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Photo-Acoustics

Inverse Transport and Diffusion theories for PAT

Inverse Scattering theory for TAT

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Outline

1. Optics, Tomography, and Inverse Transport

2. Photoacoustic effect

3. Inverse transport theory (ITID) for PAT

4. Inverse diffusion theory (IDID) for PAT

5. Inverse scattering theory (IScatID) for TAT.

Optics and Tomography

Optical tomography consists of sending **Photons** (typically NIR with $\lambda \sim 1\mu m$) into tissues and measuring **outgoing densities** of photons.

The photon density $u(t,x,v)$ solves the following transport equation

$$\frac{1}{c}\frac{\partial u}{\partial t} + v \cdot \nabla u + \sigma(x) u = \int_V k(x,v',v) u(t,x,v') dv',$$
$$u(t,x,v) = f(t,x,v), \qquad \Gamma_- = \{(x,v) \in \partial X \times V,\, \nu(x) \cdot v < 0\}.$$

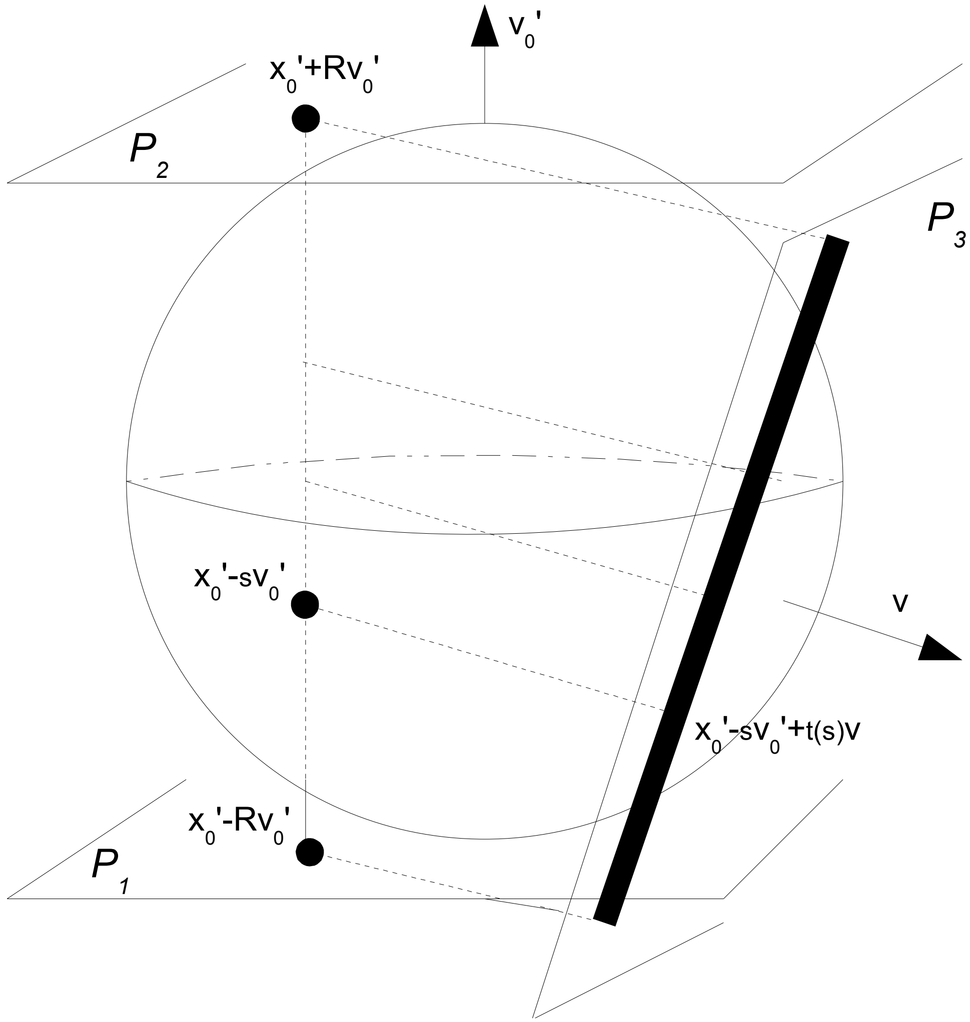
Measurements are $u(t,x,v)$ on Γ_+, the set of outgoing conditions $\{(x,v) \in \partial X \times V$ with $v \cdot \nu(x) > 0\}$.

Measurements (without time) are described by the **albedo operator**:

$$\mathcal{A} : L^1(\Gamma_-) \to L^1(\Gamma_+); \quad u_{|\Gamma_-} \mapsto \mathcal{A} u_{|\Gamma_-} = u_{|\Gamma_+}.$$

Inverse Transport with **Boundary Data** (ITBD) consists of reconstructing σ and k from (possibly partial) knowledge of $\mathcal{A}$.

Geometry of singularities



Singularities and Inverse Transport

The albedo operator has singularities that can be used to obtain stable reconstructions of the optical parameters σ and k. **Ballistic photons** are more singular than *single scattering photons*, which are often more singular than multiply scattered photons.

However, **these singularities cannot be measured** in many practical settings because either

(i) ballistic photons of intensity $e^{-\int_l \sigma}$ are **attenuated** too strongly

(ii) or only **angularly averaged** measurements are available, which destroy any singularity.

In many practical settings, inverse transport with boundary measurements results in a *severely ill-posed* problem, which results in a **severe loss of resolution**.

Summary of stabilities for ITBD

For steady-state angularly averaged measurements, we can reconstruct the low frequency component of $k(x)$ with **logarithmic**-type stability (high frequencies are exponentially unstable: ILL-POSED!).

A similar conclusion holds in this regime when the transport equation is approximated by a **diffusion equation** (diffuse optical tomography).

For time dependent (or modulation dependent) angularly averaged measurements, we can reconstruct $\sigma(x)$ and the spatial component of the scattering coefficient $k(x)\phi(v.v')$ with ϕ known, with **Hölder** stability.

For steady-state angularly dependent measurements, we reconstruct the Radon transform of $\sigma(x)$ in a **stable way** in dimension $n \geq 2$. In dimension $n \geq 3$, we reconstruct $k(x, v, v')$ in a **stable way**.

For time dependent angularly dependent measurements, the above results hold for $n \geq 2$ without any smallness assumption on k.

Photo-acoustics

We have seen that optical waves suffered from **low resolution** because of multiple scattering. However, optical waves are quite useful thanks to the **large contrast** of **optical absorption** between healthy and unhealthy tissues.

For ultrasound tomography, the opposite is true. Acoustic waves have very **high resolution** because they do not scatter. However, they do not scatter precisely because different tissues have very similar sound speeds. Ultrasounds is therefore a **low-contrast**, high-resolution method.

Photo-acoustics combines both waves to offer a **large-contrast, high resolution** imaging methodology.

Physics of Photo-acoustics

In photo-acoustics, externally generated radiation (low frequency photons in TAT or high frequency photons in PAT) is sent through tissues. Absorbed energy then **heats** up the underlying tissues, which results in *mechanical expansion* and generation of acoustic signals.

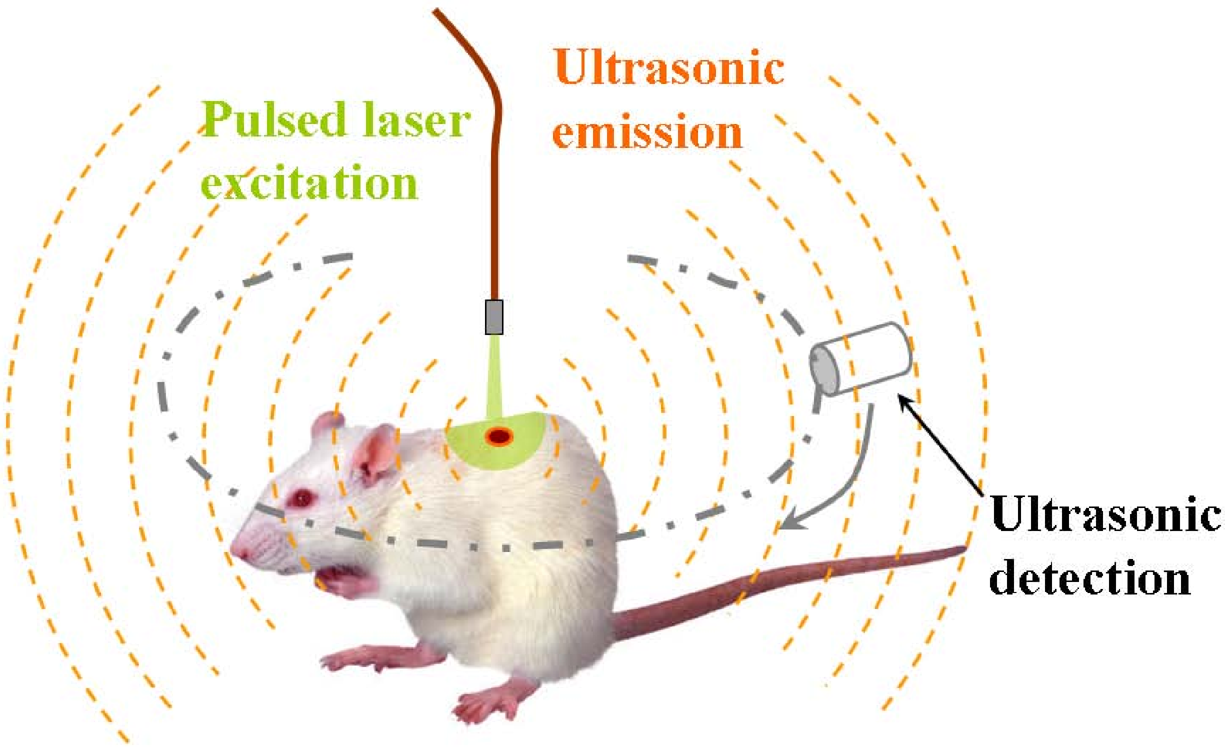
Such acoustic signals are measured by an array of transducers. They are then "sent back" (e.g. time reversed) into the (fairly homogeneous as far as sound speed is concerned) medium on a computer. This allows one to reconstruct the source of heating, which typically takes the form

$$H(x) := \sigma_a(x) I(x),$$

where σ_a is the attenuation coefficient and $I(x)$ is the radiation intensity.

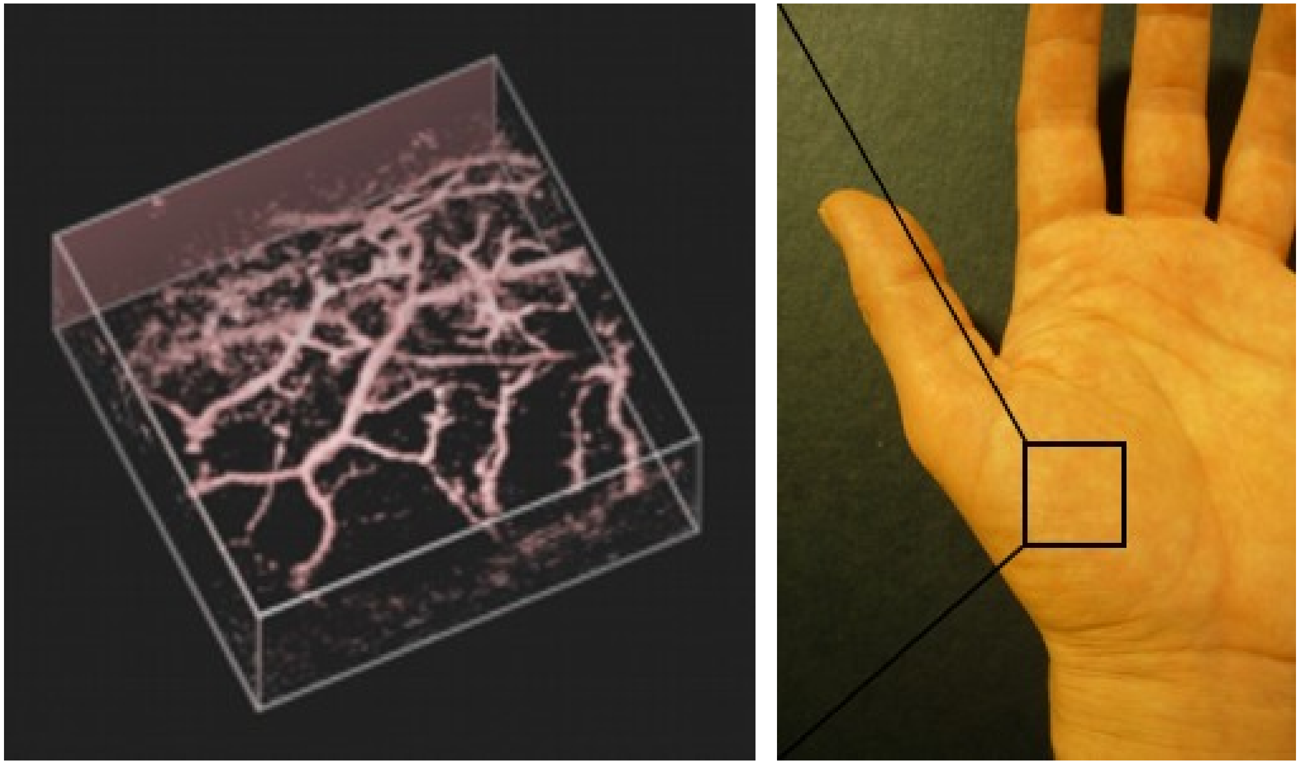
Since $I(x)$ depends on the optical parameters, it is interesting to understand which **optical parameters** may be reconstructed from knowledge of $H(x)$ pointwise.

Experiments in Photo-acoustics



Laser/ RF pulse → Absorption → Thermal expansion → Acoustic waves → Ultrasonic detection → Image formation

Experimental results



Courtesy UCL.

Transport Modeling of PAT

High frequency radiation is modeled by the following evolution equation:

$$\frac{1}{c}\frac{\partial}{\partial t}u(t,x,v) + Tu(t,x,v) = S(t,x,v), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^n,\ v \in \mathbb{S}^{n-1}$$

$$Tu = v \cdot \nabla_x u + \sigma(x,v)u - \int_{\mathbb{S}^{n-1}} k(x,v',v)u(t,x,v')dv'.$$

We define *total scattering* and *intrinsic attenuation* as

$$\sigma_s(x,v) = \int_{\mathbb{S}^{n-1}} k(x,v,v')dv', \qquad \sigma_a(x,v) = \sigma(x,v) - \sigma_s(x,v).$$

Radiation is thus modeled here by a **transport** equation.

Acoustic Modeling of PAT

Measurements are modeled by the following wave equation:

$$\frac{1}{c_s^2(x)}\frac{\partial^2 p}{\partial t^2} - \Delta p = \beta \frac{\partial}{\partial t} H(t,x), \qquad \text{where}$$

$$H(t,x) = \int_{\mathbb{S}^{n-1}} \sigma_a(x,v')u(t,x,v')dv'.$$

The measurements are typically $p(t,x)$ for $(t,x) \in \mathbb{R}_+ \times \partial X$. This is not enough to reconstruct $H(t,x)$. Modeling based on the fact that **light speed is much larger than sound speed** shows that

$$H(t,x) \sim \delta_0(t)H_0(x), \qquad H_0(x) = \int_{\mathbb{S}^{n-1}} \sigma_a(x,v)\Big(\int_{\mathbb{R}} u(t,x,v)dt\Big)dv.$$

Also, $\int_{\mathbb{R}} u(t,x,v)dt := u(x,v)$ solves a **steady-state** transport equation. The **inverse wave problem** is now "well-posed": we reconstruct $H_0(x)$ from wave measurements on $\mathbb{R}_+ \times \partial X$.

We assume here this first step **DONE** (cf. previous talks today).

Inverse Transport with Internal Data (ITID)

We can recast the transport equation as a boundary value problem

$$
\begin{aligned}
&v \cdot \nabla_x u + \sigma(x,v)u - \int_{\mathbb{S}^{n-1}} k(x,v',v)u(x,v')dv' = 0, \qquad (x,v) \in X \times \mathbb{S}^{n-1} \\
&u(x,v) = \phi(x,v) \qquad (x,v) \in \Gamma_- = \{(x,v) \in \partial X \times \mathbb{S}^{n-1}, \quad v \cdot \nu(x) < 0\},
\end{aligned}
$$

for all possible **illuminations** ϕ and consider the measurement operator

$$
\begin{aligned}
A: \ L^1(\Gamma_-, d\xi) &\rightarrow L^1(X) \\
\phi(x,v) &\mapsto A\phi(x) = H(x) := \int_{\mathbb{S}^{n-1}} \sigma_a(x,v)u(x,v)dv.
\end{aligned}
$$

Recall that $\sigma_a(x,v)$ is the intrinsic attenuation coefficient.

Assume A **known**. What can we reconstruct on $\sigma(x,v)$ and $k(x,v',v)$?

Joint work with Alexandre Jollivet and Vincent Jugnon.

Reconstruction of spatial structures

The analysis of singularities of the albedo operator A allows us to obtain the following stability results assuming the symmetry relations $\sigma_a(x,v) = \sigma_a(x,-v)$ and $\sigma(x,v) = \sigma(x,-v)$.

Theorem. The spatial structure of attenuation and scattering is stably reconstructed from knowledge of A:

$$\begin{aligned} & \|\sigma - \tilde{\sigma}\|_{L^\infty(\mathbb{S}^{n-1};W^{-1,1}(X))} + \|\sigma_a - \tilde{\sigma}_a\|_{L^\infty(\mathbb{S}^{n-1};L^1(X))} \\ \leq C \quad & \|A - \tilde{A}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}. \end{aligned}$$

For σ, $\tilde{\sigma}$ in $L^\infty(\mathbb{S}^{n-1}, W^{r,p}(X))$ for $p > 1$, $r > -1$, then for $-1 \leq s \leq r$:

$$\|\sigma - \tilde{\sigma}\|_{L^\infty(\mathbb{S}^{n-1};W^{s,p}(X))} \leq C \|A - \tilde{A}\|_{\mathcal{L}(L^1(\Gamma_-,d\xi);L^1(X))}^{\frac{1}{p}\frac{r-s}{1+r}}.$$

Anisotropy and Henyey-Greenstein (HG) kernels

The full scattering kernel $k(x, v', v)$ cannot be reconstructed stably. Assume that anisotropy is modeled by the HG kernel

$$k(x,\lambda) := \sigma_s(x)\frac{1-g^2(x)}{2\pi(1+g(x)^2-2g(x)\lambda)}, \text{ when } n=2,$$
$$k(x,\lambda) := \sigma_s(x)\frac{1-g^2(x)}{4\pi(1+g(x)^2-2g(x)\lambda)^{\frac{3}{2}}}, \text{ when } n\geq 3,$$

where $g \in C_b(X)$ and $0 \leq g(x) < 1$ for a.e. $x \in X$.

Theorem. The degree of anisotropy $g(x)$ is uniquely and stably determined by the operator A provided $\sigma_s(x) > 0$ for a.e. $x \in X$.

Summary of Inverse Transport Theory for PAT with multiple illuminations: we can reconstruct three spatial coefficients (attenuation, scattering, HG kernel) from photo-acoustic measurements.

Diffusive Regime and PAT

In the often valid regime of **large scattering**, radiation is best modeled by

$$\begin{aligned} -\nabla \cdot D(x)\nabla I(x) + \sigma_a(x)I(x) &= 0 \qquad && x \in X \\ I(x) &= \phi(x) && x \in \partial X, \end{aligned}$$

where $I(x) = \int_{S^{n-1}} u(x,v)dv$ is the spatial density of photons and $D(x)$ is the diffusion coefficient.

Recall that $H(x) = \sigma_a(x)I(x)$. When D is *known*, we can solve for $I(x)$ and get $\sigma_a(x)$ (stably if ϕ is well behaved) from a unique measurement $H(x)$.

When $D(x)$ is **not known**, then **multiple measurements** are necessary to reconstruct (σ_a, D).

Inverse Diffusion with Internal Data (ISID)

Let us assume that the domain $X \subset \mathbb{R}^n$ has smooth boundary and is sufficiently convex. Then we have the following result.

Theorem. Assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ are in $\mathcal{M}$ with $D_{|\partial X} = \tilde{D}_{|\partial X}$,

$\mathcal{M} = \{(D, \sigma_a)$ s.t. $(\sqrt{D}, \sigma_a) \in Y \times C^{k+1}(\bar{X}),\ \|\sqrt{D}\|_Y + \|\sigma_a\|_{C^{k+1}(\bar{X})} \le M\}$

and $Y = H^{\frac{n}{2}+k+2+\varepsilon}(X)$ with $k \ge 3$. Let $d = H$ and $\tilde{d} = \tilde{H}$ be internal data as above for the coefficients (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$, respectively and with boundary conditions $g = (g_j)_{j=1,2}$. Then there is an **open set of illuminations** $g \in (C^{k,\alpha}(\partial X))^2$ such that the following stability holds:

$$\|D - \tilde{D}\|_{C^{k-1}(X)} + \|\sigma_a - \tilde{\sigma}_a\|_{C^{k-1}(X)} \le C\|d - \tilde{d}\|_{(C^k(X))^2}.$$

Summary: **Two well-chosen measurements suffice to reconstruct (D, σ_a) stably**.

Joint work with Gunther Uhlmann.

Derivation and reconstructions

Using a standard Liouville change of variables, we reconstruct the pair of coefficients ($\mu = \frac{\sigma_a}{\sqrt{D}}, q = -\frac{\Delta\sqrt{D}}{\sqrt{D}} - \frac{\sigma_a}{D}$) from measurements of the form $d_k = \mu u_k$ where u_k is solution to $(\Delta + q)u_k = 0$ in X with $u_k = g_k$ on ∂X for $k = 1, 2$ (ISID). Then we find that

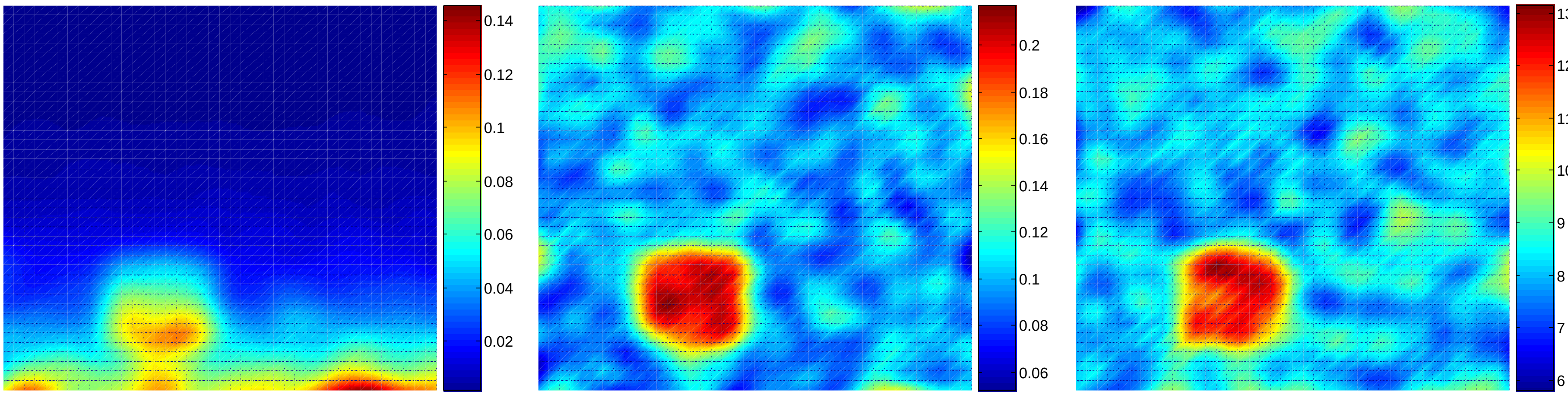
$$u_1 \Delta u_2 = u_2 \Delta u_1,$$

which after some algebra may be recast as

$$2(d_1\nabla d_2 - d_2\nabla d_1) \cdot \nabla\mu + (d_2\Delta d_1 - d_1\Delta d_2)\mu = 0.$$

Since μ is known on ∂X, we obtain a well-posed equation for μ provided that $\beta := d_1\nabla d_2 - d_2\nabla d_1$ is a **vector field** whose integral curves map any point $x \in X$ to a point $x_0 \in \partial X$.
Such "nice" vector fields are constructed by means of appropriate complex geometrical optics solutions for **well-chosen** illuminations $g_{1,2}$.

Numerical simulations in PAT by Kui Ren



Reconstructions of isotropic attenuation $\sigma_a(x)$ and scattering $\sigma_s(x)$ from **two measurements** in the transport regime.
Left: one of the data $H(x) = \sigma_a(x)I(x)$. Middle: reconstruction of σ_a.
Right: Reconstruction of σ_s. Real coefficients are square inclusions with values 0.2 and 12, respectively.

Low frequency radiation and TAT

When radiation is **low-frequency**, it is best modeled by the following system of equations for the electric field E:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E + \sigma(x)\mu\frac{\partial}{\partial t}E + \nabla\times\nabla\times E = S(t,x).$$

Here, $\sigma(x)$ is the **unknown** conductivity (attenuation). Assuming a scalar model for radiation propagation (E replaced by u) and assuming that $S = \phi(t)S(x)e^{-i\omega_c t}$ is a short pulse (typically 0.5μs) which is narrow band (typically 3GHz i.e. 0.3ns period), we can model radiation by

$$\Delta u + k^2 u + ik\sigma(x)u = 0, \qquad u = u^i + u^s, \qquad \omega_c = ck,$$

where u^i is a superposition of plane waves $e^{ik\xi\cdot x}$ with $\xi\in S^2$.

The deposited heat is then of the form $H(t,x) = \sigma(x)|u|^2(x)\phi^2(t)$.

Modeling measurements and IScatID

The internal data are then of the form $H(t,x) = \sigma(x)|u|^2(x)\phi^2(t)$.

Acoustic wave propagation is modeled by $\dfrac{1}{c_s^2(x)}\dfrac{\partial^2 p}{\partial t^2} - \Delta p = \beta \dfrac{\partial}{\partial t} H(t,x)$.

We assume the separation of scales

$$H(t,x) = H(x)\delta_0(t), \;\; H(x) = \sigma(x) \int_{\mathbb{R}} |u(t,x)|^2 dt = \sigma(x)|u(x)|^2 \int_{\mathbb{R}} \phi^2(t) dt.$$

We thus consider acoustic signals that propagate at a larger time scale than the pulse of radiation. A pulse that propagates in the medium in $0.5\mu s$ travels $0.75mm$ at (sound) speed $1.5\,10^3 m/s$. Any spatial scale below $0.75mm$ cannot be reconstructed stably in such a setting.

Here, we assume good temporal separation of scales.

The Inverse Scattering problem with Internal Data then becomes: reconstruct the **conductivity** $\sigma(x)$ from knowledge of $H(x) = \sigma(x)|u|^2(x)$, where $u = u^i + u^s$ solves the above Helmholtz equation.

IScatID: the problem

We first replace the inverse scattering problem by an **inverse boundary value** problem since traces $u_{\partial X}$ of solutions to the scattering problem are dense in $H^{\frac{1}{2}}(\partial X)$ for some smooth ∂X with supp$(\sigma) \subset X$.

Define $q(x) = k^2 + ik\sigma(x)$ on X and $Y = H^p(X)$ and $Z = H^{p-\frac{1}{2}}(\partial X)$ for $p > n$. Let $g \in Z$ be given and $u \in Y$ solution of

$$(\Delta + q)u = 0, \quad X, \qquad u = g \quad \partial X.$$

The internal data are given by

$$H(x) = \sigma(x)|u|^2.$$

The Inverse Scattering problem with Internal data is thus: **reconstruct** $\sigma(x)$ **from knowledge of** $H(x) = \sigma(x)|u|^2(x)$ **above**.

IScatID: the result

Theorem. Let σ and $\tilde{\sigma}$ be uniformly bounded functions in $Y = H^p(X)$ for $p > n$.

Then there is an **open set of illuminations** g in $Z = H^{p-\frac{1}{2}}(\partial X)$ such that $d(x) = \tilde{d}(x)$ in Y implies that $\sigma(x) = \tilde{\sigma}(x)$ in Y. Moreover, there exists a constant C independent of σ and $\tilde{\sigma}$ such that

$$\|\sigma - \tilde{\sigma}\|_Y \leq C\|d - \tilde{d}\|_Y.$$

The open set of illuminations g may be replaced by an appropriate **open set of illuminations** u^i of the initial inverse scattering problem.

The inverse scattering problem with internal data is therefore **well posed**.

IScatID: the derivation

The derivation is based on the construction of CGOs. Let $\rho \in \mathbb{C}^n$ with $|\rho|$ is large and $\rho \cdot \rho = 0$. Define $q(x) = k^2 + ik\sigma(x)$ on X appropriately extended to $x \in \mathbb{R}^n$. Then we can write the reconstruction of σ as finding the unique fixed point to the equation

$$\sigma(x) = e^{-(\rho+\bar{\rho})\cdot x} d(x) - \mathcal{H}_g[\sigma](x) \qquad \text{in } Y.$$

The functional $\mathcal{H}_g[\sigma]$ defined as

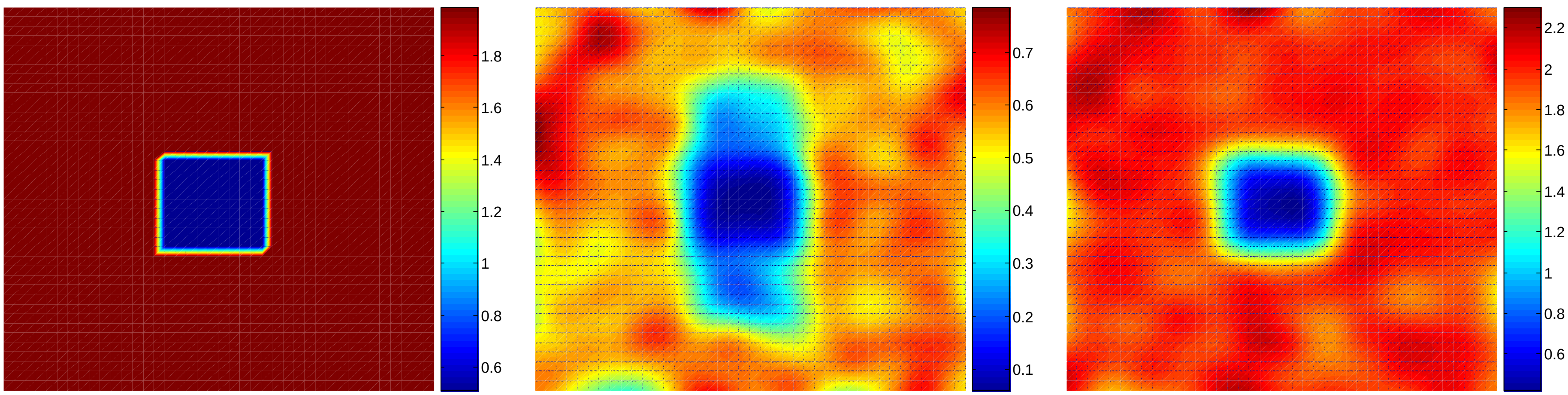
$$\mathcal{H}_g[\sigma](x) = \sigma(x)(\psi_g(x) + \overline{\psi_g}(x) + \psi_g(x)\overline{\psi_g}(x)),$$

is a contraction map for g in the open set described above, where ψ_g is defined as the solution to

$$(\Delta + 2\rho \cdot \nabla)\psi_g = -q(1 + \psi_g), \quad X, \qquad \psi_g = g \quad \partial X.$$

We first show that $\mathcal{H}_g[\sigma]$ is a contraction for CGOs $g = \psi_{\partial X}$ and then for illuminations such that $g - \psi_{|\partial X}$ is small.

Numerical simulations in TAT by Kui Ren



Left: exact attenuation. Middle: $H(x) = \sigma(x)|u|^2$. Right: Reconstructed σ (least-square).

Conclusions

Optical tomography suffers from *low resolution* and ultrasound imaging from *low contrast*. **Photo-acoustic** tomography offers the potential to combine high resolution with large contrast.

Assuming that the absorbed heat may be reconstructed accurately by solving a (well-posed) **inverse wave** problem, then the inverse problems in PAT and TAT become inverse problems with **internal data**.

We have shown uniqueness and stability results for the PAT-TAT problem in the setting of:

(i) **inverse transport** in PAT with multiple measurements
(ii) **inverse diffusion** in PAT with two well-chosen measurements
(iii) **inverse scattering** in TAT with one well-chosen illumination.