Convergence and Stability of the Inverse Scattering Series for Diffuse Waves

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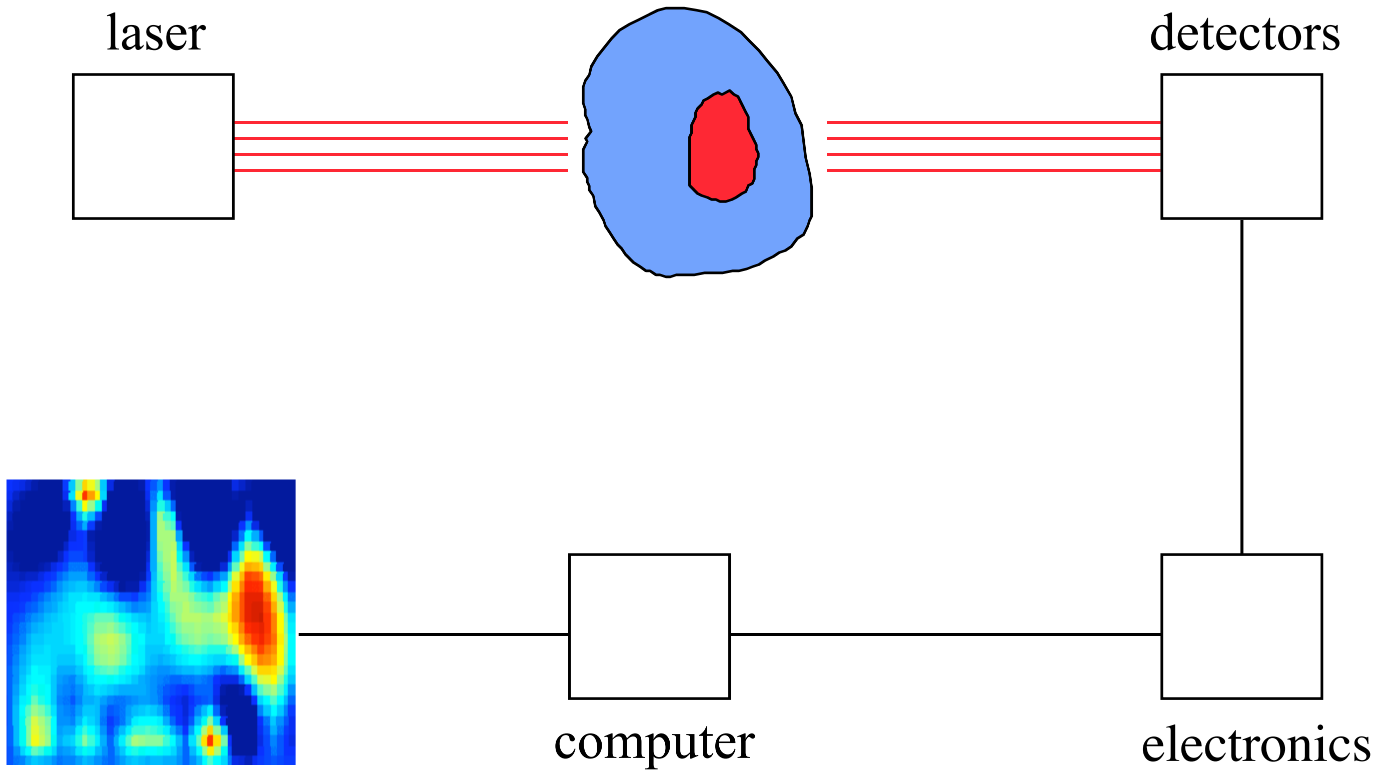
Diffusion Equation

$$-\nabla^2 u(x) + k^2(1+\eta(x))u(x) = 0 \ , \quad x \in \Omega$$

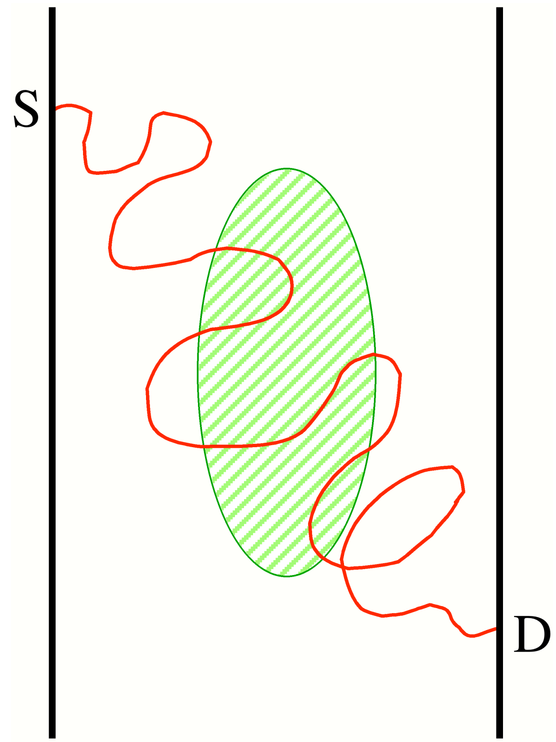
$$u(x) + \ell n(x) \cdot \nabla u(x) = 0 \ , \quad x \in \partial\Omega$$

- for optical waves in highly scattering media, such as clouds or breast tissue, the diffusion equations is an approximate model for the radiative transport equations

Optical tomography



Inverse problem



Problem: Reconstruct the optical absorption from boundary measurements.

Forward problem

- determine the energy density $u(x)$ for a given change in absorption η of compact support

Lipmann-Schwinger form

$$u(x) = u_i(x) - k^2 \int_\Omega G(x,y)u(y)\eta(y)dy \ , \quad x \in \Omega$$

- $u_i(x)$ energy density of incident wave
- G Green's function for given domain and BCs

in free space

$$G_0(x,y) = \frac{e^{-k|x-y|}}{4\pi|x-y|}$$

- G is like this plus smooth

when medium is illuminated by a point source

$$-\nabla^2 u_i(x) + k^2 u_i(x) = \delta(x - x_1) \ , \quad x \in \Omega \ , \ x_1 \in \partial\Omega$$

- gives background response at point x to a source at point x_1

Lipmann-Schwinger form

$$u(x) = u_i(x) - k^2 \int_\Omega G(x,y)u(y)\eta(y)dy \ , \quad x \in \Omega$$

- $u_i(x)$ energy density of incident wave
- G Green's function for given domain and BCs

iterate to get the well known Born series

$$u(x) = u_i(x) - k^2 \int_\Omega G(x,y)\eta(y)u_i(y)dy$$

$$+ k^4 \int_{\Omega\times\Omega} G(x,y)\eta(y)G(y,y')\eta(y')u_i(y')dydy' + \cdots \ .$$

the first term in this series is the Born approximation

$$u(x) \approx u_i(x) - k^2 \int_\Omega G(x,y)\eta(y)u_i(y)dy$$

full series yields change in the data

$$\phi = K_1\eta + K_2\eta \otimes \eta + K_3\eta \otimes \eta \otimes \eta + \cdots$$

- as a functional series in the change in the absorption.

the jth term is defined by the operator

$$(K_j f)(x_1, x_2) = (-1)^{j+1} k^{2j} \int_{B_a \times \cdots \times B_a} G(x_1, y_1) G(y_1, y_2) \cdots G(y_{j-1}, y_j)$$
$$\times G(y_j, x_2) f(y_1, \ldots, y_j) dy_1 \cdots dy_j \ , \quad x_1, x_2 \in \partial\Omega \ .(2.9)$$

- where

$$K_j : L^\infty(B_a \times \ldots \times B_a) \rightarrow L^\infty(\partial\Omega \times \partial\Omega)$$

- and B_a is a ball containing the scatterer

one can also view the operator on

$$K_j : L^2(B_a \times \ldots \times B_a) \to L^2(\partial\Omega \times \partial\Omega)$$

convergence properties of the Born series were understood previously

- for optical tomography (Markel & Schotland, Inverse Problems 2007)
- for propagating waves see Colton & Kress

Here we find the radius of convergence of the Born series by examining the norm of K_j

$$\|K_j\|_\infty \leq \nu_\infty \mu_\infty^{j-1}$$

- where

$$\mu_\infty = \sup_{x \in B_a} k^2 \|G(x, \cdot)\|_{L^1(B_a)}$$

-

$$\nu_\infty = k^2 |B_a| \sup_{x \in B_a} \sup_{y \in \partial\Omega} |G(x, y)|^2$$

or in L2...

$$\|K_j\|_2 \le \nu_2 \mu_2^{j-1}$$

- where
- $$\mu_2 = \sup_{x \in B_a} k^2 \|G(x, \cdot)\|_{L^2(B_a)}$$

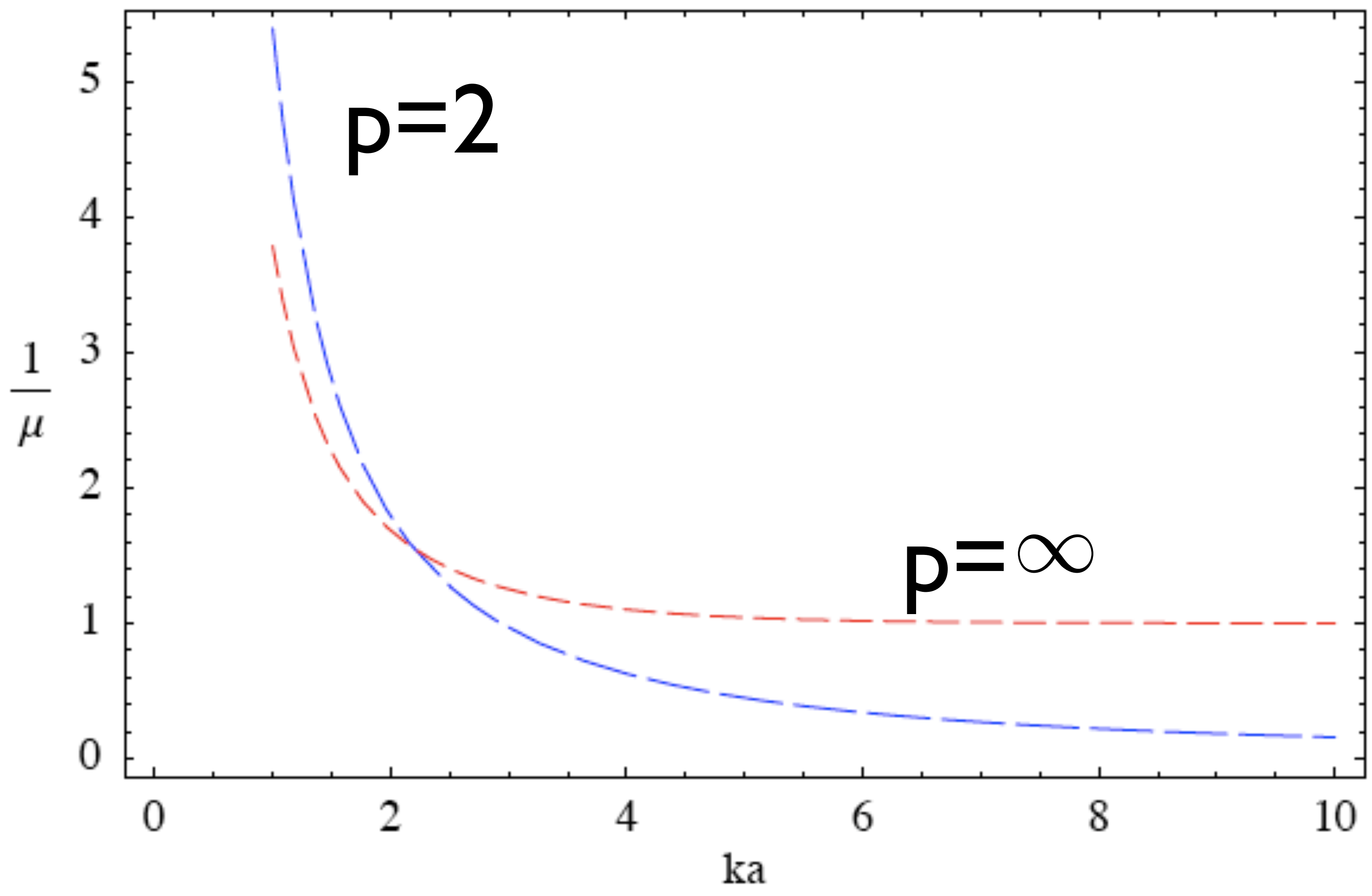
$$\nu_2 = k^2 |B_a|^{1/2} \sup_{x \in B_a} \|G(x, \cdot)\|^2_{L^2(\partial\Omega)}$$

this yields

Proposition 2.1. *The Born series converges in the L^p norm and obeys the error estimate (2.31) if the smallness condition $\|\eta\|_{L^p(B_a)} < 1/\mu_p$ holds.*

for $p = 2$ and $p = \infty$

radius of convergence as a function of ka, using free space Green's function



Inverse series for optimal tomography

V. Markel, J. O'Sullivan and J.C. Schotland, "Inverse problem in optical diffusion tomography. IV nonlinear inversion formulas," J. Opt. Soc. Am. A **20**, 903-912 (2003).

Assume we can express the change in absorption as a series in the change in data...

$$\eta = \mathcal{K}_1\phi + \mathcal{K}_2\phi \otimes \phi + \mathcal{K}_3\phi \otimes \phi \otimes \phi + \cdots$$

substitute the forward series for ϕ into this formal series for η

$$\phi = K_1 \eta + K_2 \eta \otimes \eta + K_3 \eta \otimes \eta \otimes \eta + \cdots$$

equating like tensor powers

$$
\begin{aligned}
\mathcal{K}_1 K_1 &= I \,, \\
\mathcal{K}_2 K_1 \otimes K_1 + \mathcal{K}_1 K_2 &= 0 \,, \\
\mathcal{K}_3 K_1 \otimes K_1 \otimes K_1 + \mathcal{K}_2 K_1 \otimes K_2 + \mathcal{K}_2 K_2 \otimes K_1 + \mathcal{K}_1 K_3 &= 0 \,, \\
\sum_{m=1}^{j-1} \mathcal{K}_m \sum_{i_1+\cdots+i_m=j} K_{i_1} \otimes \cdots \otimes K_{i_m} + \mathcal{K}_j K_1 \otimes \cdots \otimes K_1 &= 0 \,,
\end{aligned}
$$

This yields the formulas

$$
\begin{aligned}
\mathcal{K}_1 &= K_1^+ \ , \\
\mathcal{K}_2 &= -\mathcal{K}_1 K_2 \mathcal{K}_1 \otimes \mathcal{K}_1 \ , \\
\mathcal{K}_3 &= -\left(\mathcal{K}_2 K_1 \otimes K_2 + \mathcal{K}_2 K_2 \otimes K_1 + \mathcal{K}_1 K_3\right) \mathcal{K}_1 \otimes \mathcal{K}_1 \otimes \mathcal{K}_1 \ , \\
\mathcal{K}_j &= -\left(\sum_{m=1}^{j-1} \mathcal{K}_m \sum_{i_1+\cdots+i_m=j} K_{i_1} \otimes \cdots \otimes K_{i_m} \right) \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_1
\end{aligned}
$$

This gives us the inverse scattering series

$$\eta = \mathcal{K}_1 \phi + \mathcal{K}_2 \phi \otimes \phi + \mathcal{K}_3 \phi \otimes \phi \otimes \phi + \cdots$$

Remarks

- K_1 does not have a bounded inverse, so its inversion is ill-posed. We assume K_1^+ is some regularized pseudo-inverse.
- this inversion of the linearized problem K_1 is the only inversion necessary for this series.

The coefficients can be viewed as operators

$$\mathcal{K}_j : L^p(\partial\Omega \times \cdots \times \partial\Omega) \to L^p(B_a)$$

- for p=2 or ∞

Their norms can be bounded:

- If $(\mu_p + \nu_p)\|K_1^+\|_p < 1$ then

$$\|\mathcal{K}_j\|_p \leq ((\mu_p + \nu_p)\|K_1^+\|_p)^j$$

which tells us that the inverse scattering series converges when

$$\|K_1^+\|_p < 1/(\mu_p + \nu_p)$$

- and

$$\|K_1^+\phi\|_{L^p(B_a)} < 1/(\mu_p + \nu_p)$$

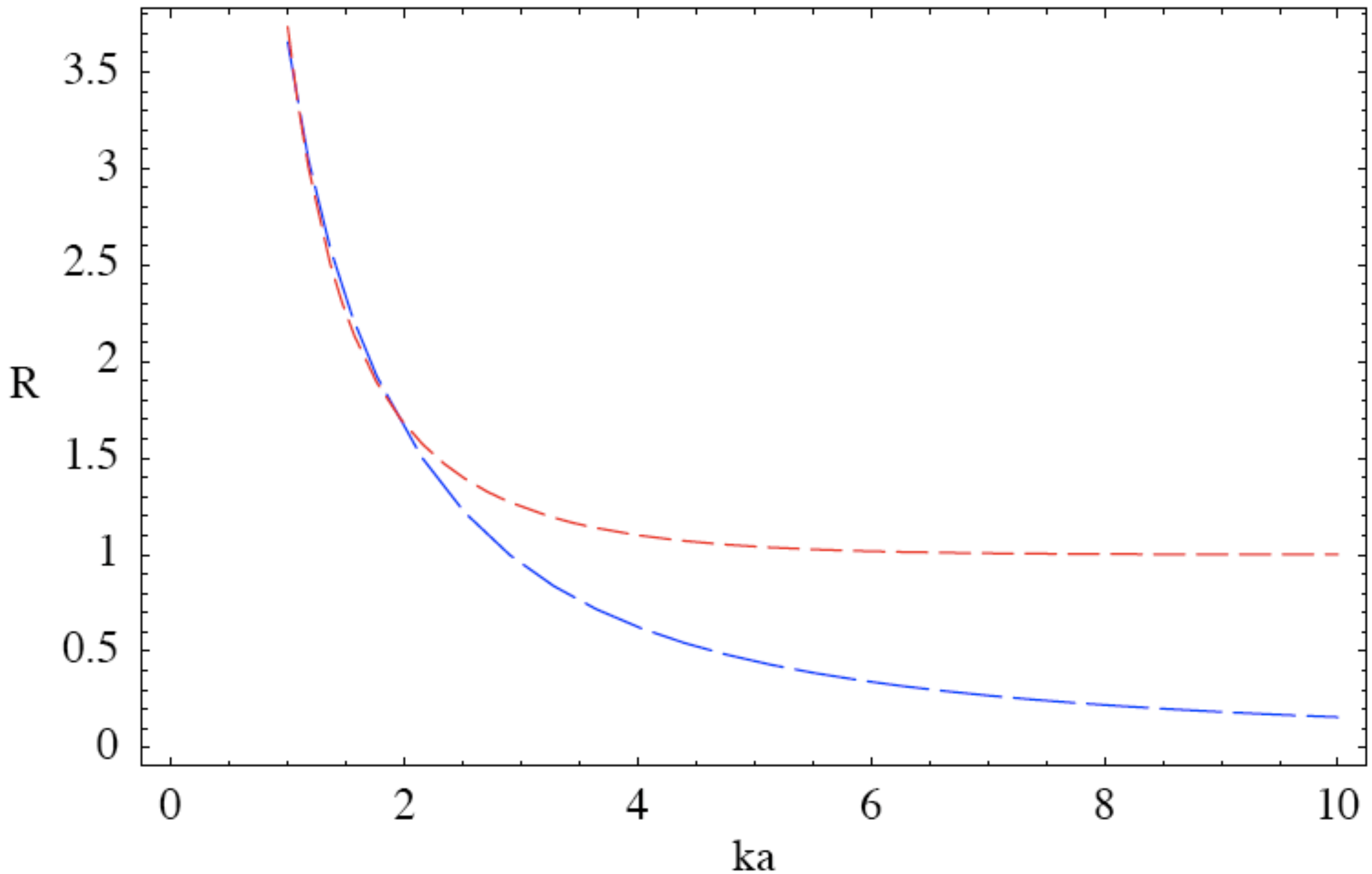


FIGURE 3. Radius of convergence of the inverse scattering series in the L^2 (— — —) and L^∞ (– – –) norms.

which also allows us to estimate the tail of the series

$$\left\|\tilde{\eta} - \sum_{j=1}^{N} \mathcal{K}_j \phi \otimes \cdots \otimes \phi \right\|_{L^p(B_a)} \leq C \frac{\left((\mu_p + \nu_p)\|\mathcal{K}_1\|_p \|\phi\|_{L^p(\partial\Omega \times \partial\Omega)}\right)^{N+1}}{1 - (\mu_p + \nu_p)\|\mathcal{K}_1\|_p \|\phi\|_{L^p(\partial\Omega \times \partial\Omega)}} \, ,$$

but the series converges to what exactly?

$$\|\eta - \tilde{\eta}\|_{L^p(B_a)} \leq C\|(I - K_1^+ K_1)\eta\|_{L^p(B_a)}$$

So one can characterize the error

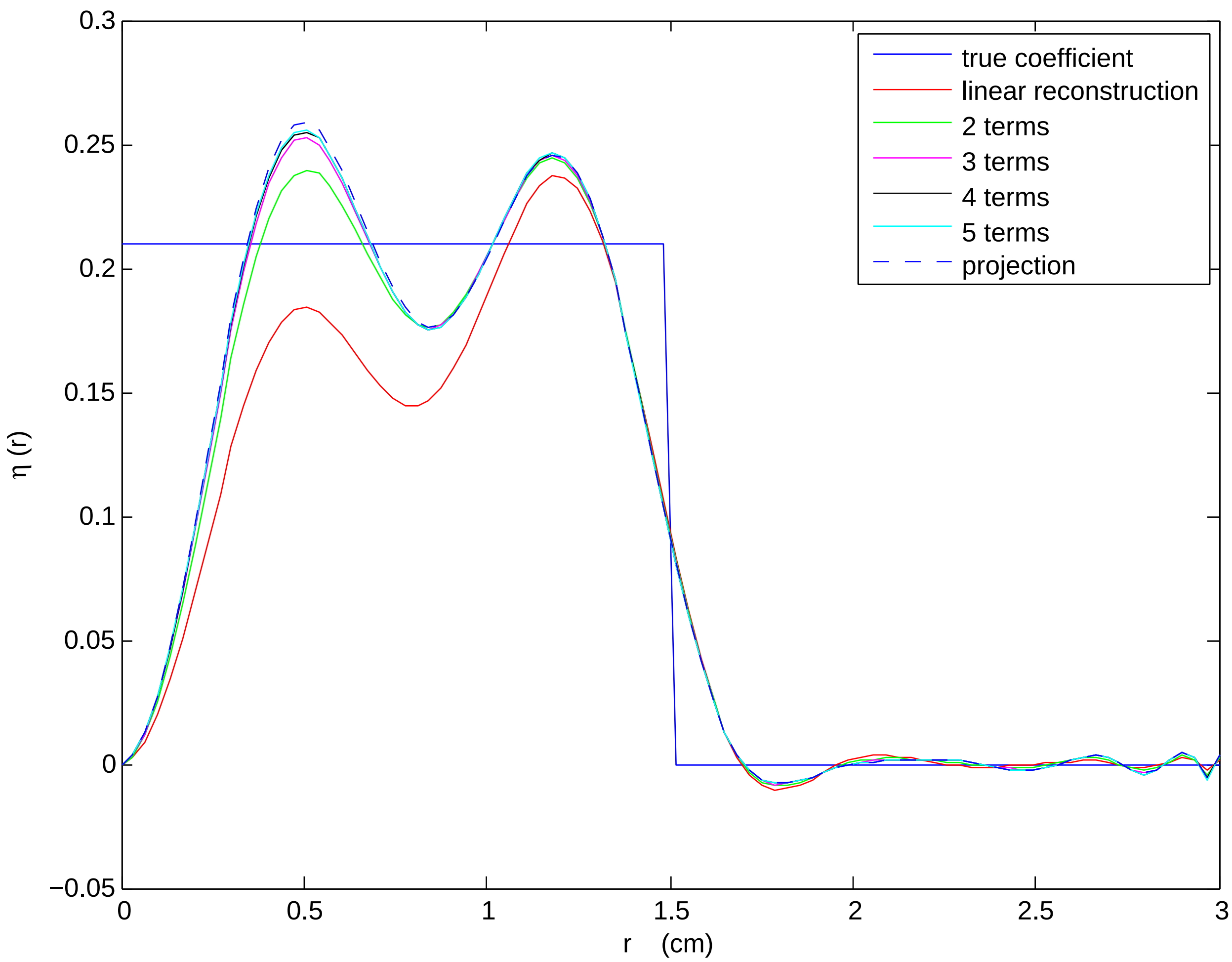
$$\left\|\eta - \sum_{j=1}^{N} \mathcal{K}_j \phi \otimes \cdots \otimes \phi \right\|_{L^p(B_a)} \leq C \|(I - \mathcal{K}_1 K_1)\eta\|_{L^p(B_a)} + \tilde{C}[(\mu_p + \nu_p)\|\mathcal{K}_1\|_p \|\phi\|]^N$$

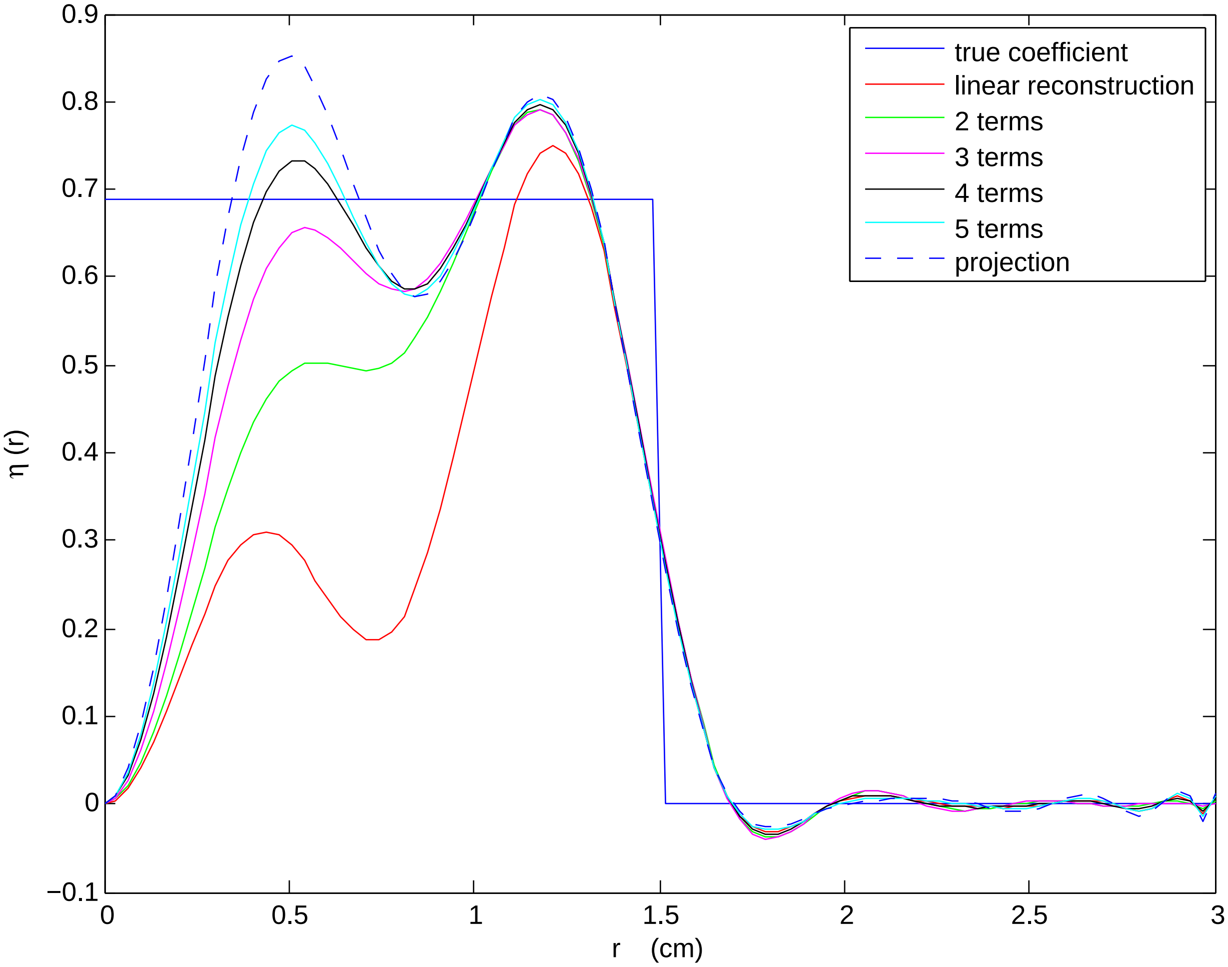
Stability

- For a fixed regularization, there is stability in the sum of the series with respect to perturbations in the data:

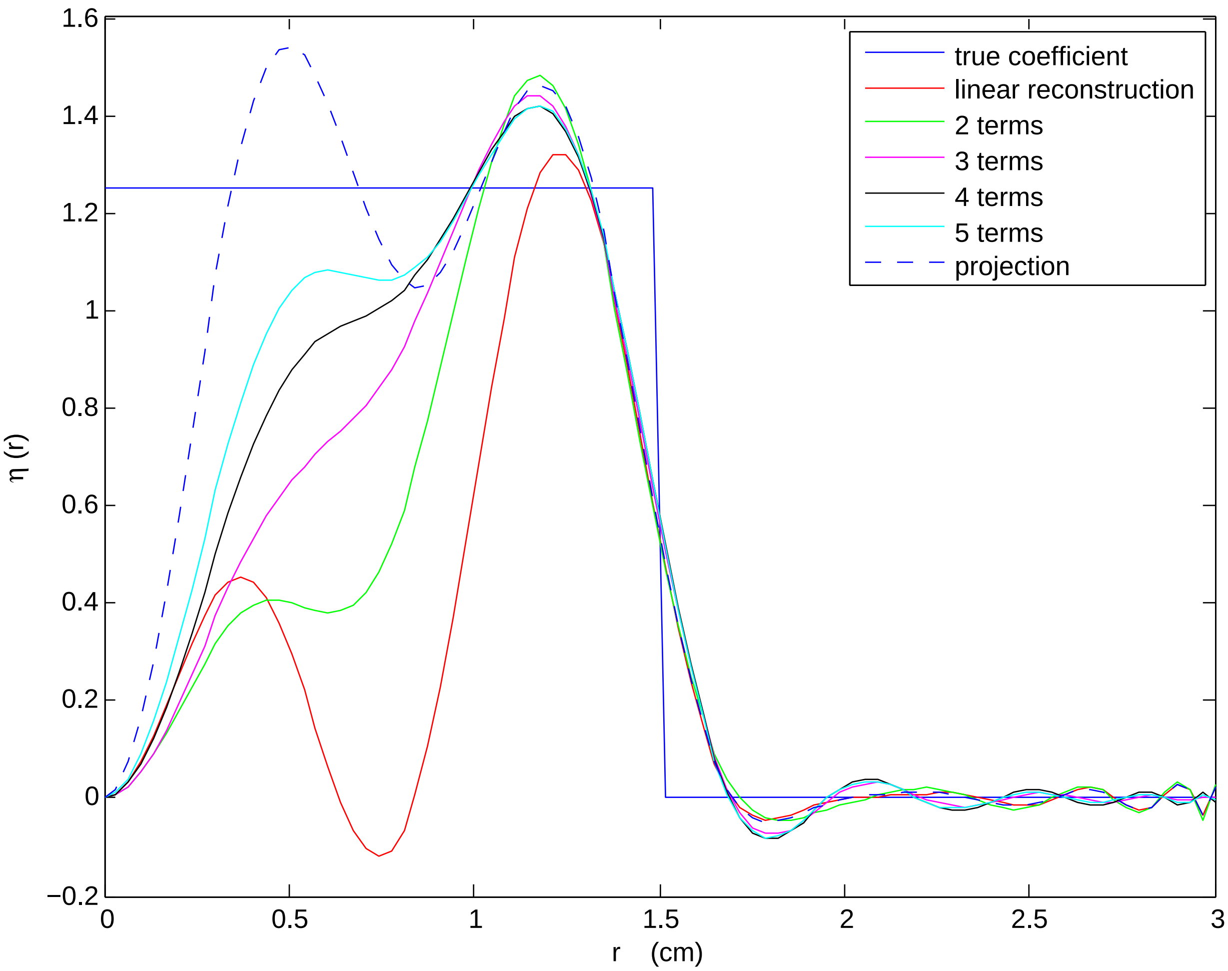
$$\|\eta_1 - \eta_2\|_{L^p(B_a)} \leq C\|\phi_1 - \phi_2\|_{L^p(\partial\Omega\times\partial\Omega)}$$

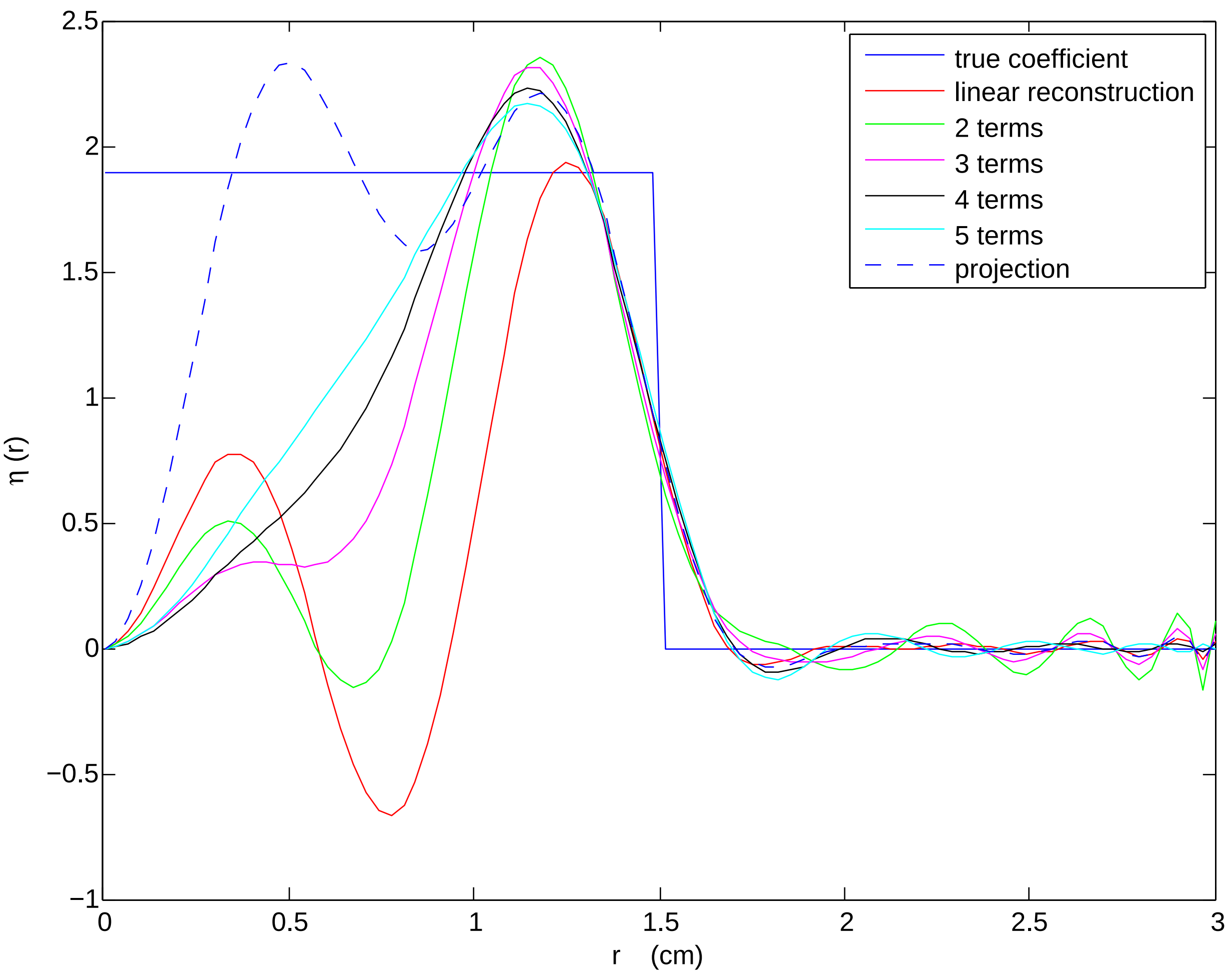
- if both series separately converge

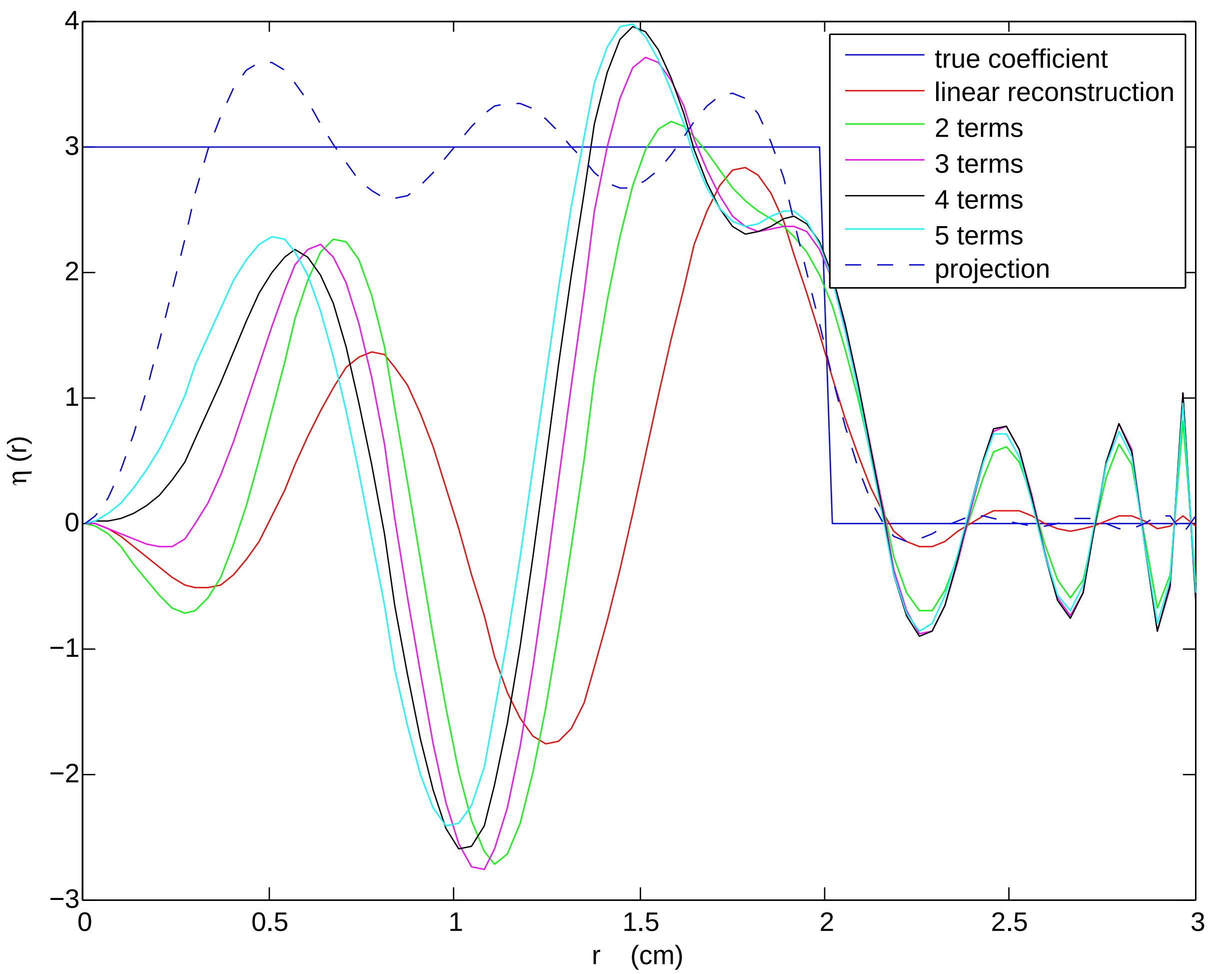

true coefficient
linear reconstruction
2 terms
3 terms
4 terms
5 terms
projection
η (r)
r (cm)
0.3
0.25
0.2
0.15
0.1
0.05
0
−0.05
0
0.5
1
1.5
2
2.5
3



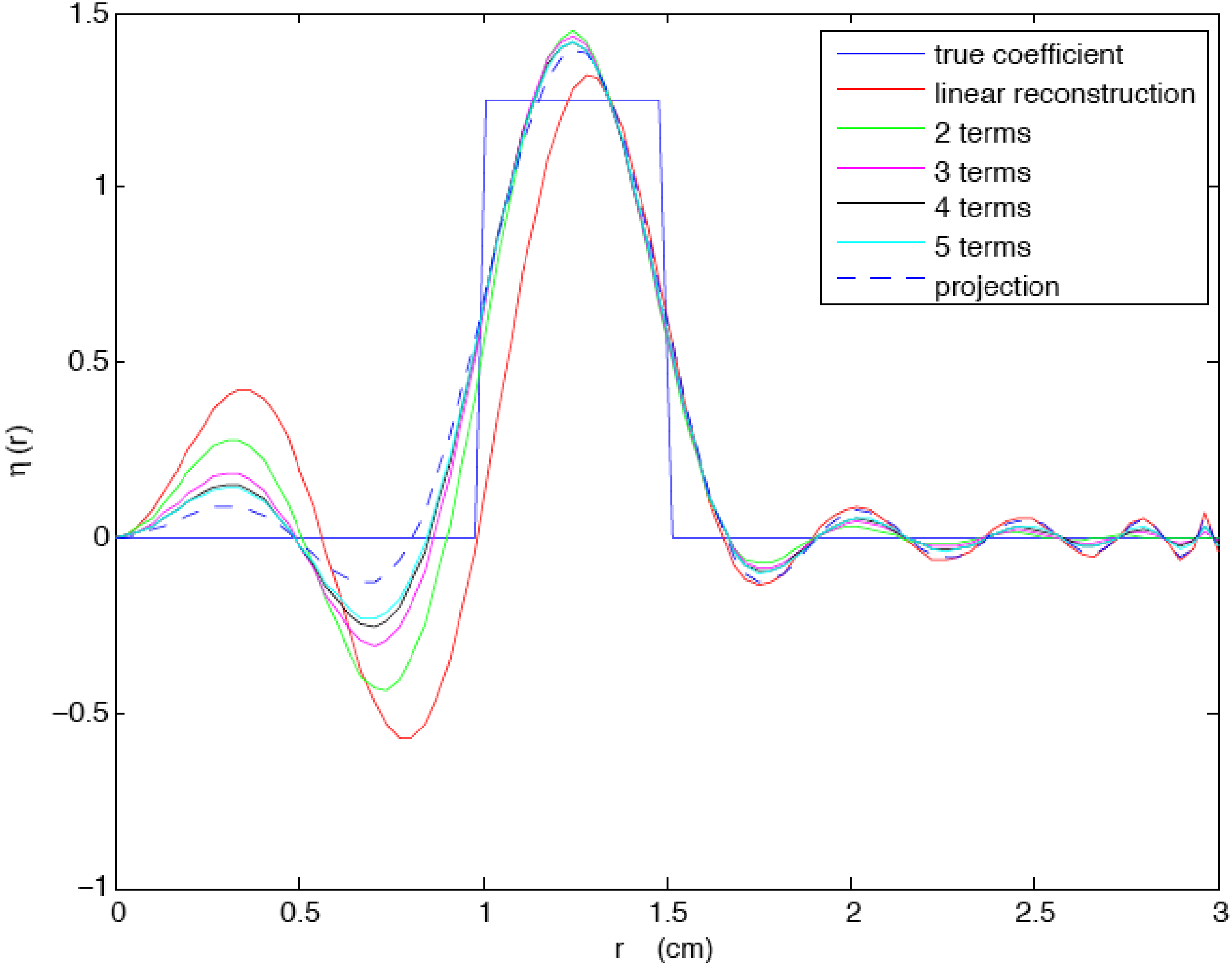
true coefficient
linear reconstruction
2 terms
3 terms
4 terms
5 terms
projection
η (r)
r (cm)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
0
0.5
1
1.5
2
2.5
3


true coefficient
linear reconstruction
2 terms
3 terms
4 terms
5 terms
projection
η (r)
r (cm)
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
0.5
1
1.5
2
2.5
3


true coefficient
linear reconstruction
2 terms
3 terms
4 terms
5 terms
projection
η (r)
r (cm)
2.5
2
1.5
1
0.5
0
−0.5
−1
0
0.5
1
1.5
2
2.5
3



true coefficient
linear reconstruction
2 terms
3 terms
4 terms
5 terms
projection
η (r)
r (cm)
4
3
2
1
0
−1
−2
−3
0
0.5
1
1.5
2
2.5
3



true coefficient
linear reconstruction
2 terms
3 terms
4 terms
5 terms
projection
η (r)
r (cm)
1.5
1
0.5
0
−0.5
−1
0
0.5
1
1.5
2
2.5
3

Electrical Impedance Tomography

- Can we apply the inverse Born series to the Calderon problem?

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in} \quad \Omega$$

- Apply currents and read potentials to determine interior conductivity

- Assume background conductivity

$$\sigma = 1$$

- Rewrite data in integral equation form

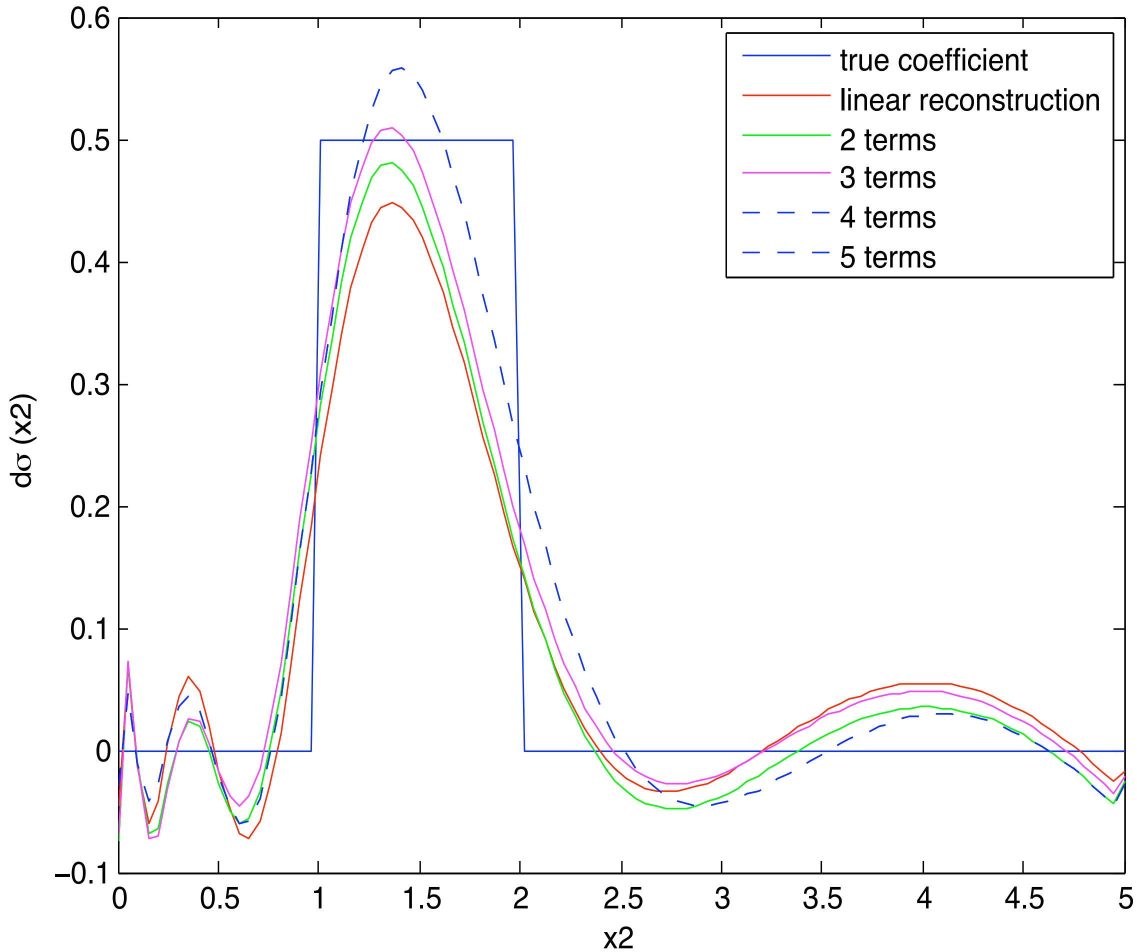
$$u(x) = u_0(x) - \int_\Omega \nabla_y G(x,y) \nabla u_0(y) \delta\sigma(y) dy$$

$$\delta\sigma = \sigma - 1$$

- Background potential in place of incident wave
- Linear term commonly used
- Born series more complex due to gradients
- Higher order forward operators involve powers of a singular integral operator.

Preliminary EIT results

- Domain 2-d halfspace
- Assume layered medium


true coefficient
linear reconstruction
2 terms
3 terms
4 terms
5 terms
dσ (x2)
x2
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Scalar Waves

$$\nabla^2 u(x) + k^2(1+\eta(x))u(x) = 0\ , \quad x \in \mathbb{R}^3$$

- where u obeys the outgoing Sommerfeld radiation condition
- Lippman-Schwinger form

$$u(x) = u_i(x) + k^2 \int_{\mathbb{R}^3} G(x,y)u(y)\eta(y)dy$$

Algebraically things are the same, but...

$$G(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

- Green's function is different, before

$$G_0(x, y) = \frac{e^{-k|x-y|}}{4\pi|x-y|}$$

convergence of Born series for scalar waves (Colton & Kress)

$$\|\eta\|_p \leq C_p/(ka)^2$$

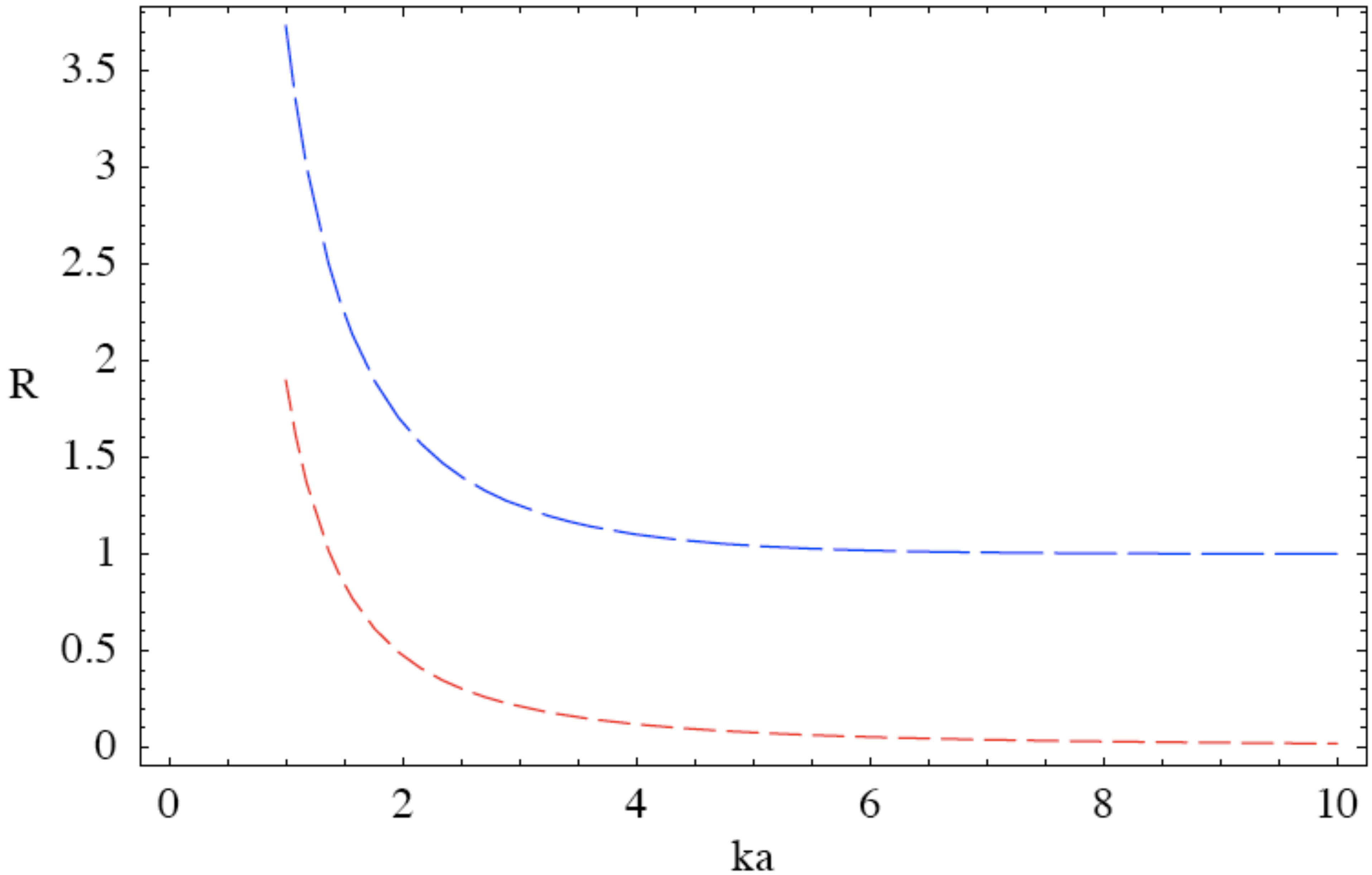


FIGURE 4. Radius of convergence of the inverse scattering series in the L^∞ norm for diffuse (— — —) and propagating waves (– – –).

Conclusions and future work...

- Inverse series appears to be well suited for diffuse waves in optimal tomography
- Maxwell in the near field?
- How pessimistic are the convergence requirements?
- More numerical studies