

(iv) $0 \in \operatorname{sqri}(\operatorname{dom} f - \operatorname{dom} g)$ (Attouch, Brézis, 1986, Zălinescu, 1987).

Closedness-type regularity condition:

(v) $\operatorname{epi} f^* + \operatorname{epi} g^*$ is closed in the product topology of $(X^*, \omega(X^*, X)) \times \mathbb{R}$ (B., Wanka, 2006, Burachik, Jeyakumar, 2006).

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We have that

- condition $(i) \Rightarrow$ stable Fenchel duality;
- if f, g are lower semicontinuous and X is a Fréchet space, then $(ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow$ stable Fenchel duality;
- if f, g are lower semicontinuous, then $(v) \Leftrightarrow$ stable Fenchel duality.

Example 1. Let $X = \mathbb{R}$, $f(x) = \frac{1}{2}x^2$, if $x \geq 0$, and $f(x) = +\infty$, otherwise, and $g = \delta_{(-\infty,0]}$. Then $(i) - (iv)$ are not fulfilled, while (v) is valid.

Consider the following regularity condition for Fenchel duality:

(vi) $f^* \Box g^*$ is lower semicontinuous and exact at 0 (B., Wanka, 2006).

If f, g are lower semicontinuous, then $(v) \Rightarrow (vi) \Rightarrow$ Fenchel duality.

Example 2. Let $X = \mathbb{R}^2$, $C = \left\{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0\right\}$, $D = \left\{(x_1, x_2)^T \in \mathbb{R}^2 : 2x_1 + x_2^2 \leq 0\right\}, f = \delta_C$ and $g = \delta_D$.

Thus f, g satisfy Fenchel duality, f, g doesn't satisfy stable Fenchel duality and the pair f, g is not totally Fenchel unstable.

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Example 1. Let $X = \mathbb{R}$, $f(x) = \frac{1}{2}x^2$, if $x \geq 0$, and $f(x) = +\infty$, otherwise, and $g = \delta_{(-\infty,0]}$. Then $(i) - (iv)$ are not fulfilled, while (v) is valid.

Consider the following regularity condition for Fenchel duality:

(vi) $f^* \square g^*$ is lower semicontinuous and exact at 0 (B., Wanka, 2006).

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Example 2. Let $X = \mathbb{R}^2$, $C = \left\{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0\right\}$, $D = \left\{(x_1, x_2)^T \in \mathbb{R}^2 : 2x_1 + x_2^2 \leq 0\right\}, f = \delta_C$ and $g = \delta_D$.

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Algebraic result:

$$(f+g)^*(x^*) = \min_{y^* \in X^*} \{f^*(x^* - y^*) + g^*(y^*)\} \ \forall x^* \in X^* \tag{1}$$

if and only if

$$\inf_{x \in X}[f(x)+g(x)-\langle x^*, x\rangle] = \max_{y^* \in X^*} \{-f^*(x^*-y^*)-g^*(y^*)\} \ \forall x^* \in X^* \tag{2}$$

if and only if

$$\partial_\varepsilon (f+g)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} (\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x)) \ \forall x \in X \ \forall \varepsilon \geq 0. \tag{3}$$

On the other hand, (3) implies (take $\varepsilon = 0$)

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Two convex regularization schemes

- (Burger, Osher, 2004) Take $\mathcal{U}$ a Banach space, $\mathcal{H}$ a Hilbert space, $K : \mathcal{U} \to \mathcal{H}$ a linear continuous operator and the ill-posed operator equation

$$Ku = f, \tag{5}$$

where $f \in R(\mathcal{K})$.

Let $J : \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function. Then $\bar{u} \in \mathcal{U}$ is called J-minimizing solution for (5) if it is an optimal solution of

$$\inf_{Ku=f} J(u).$$

Source condition: the existence of Lagrange multiplier, i.e. $\exists \bar{w} \in \mathcal{H}$ with $K^*w \in \partial J(\bar{u}) \Rightarrow \bar{u}$ is a J-minimizing solution for (5).

Viceversa, if $\bar{u}$ is a J-minimizing solution for (5) and

$$f \in \operatorname{sqri}(K(\operatorname{dom} J)) \text{ (interior-point regularity condition)},$$

then there exists a Lagrange multiplier $\bar{w} \in \mathcal{H}$ with $\langle \bar{w}, f - K\bar{u} \rangle = 0$ and

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▶ (Chambolle, Lions, 1997) On $\Omega \subseteq \mathbb{R}^2$ a bounded and piecewise smooth open set consider the image recovery problem

$$u_0 = Au + n.$$

Here:

- ▶ u_0 is the image;
- ▶ u is the transformed image;
- ▶ n is the random noise. It fulfills $\int_\Omega n = 0$ and $\int_\Omega |n|^2 = \sigma^2$;
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Problem: Knowing u_0, one has to recover u.

▶ (Rudin, Osher, Fatemi, 1992): Solve the constrained minimization problem:

$$\inf_{\substack{\int_\Omega Au = \int_\Omega u_0, \\ \int_\Omega |Au - u_0|^2 = \sigma^2}} |\mathcal{D}u|(\Omega). \tag{6}$$

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Under some natural assumptions one can prove that (Chambolle, Lions, 1997) (6) is equivalent to

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- (Chambolle, Lions, 1997) Assume that $\bar{u}$ is an optimal solution of (7) and that $u_0 \in \mathrm{cl}(L^2(\Omega) \cap A(BV(\Omega)))$. For $C = \mathbb{R}_+$ and $g(u) = \|Au - u_0\|^2 - \sigma^2$ the latter condition means in fact that

$$\exists u' \in \mathrm{dom}\, J : g(u') < 0 \text{ (Slater regularity condition).}$$

Thus there exists a Lagrange multiplier $\bar{\lambda} \geq 0$ such that $\bar{\lambda}(\|A\bar{u} - u_0\| - \sigma) = 0$ and

$$0 \in \partial(J + \bar{\lambda}(\|A\cdot - u_0\|^2 - \sigma^2))(\bar{u}) = \partial J(\bar{u}) + \bar{\lambda}\partial(\|A\cdot - u_0\|^2 - \sigma^2)(\bar{u})$$

$$\Leftrightarrow -\bar{\lambda}A^*(A\bar{u} - u_0) \in \partial J(\bar{u}).$$

Define $J : L^p(\Omega) \to \mathbb{R} \cup \{+\infty\}$,

$$J(u) = |\mathcal{D}u|(\Omega), \text{ if } u \in BV(\Omega), J(u) = +\infty, \text{ otherwise.}$$

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Totally Fenchel unstable functions

Consider X a nontrivial real Banach space, X^* its topological dual space and X^{**} its bidual space. We have

- the canonical embedding of X into X^{**}, $\widehat{\cdot} : X \to X^{**}$, $\langle x^*, \widehat{x} \rangle := \langle x, x^* \rangle$, for all $x \in X$ and $x^* \in X^*$
- if $C \subseteq X$ is convex, then $x \in C$ is a support point of C if there exists $x^* \in X^* \setminus \{0\}$, such that $\sup\langle C, x^* \rangle = \langle x, x^* \rangle$.

Example 3 (totally Fenchel unstable functions). (Simons, 2007) Let $C \subset X$ be nonempty, bounded, closed and convex such that there exists an extreme point x_0 of C which is not a support point of C. Take $f := \delta_{x_0 - C}$ and $g := \delta_{C - x_0}$. Then f, g satisfy Fenchel duality and the pair f, g is totally Fenchel unstable.

Example 4. (Borwein, 2007) Let $X = l_2$, $1 < p < 2$ and $C = \{x \in l_2 : \|x\|_p \leq 1\}$. Then x is an extreme point of $C \Leftrightarrow \|x\|_p = 1$. An extreme point of C is a support point of $C \Leftrightarrow x \in l_{2(p-1)}$. Thus there are a plenty of extreme points of C which are not support points.

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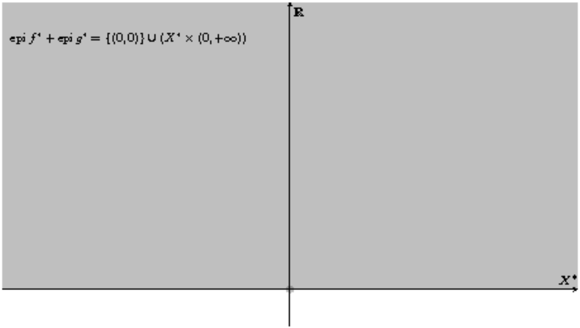
Regarding the functions defined in Example 3, Simons asks whether,

$$\operatorname{epi} f^* + \operatorname{epi} g^* \supset X^* \times (0, +\infty)$$

or, equivalently,

$$\operatorname{epi} f^* + \operatorname{epi} g^* = \{(0,0)\} \cup (X^* \times (0, +\infty)).$$



The reflexive case (B, 2007)

Let $y^* \in X^*$ be arbitrary and $h, k : X^* \to \mathbb{R}$, $h(z^*) := f^*(z^*)$ and $k(z^*) := g^*(y^* - z^*)$. Since h and k are continuous, by the Fenchel duality theorem,

$$-\inf_{X^*}[h + k] = \min_{z \in X}[h^*(z) + k^*(-z)] = \min_{X}[\delta_{\{0\}} - y^*] = 0,$$

so, for all $\varepsilon > 0$, there exists $z^* \in X^*$ such that $h(z^*) + k(z^*) \leq \varepsilon$, thus $(y^*, \varepsilon) \in \operatorname{epi} f^* + \operatorname{epi} g^*$.

The nonreflexive case

Problem 1. (raised by Stephen Simons in his book "From Hahn-Banach to Monotonicity", Springer-Verlag, 2008)
Let C be a nonempty, bounded, closed and convex subset of a nonreflexive Banach space X, x_0 be an extreme point of C, $y^* \in X^*$ and $\varepsilon > 0$. Then does there always exist $M \geq 0$ such that, for all $u, v \in C$, $M\|u + v - 2x_0\| \geq \langle v - x_0, y^* \rangle - \varepsilon$? The answer to this question is in the affirmative if and only if

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Weak*-extreme points

- We recall that x_0 is a weak*-extreme point of the bounded, closed and convex set $C \subseteq X$ if $\widehat{x_0}$ is an extreme point of $\operatorname{cl} \widehat{C}$, where the closure is taken with respect to the weak* topology $w(X^{**}, X^*)$.
- If x_0 is a weak*-extreme point of C, then x_0 is an extreme point of C.
- (Phelps, 1961): must the image $\widehat{x}$ of an extreme point of $x \in B_X$ (the unit ball of X) be an extreme point of $B_{X^{**}}$ (the unit ball of the bidual)? We recall that by the Goldstine Theorem the closure of $\widehat{B_X}$ in the weak* topology $w(X^{**}, X^*)$ is $B_{X^{**}}$ (hence the generalization to a bounded, closed and convex set is natural).
- The first example of a Banach space and a point of its unit ball which is not weak*-extreme was suggested by K. de Leeuw and proved in (Y. Katznelson, 1961).
- In the spaces $C(X)$, $L^p(1 \leq p \leq \infty)$, all the extreme points of the corresponding unit balls are weak*-extreme points.

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The solution of the Problem 1 (B., Csetnek, Proc. of AMS, 2009)

For $f : X \to \overline{\mathbb{R}}$ we define $\widehat{f} : X^{**} \to \overline{\mathbb{R}}$ by $\widehat{f}(x^{**}) = f(x)$, if $x^{**} = \widehat{x} \in \widehat{X}$ and $\widehat{f}(x^{**}) = +\infty$, otherwise.

Lemma 1. We assume that f is convex with $\operatorname{dom} f \neq \emptyset$ and that $\operatorname{cl}(\widehat{f})$ is proper, where the lower semicontinuous hull is considered with respect to the topology $w(X^{**}, X^*)$. Then $f^{**} = \operatorname{cl}(\widehat{f})$.

Remark 2. If $C \subseteq X$ is a nonempty convex set, then by Lemma 1 follows that $\delta_C^{**} = \delta_{\operatorname{cl}(\widehat{C})}$, where the closure is considered in the topology $\omega(X^{**}, X^*)$. Thus Lemma 1 generalizes a result obtained in (Chakrabarty, Shunmugaraj, Zălinescu, 2007).

Consider $f, g : X \to \overline{\mathbb{R}}$ proper convex functions with the following properties

- $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$
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Theorem 1. We have $X^* \times (0,\infty) \subset \operatorname{epi} f^* + \operatorname{epi} g^*$ if and only if $\operatorname{dom}(\operatorname{cl}(\widehat{f})) \cap \operatorname{dom}(\operatorname{cl}(\widehat{g})) = \{0\}$.

Now consider

- C a nonempty, bounded and convex subset of the Banach space X and $x_0 \in C$
- $f := \delta_A$, $g := \delta_B$, where $A := x_0 - C$, $B := C - x_0$.

In this case we have

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Theorem 2. We have $X^* \times (0,\infty) \subset \operatorname{epi} f^* + \operatorname{epi} g^*$ if and only if x_0 is a weak*-extreme point of C.

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Now consider

- C a nonempty, bounded and convex subset of the Banach space X and $x_0 \in C$
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The finite dimensional case

Problem 2. (raised by Stephen Simons in his book "From Hahn-Banach to Monotonicity", Springer-Verlag, 2008)
Do there exist a nonzero finite dimensional Banach space X and $f, g : X \to \overline{\mathbb{R}}$ proper and convex functions such that the pair f, g is totally Fenchel unstable?

The solution of the Problem 2 (B., Löhne, Math. Prog., to appear)

For all $x^*, y^* \in X^*$ it holds

$$(f+g)^*(x^*) \leq f^*(x^* - y^*) + g^*(y^*). \quad (8)$$

Therefore, a pair f, g of proper and convex functions is totally Fenchel unstable if and only if

$$\exists y^* \in X^* : \ (f+g)^*(0) = f^*(-y^*) + g^*(y^*). \quad (9)$$

$$\forall x^* \in X^* \setminus \{0\}, \forall y^* \in X^* : \ (f+g)^*(x^*) < f^*(x^* - y^*) + g^*(y^*). \quad (10)$$

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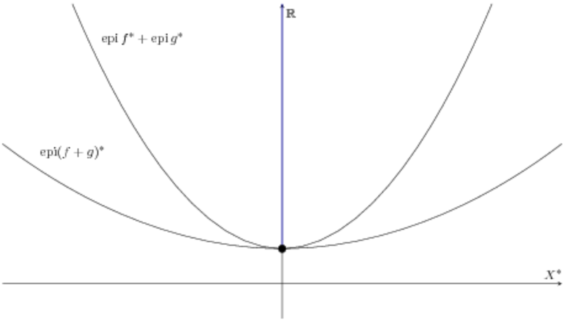
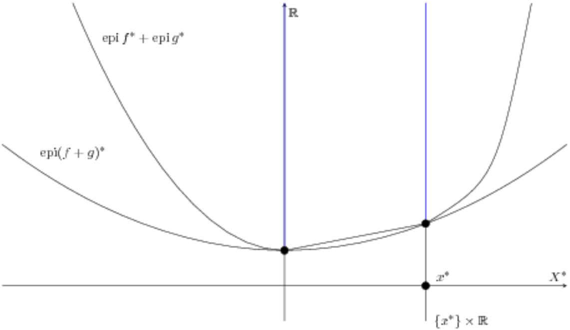
Theorem 2. There are no proper convex functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that the pair f, g is totally Fenchel unstable.

Comment. The situation below is not possible:



Interpretation of the result. If two proper and convex functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ satisfy Fenchel duality, then there exists at least one element $x^* \in \mathbb{R}^n \setminus \{0\}$, such that $f - \langle x^*, \cdot \rangle$ and g (or f and $g - \langle x^*, \cdot \rangle$) satisfy Fenchel duality, too.

Comment. We must have something like:



Comment. More precisely, for the concrete situation considered in the previous picture the following behavior can be noticed: