Can One Hear the Heat of a Body?
A Survey of Mathematics of Thermo- and Photoacoustic Tomography

David Finch (Oregon State), Peter Kuchment (Texas A&M),
Leonid Kunyansky (Univ. Arizona)

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An outline

- Mathematical model
- Uniqueness of reconstruction
- Inversion formulas and procedures
- Stability
- Range description
- Limited view reconstruction
- Speed of sound recovery
Surveys of mathematics of TAT

- M. Agranovsky, P. K., L. Kunyansky, On reconstruction formulas and algorithms for the thermoacoustic tomography, Ch. 8 in "Photoacoustic imaging and spectroscopy," CRC Press. 2009
- D. Finch and Rakesh, The spherical mean value operator with centers on a sphere, Inv. Problems 23(2007), No. 6, S37–S50.
- D. Finch and Rakesh, Recovering a function from its spherical mean values in two and three dimensions, Ch. 7 in "Photoacoustic imaging and spectroscopy," CRC Press. 2009
1. **Thermoacoustic/Photoacoustic Tomography (TAT/PAT)**

Goal: recover EM energy absorption $f(x)$. Cancerous cells absorb several times more energy than the healthy ones $\Rightarrow$ high contrast. Also high resolution of ultrasound.

Contributors present: Ambartsoumian, Arridge, Bal, Burgholzer, Finch, Haltmeier, Hristova, Kuchment, Kunyansky, Li, Nguyen, Palamodov, Quinto, Scherzer, Stefanov, Uhlmann, Xu.
2. Mathematical model

\[
\begin{cases}
    p_{tt} = c^2(x) \Delta_x p, & t \geq 0, \quad x \in \mathbb{R}^3 \\
    p(x, 0) = f(x), & p_t(x, 0) = 0, \\
    p(y, t) = g(y, t) & \text{for } y \in S, \quad t \geq 0.
\end{cases}
\]

\(c(x)\) - sound speed; \(g(y, t)\) - data, \(S\) - observation surface.

- Given \(g\), find \(f\). Why is this problem solvable?
- \(c = 1\), \(\Leftrightarrow\) inversion of restricted spherical mean transform:

\[
R_S f(p, r) = \omega^{-1} \int_{|y-p|=r} f(y) d\sigma(y), \quad p \in S, \quad r \geq 0.
\]
• \( f(x) \) is NOT what’s needed (hear talks by Arridge and Bal).
• Ultrasound attenuation is neglected (hear Burgholzer’s talk).
• Detectors are assumed small omnidirectional. Other designs (Burgholzer, Haltmeier, Scherzer, et al) are integrating planar, line, and circular detectors. Mathematical models here are somewhat different.
• No free space outside \( S \) - see below.
3. **Uniqueness of reconstruction**

Assume $\text{supp } f$ - compact, $S$ - observation surface. Image to measured data operator $R_S : f \mapsto g$.

**Q.** Can one uniquely reconstruct $f$ from $g = R_S f$?

**A.1:** If $S$ is closed - **yes**, even for a variable speed (PK '93, Agranovsky-Berenstein-PK '96, Agranovsky-PK '07).

**A.2:** $2D$ and constant speed - **yes** for $S$ that does not fit into a Coxeter cross $\bigcup$ finite set (Lin-Pinkus’ conjecture '03, proof Agranovsky-Quinto '96):
Open: Any dim. and constant speed - conjectured yes, unless $S \subset$ zeros of a homogeneous harmonic polynomial $\cup$ algebraic set of codim $\geq 2$. (Partial results Agranovsky-Quinto, Ambartsoumian-P.K. '05 (using Finch-Patch-Rakesh))

Open: Analog (even in 2D) of A.-Q. for decaying functions. ($S$ closed - Agranovsky-Berenstein-P.K. '96)
Open: Hyperbolic plane analog (even for compactly supported functions).
4. **Inversion formulas and procedures**

- **Filtered backprojection (FBP)** (Finch-Patch-Rakesh ’04, Xu-Wang ’05, Finch-Haltmeier-Rakesh ’06, Kunyansky ’06, Nguyen ’09)

\[
f(y) = -\frac{1}{8\pi^2} \Delta y \int_S \frac{g(z,|z-y|)}{|z-y|} dA(z),
\]

\[
f(y) = -\frac{1}{8\pi^2} \int_S \left( \frac{1}{t} \frac{d^2}{dt^2} g(z, t) \right) \bigg|_{t=|z-y|} dA(z).
\]

Known for **constant speed** and **$S$-sphere** (cylinder and plane analogs exist).

Various formulas disagree outside the range.
A unified approach to them is given by Nguyen ’09.
• **Eigenfunctions series expansions** (Norton’s precursors ’80, ’81, Kunyansky ’06, Agranovsky-PK ’07)

$S = \partial B$, $\psi_k(x)$, $\lambda_k^2$ - eigenfunctions/eigenvalues of $A = -c^2(x)\Delta_D$

in $B$, speed $c$-non-trapping, $g$ - data.

Series expansion

$$f(x)|_B = \sum_k f_k \psi_k(x),$$

$$f_k = -\lambda_k^{-1} \int_0^\infty \int_S \sin(\lambda_k t) g(x,t) \frac{\partial \psi_k}{\partial \nu}(x) dx dt.$$

Works wonderfully when $S$ - cube (Kunyansky '06).
• **Time reversal** (Fink, Finch-Patch-Rakesh ’04, Xu-Wang ’04, Burgholzer et al ’07, Hristova-PK-Nguyen ’08, Hristova ’09, Stefanov-Uhlmann ’09)

$T$ – large, $p(x,T)$ – small inside $S$. Solve back in time:

\[
\begin{cases}
    p_{tt} = c^2(x) \Delta_x p, & t \geq 0, \quad x \in \mathbb{R}^3 \\
    p(x,T) = 0, & p_t(x,T) = 0, \\
    p(y,t) = g(y,t) & \text{for } y \in S, \quad t \geq 0.
\end{cases}
\]

Find at $t = 0$ approximation for $f(x) = p(x,0)$.

Exact when dimension $n > 1$ is odd, speed is constant.

Works approximately (estimates by Hristova ’09), best in odd dimensions with non-trapping speed.

Works for any $S$, any speed; allows the support outside $S$.

Easy to implement.
• **Parametrix** (Popov-Sushko, Xu-Wang, Burgholzer-Haltmeier-...)

• **Algebraic iterative procedures** (Anastasio, ...)

<table>
<thead>
<tr>
<th>Name</th>
<th>closed form</th>
<th>exact</th>
<th>$S$</th>
<th>external sources</th>
<th>speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>FBP</td>
<td>+</td>
<td>+</td>
<td>sphere</td>
<td>-</td>
<td>constant</td>
</tr>
<tr>
<td>Series</td>
<td>-</td>
<td>+</td>
<td>any?</td>
<td>+</td>
<td>any?</td>
</tr>
<tr>
<td>Time reversal</td>
<td>-</td>
<td>-</td>
<td>any</td>
<td>+</td>
<td>any</td>
</tr>
<tr>
<td>Parametrix</td>
<td>+</td>
<td>-</td>
<td>any</td>
<td>-</td>
<td>const</td>
</tr>
<tr>
<td>Algebraic</td>
<td>-</td>
<td>-</td>
<td>any</td>
<td>+</td>
<td>any</td>
</tr>
</tbody>
</table>

- **Open:** Do closed form inversions exist for $S$ not a sphere?
- **Open:** Do closed form inversions exist that do not react to external sources?
Time reversal reconstruction of Shepp-Logan phantom
(Y. Hristova)
Parametrix reconstruction of a physical phantom (Y. Xu)
5. **Stability** Stability of reconstruction with **full data** and non-trapping speed is comparable to MRI or X-ray CT scan. (Palamodov '07, Stefanov-Uhlmann '09)

6. **Range description**

   *S* - sphere. **Range conditions** for $R_S$? **Moment conditions** (Lin& Pinkus '93, Agranovsky& Quinto '96, Patch '04) on data $g(p,r) = R_S f(p,r)$ ($p$ - center, $r$ - radius):

   \[ \forall k \in \mathbb{Z}, k \geq 0, \]

   \[
   G_k(\omega) = \int_0^\infty r^{2k} g(p,r) dr
   \]

   is a polynomial of degree at most $k$. **Incomplete set** of range conditions.

Data \( g(y,t) = R_S f = \int_S f(y + \omega t) d\omega \). \( S = \partial B \) – unit sphere. 

**Theorem** (AFK ’09) **TFAE** for a function \( g \in C_0^\infty (S \times [0,2]) \):

(a) \( g = R_S f \) for some \( f \in C_0^\infty (B) \).

(b) \( \forall (-\lambda^2, \psi_\lambda) – \text{eigenv/eigenf of } \Delta_D, \text{ one has } \int_{S \times [0,2]} g(x,t) \partial_\nu \psi_\lambda(x) j_{n/2-1}(\lambda t) t^{n-1} dx dt = 0. \)

(c) Let \( \hat{g}(x, \lambda) = \int g(x,t) j_{n/2-1}(\lambda t) t^{n-1} dt \). \( \forall m \in \mathbb{Z}, m^{th} \text{ spherical harmonic term } \hat{g}_m(x, \lambda) \text{ of } \hat{g}(x, \lambda) \text{ vanishes at non-zero zeros of Bessel function } J_{m+n/2-1}(\lambda). \)
7. **Limited view**

- **Uniqueness** (Agranovsky & Quinto '96, Finch-Patch-Rakesh '04, Stefanov & Uhlmann '09, Steinhauser '09)
- "Visible" singularities & instability. (Louis-Quinto '00, Xu-Wang-Ambartsumian-PK '04, '09, Hristova-PK-Nguyen '08, Stefanov-Uhlmann '09, Nguyen '09)

The "invisible" parts are blurred.
8. **Quality reconstructions from partial data**

Q.: Can one obtain quality reconstructions if the whole object is in “visible” zone?

A.: Yes, at least for constant sound speed (Kunyansky 2008; Patch ’04 used range conditions with less satisfactory results)

![Phantom (left) in the visible zone and its reconstruction (right) (Kunyansky ’08).](image)

Open: Variable speed case.
9. **Trapping:**
\[ H = \frac{c^2(x)}{2} |\xi|^2 \], bicharacteristics:
\[
\begin{cases}
  x'_t = \frac{\partial H}{\partial \xi} = c^2(x)\xi \\
  \xi'_t = -\frac{\partial H}{\partial x} = -\frac{1}{2} \nabla \left( c^2(x) \right) |\xi|^2 \\
  x|_{t=0} = x_0, \xi|_{t=0} = \xi_0.
\end{cases}
\]

Their projections to \( \mathbb{R}^n_x - \text{rays}. \)

**Non-trapping condition:** Rays (with \( \xi_0 \neq 0 \)) tend to \( \infty \) when \( t \to \infty \). Non-trapping \( \Rightarrow \) decay and eventual smoothing in any compact region.

Trapping “crater” speed \( c(x) = |x| \) for \( r_1 < |x| < r_2 \).

Worse is a parabolic crater.
Variable speed and “full view” (Hristova-PK-Nguyen ’08)

“Limited view” blurring effect due to trapping (crater (top) and parabolic (bottom) speeds).
10. **Finding the sound speed**

What if we get the speed wrong?

Phantom, sound speed, and their overlap.

Reconstructions: correct (left) and average (right) speed.
Can one find the speed?
Transmission ultrasound tomography before TAT (Xu-Wang).
Open: Is the speed $c$ uniquely determined by TAT data?
Can one recover it? (analog of a SPECT problem)
Successful numerical experiments (Anastasio-Zhang ’06, Yuan&Jiang ’05-...).
$f$ is supported strictly inside $S \Rightarrow$ constant speed is determined uniquely (PK-Nguyen ’08).
$f$ is supported inside $S \Rightarrow$ range conditions locally uniquely determine coefficient $\alpha$ in $\alpha c(x)$ (PK-Nguyen ’08).
Relation to the transmission eigenvalue problem (Finch ’08).
If $c_1(x) \geq c_2(x)$, TAT data coincide only if $c_1(x) = c_2(x)$ (Finch&Hickmann, Agranovsky ’09).
Linearized uniqueness result in $1D$ (Nguyen ’09), detection of constant speed in $1D$ (Finch&Hickman).