On Fenchel Duality and Some of Its Variants

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Preliminary notions and results

Consider

- $X$ a separated locally convex space and its topological dual space $X^*$ endowed with the weak* topology $\omega(X^*, X)$;
- for $C \subseteq X$ convex, $\text{core}(C)$, the algebraic interior of $C$. One has $x \in \text{core}(C)$ if and only if $\bigcup_{\lambda > 0} \lambda (C - x) = X$;
- for $C \subseteq X$ convex, $\text{sqri}(C)$, the strong-quasi relative interior of $C$. One has $x \in \text{sqri}(C)$ if and only if $\bigcup_{\lambda > 0} \lambda (C - x)$ is a closed linear subspace of $X$;
- for a given set $C \subseteq X$, the indicator function of $C$, $\delta_C : X \to \overline{\mathbb{R}}$, defined as $\delta_C(x) = 0$, if $x \in C$ and $\delta_C(x) = +\infty$, otherwise.
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For $f : X \to \overline{\mathbb{R}}$ we consider the following notions

- **domain:** $\text{dom } f = \{x \in X : f(x) < +\infty\}$;
- **$f$ is proper:** $f(x) > -\infty \ \forall x \in X$ and $\text{dom } f \neq \emptyset$;
- **epigraph:** $\text{epi } f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$;
- **lower semicontinuous envelope of $f$:** the function $\text{cl}(f) : X \to \overline{\mathbb{R}}$ defined by $\text{epi}(\text{cl}(f)) = \text{cl}(\text{epi } f)$;
- **conjugate function of $f$:** $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup \{\langle x, x^* \rangle - f(x) : x \in X\}$;
- **for $\varepsilon \geq 0$ and $\bar{x} \in X$ with $f(\bar{x}) \in \mathbb{R}$ the (convex) $\varepsilon$-subdifferential of $f$ at $\bar{x}$:**
  $$\partial_\varepsilon f(\bar{x}) = \{x^* \in X^* : f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - \varepsilon \ \forall x \in X\}$$
  otherwise, $\partial_\varepsilon f(\bar{x}) = \emptyset$;
- **the (convex) subdifferential of $f$ at $\bar{x} \in X$:** $\partial f(\bar{x}) := \partial_0 f(\bar{x})$.

- **When $f, g : X \to \overline{\mathbb{R}}$ are proper functions, their infimal convolution is defined by $f \Box g : X \to \overline{\mathbb{R}}$, $f \Box g(x) = \inf \{f(x - y) + g(y) : y \in X\}$.

- We say that $f \Box g$ is exact at $x \in X$ if there exists some $y \in X$ for which the infimum is attained.
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- We say that $f\Box g$ is exact at $x \in X$ if there exists some $y \in X$ for which the infimum is attained.
Consider \( f, g : X \to \overline{\mathbb{R}} \) two arbitrary proper convex functions and the following convex optimization problem

\[
(P) \quad \inf_{x \in X} \{ f(x) + g(x) \}.
\]

The Fenchel dual problem to \((P)\) is

\[
(D) \quad \sup_{z^* \in X^*} \{-f^*(-z^*) - g^*(z^*)\}.
\]

We say that

- the pair \( f, g \) satisfy **stable Fenchel duality** if for all \( x^* \in X^* \), there exists \( z^* \in X^* \) such that \((f + g)^*(x^*) = f^*(x^* - z^*) + g^*(z^*)\)

- the pair \( f, g \) satisfy the **classical Fenchel duality** if there exists \( z^* \in X^* \) such that \((f + g)^*(0) = f^*(-z^*) + g^*(z^*)\)

- the pair \( f, g \) is **totally Fenchel unstable** if \( f, g \) satisfy Fenchel duality but \( y^*, z^* \in X^* \) and

\[
(f + g)^*(y^* + z^*) = f^*(y^*) + g^*(z^*) \implies y^* + z^* = 0.
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If the pair \( f, g \) satisfy stable Fenchel duality, then \( f, g \) satisfy Fenchel duality.
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If the pair $f, g$ satisfy stable Fenchel duality, then $f, g$ satisfy Fenchel duality.
One always has

$$\text{epi } f^* + \text{epi } g^* \subseteq \text{epi}(f + g)^*.$$
The pair $f, g$ satisfy **stable Fenchel duality** if and only if

$$\text{epi}(f + g)^* = \text{epi} f^* + \text{epi} g^*.$$
The pair $f, g$ satisfy **Fenchel duality** if and only if

$$\text{epi}(f + g)^* \cap \{0\} \times \mathbb{R} = (\text{epi} f^* + \text{epi} g^*) \cap \{0\} \times \mathbb{R}.$$
The pair $f, g$ is **totally Fenchel unstable** if and only if

$$\text{epi}(f + g)^* \cap (\{0\} \times \mathbb{R}) = (\text{epi } f^* + \text{epi } g^*) \cap (\{0\} \times \mathbb{R})$$

and there is no $x^* \in X^* \setminus \{0\}$ such that

$$\text{epi}(f + g)^* \cap (\{x^*\} \times \mathbb{R}) = (\text{epi } f^* + \text{epi } g^*) \cap (\{x^*\} \times \mathbb{R}).$$
Regularity conditions for Fenchel duality

Assume that $f, g : X \to \overline{\mathbb{R}}$ are proper convex functions such that $\text{dom } f \cap \text{dom } g \neq \emptyset$. In the literature there exist different classes of regularity conditions for stable Fenchel duality:

(i) $f$ is continuous at $x' \in \text{dom } f \cap \text{dom } g$;

Interior point regularity conditions:

(ii) $0 \in \text{int}(\text{dom } f - \text{dom } g)$;

(iii) $0 \in \text{core}(\text{dom } f - \text{dom } g)$ (Rockafellar, 1974);

(iv) $0 \in \text{sqri}(\text{dom } f - \text{dom } g)$ (Attouch, Brézis, 1986, Zălinescu, 1987).

Closedness-type regularity condition:

(v) $\text{epi } f^* + \text{epi } g^*$ is closed in the product topology of $(X^*, \omega(X^*, X)) \times \mathbb{R}$ (B., Wanka, 2006, Burachik, Jeyakumar, 2006).
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Closedness-type regularity condition:

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We have that

- condition \((i)\) \(\Rightarrow\) stable Fenchel duality;
- if \(f, g\) are lower semicontinuous and \(X\) is a Fréchet space, then \((ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow\) stable Fenchel duality;
- if \(f, g\) are lower semicontinuous, then \((v) \Leftrightarrow\) stable Fenchel duality.

**Example 1.** Let \(X = \mathbb{R}, f(x) = \frac{1}{2}x^2, \) if \(x \geq 0,\) and \(f(x) = +\infty,\) otherwise, and \(g = \delta_{(-\infty, 0]}\). Then \((i) - (iv)\) are not fulfilled, while \((v)\) is valid.

Consider the following regularity condition for Fenchel duality:

\[(vi)\]  \(f^* \square g^*\) is lower semicontinuous and exact at 0 (B., Wanka, 2006).

If \(f, g\) are lower semicontinuous, then \((v) \Rightarrow (vi) \Rightarrow\) Fenchel duality.

**Example 2.** Let \(X = \mathbb{R}^2, C = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0\},\)
\(D = \{(x_1, x_2)^T \in \mathbb{R}^2 : 2x_1 + x_2^2 \leq 0\}, f = \delta_C\) and \(g = \delta_D\).

Thus \(f, g\) satisfy Fenchel duality, \(f, g\) doesn’t satisfy stable Fenchel duality and the pair \(f, g\) is not totally Fenchel unstable.
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- condition (i) ⇒ stable Fenchel duality;
- if \( f, g \) are lower semicontinuous and \( X \) is a Fréchet space, then (ii) ⇔ (iii) ⇒ (iv) ⇒ stable Fenchel duality;
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**Example 2.** Let \( X = \mathbb{R}^2 \), \( C = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0\} \), \( D = \{(x_1, x_2)^T \in \mathbb{R}^2 : 2x_1 + x_2^2 \leq 0\} \), \( f = \delta_C \) and \( g = \delta_D \). Thus \( f, g \) satisfy Fenchel duality, \( f, g \) doesn’t satisfy stable Fenchel duality and the pair \( f, g \) is not totally Fenchel unstable.
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- condition (i) $\Rightarrow$ stable Fenchel duality;

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Consider the following regularity condition for Fenchel duality:

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11 Radu Ioan Boț

*On Fenchel Duality and Some of Its Variants*
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Let \( f, g : X \to \overline{\mathbb{R}} \) be proper functions with \( \text{dom } f \cap \text{dom } g \neq \emptyset \).

Algebraic result:

\[
(f + g)^*(x^*) = \min_{y^* \in X^*} \{ f^*(x^* - y^*) + g^*(y^*) \} \quad \forall x^* \in X^* \tag{1}
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if and only if

\[
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if and only if

\[
\partial_{\varepsilon}(f + g)(x) = \bigcup_{\varepsilon_1, \varepsilon_2 \geq 0, \varepsilon_1 + \varepsilon_2 = \varepsilon} (\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x)) \quad \forall x \in X \quad \forall \varepsilon \geq 0. \tag{3}
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On the other hand, (3) implies (take \( \varepsilon = 0 \))

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Two convex regularization schemes

- (Burger, Osher, 2004) Take $\mathcal{U}$ a Banach space, $\mathcal{H}$ a Hilbert space, $K : \mathcal{U} \rightarrow \mathcal{H}$ a linear continuous operator and the ill-posed operator equation

$$Ku = f,$$

where $f \in R(K)$.

Let $J : \mathcal{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function. Then $\bar{u} \in \mathcal{U}$ is called $J$-minimizing solution for (5) if it is an optimal solution of

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Source condition: the existence of Lagrange multiplier, i.e. $\exists \bar{w} \in \mathcal{H}$ with $K^*w \in \partial J(\bar{u}) \Rightarrow \bar{u}$ is a $J$-minimizing solution for (5).

Viceversa, if $\bar{u}$ is a $J$-minimizing solution for (5) and $f \in \text{sqri}(K(\text{dom } J))$ (interior-point regularity condition), then there exists a Lagrange multiplier $\bar{w} \in \mathcal{H}$ with $\langle \bar{w}, f - Ku \rangle = 0$ and $0 \in \partial (J - K^*\bar{w})(\bar{u}) \Leftrightarrow K^*\bar{w} \in \partial J(\bar{u})$. 
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(Chambolle, Lions, 1997) On $\Omega \subseteq \mathbb{R}^2$ a bounded and piecewise smooth open set consider the image recovery problem

$$u_0 = Au + n.$$ 

Here:

- $u_0$ is the image;
- $u$ is the transformed image;
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- $A : L^2(\Omega) \to L^2(\Omega)$ is a linear and continuous operator.

Problem: Knowing $u_0$, one has to recover $u$.

(Rudin, Osher, Fatemi, 1992): Solve the constrained minimization problem:

$$\inf \int_{\Omega} |Au| (\Omega). \quad \text{subject to} \quad \int_{\Omega} Au = \int_{\Omega} u_0,$$

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Define $J : L^p(\Omega) \to \mathbb{R} \cup \{+\infty\}$,
\[ J(u) = |Du|(\Omega), \text{ if } u \in BV(\Omega), \quad J(u) = +\infty, \text{ otherwise.} \]

Under some natural assumptions one can prove that (Chambolle, Lions, 1997) (6) is equivalent to
\[ \inf_{\int_{\Omega} |Au - u_0|^2 \leq \sigma^2} J(u). \quad (7) \]

(Chambolle, Lions, 1997) Assume that $\tilde{u}$ is an optimal solution of (7) and that $u_0 \in \text{cl}(L^2(\Omega) \cap A(BV(\Omega)))$. For $C = \mathbb{R}_+$ and $g(u) = \|Au - u_0\|^2 - \sigma^2$ the latter condition means in fact that
\[ \exists u' \in \text{dom } J : g(u') < 0 \text{ (Slater regularity condition).} \]

Thus there exists a Lagrange multiplier $\bar{\lambda} \geq 0$ such that
\[ \bar{\lambda}(\|A\tilde{u} - u_0\| - \sigma) = 0 \text{ and} \]
\[ 0 \in \partial(J + \bar{\lambda}(\|A \cdot - u_0\|^2 - \sigma^2))(\tilde{u}) = \partial J(\tilde{u}) + \bar{\lambda} \partial(\|A \cdot - u_0\|^2 - \sigma^2)(\tilde{u}) \]
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$$J(u) = |Du|(\Omega), \text{ if } u \in BV(\Omega), J(u) = +\infty, \text{ otherwise.}$$

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Totally Fenchel unstable functions

Consider $X$ a nontrivial real Banach space, $X^*$ its topological dual space and $X^{**}$ its bidual space. We have

- the canonical embedding of $X$ into $X^{**}$, $\hat{\cdot} : X \rightarrow X^{**}$, $\langle x^*, \hat{x} \rangle := \langle x, x^* \rangle$, for all $x \in X$ and $x^* \in X^*$
- if $C \subseteq X$ is convex, then $x \in C$ is a support point of $C$ if there exists $x^* \in X^* \setminus \{0\}$, such that $\sup \langle C, x^* \rangle = \langle x, x^* \rangle$.

**Example 3 (totally Fenchel unstable functions).** (Simons, 2007) Let $C \subseteq X$ be nonempty, bounded, closed and convex such that there exists an extreme point $x_0$ of $C$ which is not a support point of $C$. Take $f := \delta_{x_0-C}$ and $g := \delta_{C-x_0}$. Then $f, g$ satisfy Fenchel duality and the pair $f, g$ is totally Fenchel unstable.

**Example 4.** (Borwein, 2007) Let $X = l_2$, $1 < p < 2$ and $C = \{x \in l_2 : \|x\|_p \leq 1\}$. Then $x$ is an extreme point of $C$ $\iff \|x\|_p = 1$. An extreme point of $C$ is a support point of $C$ $\iff x \in l_2(p-1)$. Thus there are a plenty of extreme points of $C$ which are not support points.
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Regarding the functions defined in Example 3, Simons asks whether,

\[ \text{epi } f^* + \text{epi } g^* \supset X^* \times (0, +\infty) \]

or, equivalently,

\[ \text{epi } f^* + \text{epi } g^* = \{(0, 0)\} \cup (X^* \times (0, +\infty)) \].

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The reflexive case (B, 2007)

Let \( y^* \in X^* \) be arbitrary and \( h, k : X^* \to \mathbb{R} \), \( h(z^*) := f^*(z^*) \) and \( k(z^*) := g^*(y^* - z^*) \). Since \( h \) and \( k \) are continuous, by the Fenchel duality theorem,

\[
- \inf_{X^*} [h + k] = \min_{z \in X} [h^*(z) + k^*(-z)] = \min_{X} [\delta_{\{0\}} - y^*] = 0
\]

so, for all \( \varepsilon > 0 \), there exists \( z^* \in X^* \) such that \( h(z^*) + k(z^*) \leq \varepsilon \), thus \((y^*, \varepsilon) \in \text{epi } f^* + \text{epi } g^*\).

The nonreflexive case

**Problem 1.** (raised by Stephen Simons in his book "From Hahn-Banach to Monotonicity", Springer-Verlag, 2008)

Let \( C \) be a nonempty, bounded, closed and convex subset of a nonreflexive Banach space \( X \), \( x_0 \) be an extreme point of \( C \), \( y^* \in X^* \) and \( \varepsilon > 0 \). Then does there always exist \( M \geq 0 \) such that, for all \( u, v \in C \),

\[
M \|u + v - 2x_0\| \geq \langle v - x_0, y^* \rangle - \varepsilon
\]

The answer to this question is in the affirmative if and only if

\[
\text{epi } \delta^*_{x_0 - C} + \text{epi } \delta^*_{C - x_0} \supset X^* \times (0, +\infty).
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Weak*-extreme points

- We recall that $x_0$ is a weak*-extreme point of the bounded, closed and convex set $C \subseteq X$ if $\hat{x}_0$ is an extreme point of $\text{cl} \hat{C}$, where the closure is taken with respect to the weak* topology $w(X^{**}, X^*)$.

- If $x_0$ is a weak*-extreme point of $C$, then $x_0$ is an extreme point of $C$.

- (Phelps, 1961): must the image $\hat{x}$ of an extreme point of $x \in B_X$ (the unit ball of $X$) be an extreme point of $B_{X^{**}}$ (the unit ball of the bidual)? We recall that by the Goldstine Theorem the closure of $\hat{B}_X$ in the weak* topology $w(X^{**}, X^*)$ is $B_{X^{**}}$ (hence the generalization to a bounded, closed and convex set is natural).

- The first example of a Banach space and a point of its unit ball which is not weak*-extreme was suggested by K. de Leeuw and proved in (Y. Katznelson, 1961).

- In the spaces $C(X), L^p(1 \leq p \leq \infty)$, all the extreme points of the corresponding unit balls are weak*-extreme points.
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- (Phelps, 1961): must the image $\hat{x}$ of an extreme point of $x \in B_X$ (the unit ball of $X$) be an extreme point of $B_{X^{**}}$ (the unit ball of the bidual) if $X$ is not completely regular? We recall that by the Goldstine Theorem the closure of $\hat{B}_X$ in the weak* topology $w(X^{**}, X^*)$ is $B_{X^{**}}$ (hence the generalization to a bounded, closed and convex set is natural).

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The solution of the Problem 1 (B., Csetnek, Proc. of AMS, 2009)

For \( f : X \to \overline{\mathbb{R}} \) we define \( \hat{f} : X^{**} \to \overline{\mathbb{R}} \) by \( \hat{f}(x^{**}) = f(x) \), if \( x^{**} = \hat{x} \in \hat{X} \) and \( \hat{f}(x^{**}) = +\infty \), otherwise.

**Lemma 1.** We assume that \( f \) is convex with \( \text{dom} \ f \neq \emptyset \) and that \( \text{cl}(\hat{f}) \) is proper, where the lower semicontinuous hull is considered with respect to the topology \( w(X^{**}, X^*) \). Then \( f^{**} = \text{cl}(\hat{f}) \).

**Remark 2.** If \( C \subseteq X \) is a nonempty convex set, then by Lemma 1 follows that \( \delta^{**}_C = \delta_{\text{cl}(\hat{C})} \), where the closure is considered in the topology \( \omega(X^{**}, X^*) \). Thus Lemma 1 generalizes a result obtained in (Chakrabarty, Shunmugaraj, Zălinescu, 2007).

Consider \( f, g : X \to \overline{\mathbb{R}} \) proper convex functions with the following properties

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**Remark 2.** If \( C \subseteq X \) is a nonempty convex set, then by Lemma 1 follows that \( \delta^{**} = \delta_{\text{cl}(\hat{C})} \), where the closure is considered in the topology \( \omega(X^{**}, X^*) \). Thus Lemma 1 generalizes a result obtained in (Chakrabarty, Shunmugaraj, Zălinescu, 2007).

Consider \( f, g : X \to \overline{\mathbb{R}} \) proper convex functions with the following properties

- \( \text{dom} \ f \cap \text{dom} \ g \neq \emptyset \)
- \( \text{cl}(\hat{f}) \) and \( \text{cl}(\hat{g}) \) are proper
- \( f^{**}(0) + g^{**}(0) \geq 0 \)
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The solution of the Problem 1 (B., Csetnek, Proc. of AMS, 2009)

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On Fenchel Duality and Some of Its Variants
Theorem 1. We have $X^* \times (0, \infty) \subset \text{epi } f^* + \text{epi } g^*$ if and only if $\text{dom}(\text{cl}(\hat{f})) \cap \text{dom}(\text{cl}(\hat{g})) = \{0\}$.

Now consider

- $C$ a nonempty, bounded and convex subset of the Banach space $X$ and $x_0 \in C$
- $f := \delta_A$, $g := \delta_B$, where $A := x_0 - C$, $B := C - x_0$.

In this case we have

- $f^* = \sup\langle A, \cdot \rangle$, $g^* = \sup\langle B, \cdot \rangle$, $\text{dom}(f^*) = \text{dom}(g^*) = X^*$
- $\hat{f} = \delta_{\text{cl}(\hat{A})}$, $\text{cl}(\hat{f}) = \delta_{\text{cl}(\hat{A})}$, thus $f^{**} = \delta_{\text{cl}(\hat{A})}$. By the same argument, $g^{**} = \delta_{\text{cl}(\hat{B})}$

Theorem 2. We have $X^* \times (0, \infty) \subset \text{epi } f^* + \text{epi } g^*$ if and only if $x_0$ is a weak*-extreme point of $C$.

Remark 3. The closedness of the set $C$, requested in (Simons, 2008), is not needed anymore for this result.
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The finite dimensional case

**Problem 2.** (raised by Stephen Simons in his book "From Hahn-Banach to Monotonicity", Springer-Verlag, 2008)
Do there exist a nonzero finite dimensional Banach space \( X \) and \( f, g : X \to \mathbb{R} \) proper and convex functions such that the pair \( f, g \) is totally Fenchel unstable?

The solution of the Problem 2 (B., Löhne, Math. Prog., to appear)
For all \( x^*, y^* \in X^* \) it holds

\[
(f + g)^*(x^*) \leq f^*(x^* - y^*) + g^*(y^*). \tag{8}
\]

Therefore, a pair \( f, g \) of proper and convex functions is totally Fenchel unstable if and only if

\[
\exists y^* \in X^* : (f + g)^*(0) = f^*(-y^*) + g^*(y^*). \tag{9}
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\forall x^* \in X^* \setminus \{0\}, \forall y^* \in X^* : (f + g)^*(x^*) < f^*(x^* - y^*) + g^*(y^*). \tag{10}
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**Theorem 2.** There are no proper convex functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that the pair $f, g$ is totally Fenchel unstable.

**Comment.** The situation below is not possible:
Interpretation of the result. If two proper and convex functions $f, g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ satisfy Fenchel duality, then there exists at least one element $x^* \in \mathbb{R}^n \setminus \{0\}$, such that $f - \langle x^*, \cdot \rangle$ and $g$ (or $f$ and $g - \langle x^*, \cdot \rangle$) satisfy Fenchel duality, too.

Comment. We must have something like:
Comment. More precisely, for the concrete situation considered in the previous picture the following behavior can be noticed: